1

NUMBER SYSTEMS

From time immemorial human beings have been trying to have a count of their belongings- goods, ornaments, jewels, animals, trees, sheeps/goats, etc. by using various techniques

- putting scratches on the ground/stones

- by storing stones - one for each commodity kept/taken out.

This was the way of having a count of their belongings without having any knowledge of counting.

One of the greatest inventions in the history of civilization is the creation of numbers. You can imagine the confusion when there were no answers to questions of the type "How many?", "How much?" and the like in the absence of the knowledge of numbers. The invention of number system including zero and the rules for combining them helped people to reply questions of the type:

(i) How many apples are there in the basket?

(ii) How many speakers have been invited for addressing the meeting?

(iii) What is the number of toys on the table?

(iv) How many bags of wheat have been the yield from the field?

The answers to all these situations and many more involve the knowledge of numbers and operations on them. This points out to the need of study of number system and its extensions in the curriculum. In this lesson, we will present a brief review of natural numbers, whole numbers and integers. We shall then introduce you about rational and irrational numbers in detail. We shall end the lesson after discussing about real numbers.

OBJECTIVES

After studying this lesson, you will be able to

- *illustrate the extension of system of numbers from natural numbers to real (rationals and irrational) numbers*

- *identify different types of numbers;*
- *express an integer as a rational number;*
- *express a rational number as a terminating or non-terminating repeating decimal, and vice-versa;*
- *find rational numbers between any two rationals;*
- *represent a rational number on the number line;*
- *cites examples of irrational numbers;*
- *represent* $\sqrt{2}, \sqrt{3}, \sqrt{5}$ *on the number line;*
- *find irrational numbers betwen any two given numbers;*
- *round off rational and irrational numbers to a given number of decimal places;*
- *perform the four fundamental operations of addition, subtraction, multiplication and division on real numbers.*

1.1 EXPECTED BACKGROUND KNOWLEDGE

Basic knowledge about counting numbers and their use in day-to-day life.

1.2 RECALL OF NATURAL NUMBERS, WHOLE NUMBERS AND INTEGERS

1.2.1 Natural Numbers

Recall that the counting numbers 1, 2, 3, ... constitute the system of natural numbers. These are the numbers which we use in our day-to-day life.

Recall that there is no greatest natural number, for if 1 is added to any natural number, we get the next higher natural number, called its successor.

We have also studied about four-fundamental operations on natural numbers. For, example,

$4 + 2 = 6$, again a natural number;

$6 + 21 = 27$, again a natural number;

$22 - 6 = 16$, again a natural number, but

$2 - 6$ is not defined in natural numbers.

Similarly, $4 \times 3 = 12$, again a natural number

$12 \times 3 = 36$, again a natural number

$\frac{12}{2} = 6$ is a natural number but $\frac{6}{4}$ is not defined in natural numbers. Thus, we can say that

i) a) addition and multiplication of natural numbers again yield a natural number but

   b) subtraction and division of two natural numbers may or may not yield a natural number

ii) The natural numbers can be represented on a number line as shown below.

iii) Two natural numbers can be added and multiplied in any order and the result obtained is always same. This does not hold for subtraction and division of natural numbers.

1.2.2 Whole Numbers

(i) When a natural number is subtracted from itself we can not say what is the left out number. To remove this difficulty, the natural numbers were extended by the number zero (0), to get what is called the system of whole numbers

Thus, the whole numbers are

0, 1, 2, 3, ...

Again, like before, there is no greatest whole number.

(ii) The number 0 has the following properties:

$$a + 0 = a = 0 + a$$

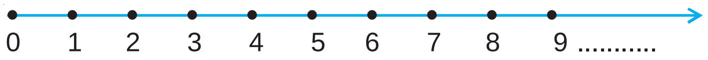
$a - 0 = a$ but $(0 - a)$ is not defined in whole numbers

$$a \times 0 = 0 = 0 \times a$$

Division by zero (0) is not defined.

(iii) Four fundamental operations can be performed on whole numbers also as in the case of natural numbers (with restrictions for subtraction and division).

(iv) Whole numbers can also be represented on the number line as follows:



1.2.3 Integers

While dealing with natural numbers and whole numbers we found that it is not always possible to subtract a number from another.

For example, (2 – 3), (3 – 7), (9 – 20) etc. are all not possible in the system of natural numbers and whole numbers. Thus, it needed another extension of numbers which allow such subtractions.

Thus, we extend whole numbers by such numbers as –1 (called negative 1), – 2 (negative 2) and so on such that

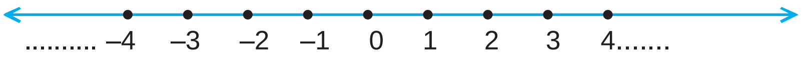
$$1 + (-1) = 0,\ 2 + (-2) = 0,\ 3 + (-3) = 0...,\ 99 + (-99) = 0,\ ...$$

Thus, we have extended the whole numbers to another system of numbers, called integers. The integers therefore are

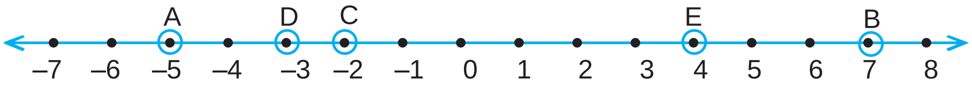
$$..., -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, ...$$

1.2.4 Representing Integers on the Number Line

We extend the number line used for representing whole numbers to the left of zero and mark points – 1, – 2, – 3, – 4, ... such that 1 and – 1, 2 and – 2, 3 and – 3 are equidistant from zero and are in opposite directions of zero. Thus, we have the integer number line as follows:



We can now easily represent integers on the number line. For example, let us represent – 5, 7, – 2, – 3, 4 on the number line. In the figure, the points A, B, C, D and E respectively represent – 5, 7, – 2, – 3 and 4.



We note here that if an integer a > b, then 'a' will always be to the right of 'b', otherwise vise-versa.

For example, in the above figure 7 > 4, therefore B lies to the right of E. Similarly, – 2 > – 5, therefore C (– 2) lies to the right of A (–5).

Conversely, as 4 < 7, therefore 4 lies to the left of 7 which is shown in the figure as E is to the left of B

∴ For finding the greater (or smaller) of the two integers a and b, we follow the following rule:

i) a > b, if a is to the right of b

ii) a < b, if a is to the left of b

Example 1.1: Identify natural numbers, whole numbers and integers from the following:-

15, 22, – 6, 7, – 13, 0, 12, – 12, 13, – 31

Solution: Natural numbers are: 7, 12, 13, 15 and 22

whole numbers are: 0, 7, 12, 13, 15 and 22

Integers are: – 31, – 13, – 12, – 6, 0, 7, 12, 13, 15 and 22

Example 1.2: From the following, identify those which are (i) not natural numbers (ii) not whole numbers

– 17, 15, 23, – 6, – 4, 0, 16, 18, 22, 31

Solution: i) – 17, – 6, – 4 and 0 are not natural numbers

ii) – 17, – 6, – 4 are not whole numbers

Note: From the above examples, we can say that

i) all natural numbers are whole numbers and integers also but the vice-versa is not true

ii) all whole numbers are integers also

You have studied four fundamental operations on integers in earlier classes. Without repeating them here, we will take some examples and illustrate them here

Example 1.3: Simplify the following and state whether the result is an integer or not

$12 \times 4,\ 7 \div 3,\ 18 \div 3,\ 36 \div 7,\ 14 \times 2,\ 18 \div 36,\ 13 \times (-3)$

Solution: $12 \times 4 = 48$; it is an integer

$7 \div 3 = \frac{7}{3}$; It is not an integer

$18 \div 3 = 6$; It is an integer

$36 \div 7 = \frac{36}{7}$; It is not an integer.

$14 \times 2 = 28$, It is an integer

$18 \div 36 = \frac{18}{36}$; It is not an integer

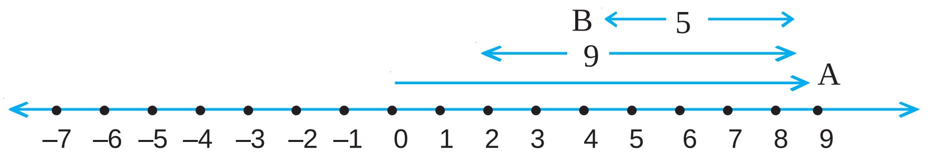
$13 \times (-3) = -39$; It is an integer

Example 1.4: Using number line, add the following integers:

(i) $9 + (-5)$ (ii) $(-3) + (-7)$

Solution:

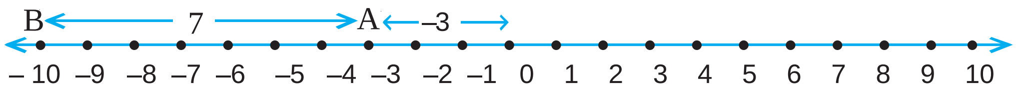
(i)



A represents 9 on the number line. Going 5 units to the left of A, we reach the point B, which represents 4.

$\therefore\ 9 + (-5) = 4$

(ii)



Starting from zero (0) and going three units to the left of zero, we reach the point A, which represents – 3. From A going 7 units to the left of A, we reach the point B which represents – 10.

$\therefore (-3) + (-7) = -10$

1.3 RATIONAL NUMBERS

Consider the situation, when an integer a is divided by another non-zero integer b. The following cases arise:

(i) When ‘a’ is a multiple of ‘b’

Suppose a = mb, where m is a natural number or integer, then $\frac{a}{b} = m$

(ii) When a is not a multiple of b

In this case $\frac{a}{b}$ is not an integer, and hence is a new type of number. Such a number is called a rational number.

Thus, a number which can be put in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, is called a rational number

Thus, $-\frac{2}{3}, \frac{5}{-8}, \frac{6}{2}, \frac{11}{7}$ are all rational numbers.

1.3.1 Positive and Negative Rational Numbers

(i) A rational number $\frac{p}{q}$ is said to be a positive rational number if p and q are both positive or both negative integers

Thus $\frac{3}{4}, \frac{5}{6}, \frac{-3}{-2}, \frac{-8}{-6}, \frac{-12}{-57}$ are all positive rationals.

(ii) If the integes p and q are of different signs, then $\frac{p}{q}$ is said to be a negaive rational number.

Thus, $\frac{-7}{2}, \frac{6}{-5}, \frac{-12}{4}, \frac{16}{-3}$ are all negaive rationals.

1.3.2 Standard form of a Rational Number

We know that numbers of the form

$$\frac{-p}{q}, \frac{p}{-q}, \frac{-p}{-q} \text{ and } \frac{p}{q}$$

are all rational numbers, where p and q are positive integers

We can see that

$$\frac{-p}{q} = -\left(\frac{p}{q}\right), \frac{-p}{-q} = \frac{-(-p)}{-(-q)} = \frac{p}{q}, \frac{p}{-q} = \frac{(-p)}{-(-q)} = \frac{-p}{q},$$

In each of the above cases, we have made the denominator q as positive.

A rational number $\frac{p}{q}$, where p and q are integers and $q \neq 0$, in which q is positive (or made positive) and p and q are co-prime (i.e. when they do not have a common factor other than 1 and –1) is said to be in standard form.

Thus the standard form of the rational number $\frac{2}{-3}$ is $\frac{-2}{3}$. Similarly, $\frac{-5}{6}$ and $\frac{-3}{5}$ are rational numbers in standard form.

Note: "A rational number in standard form is also referred to as "a rational number in its lowest form". In this lesson, we will be using these two terms interchangably.

For example, rational number $\frac{18}{27}$ can be written as $\frac{2}{3}$ in the standard form (or the lowest form).

Similarly, $\frac{25}{-35}$, in standard form (or in lowest form) can be written as $\frac{-5}{7}$ (cancelling out 5 from both numerator and denominator).

Some Important Results

(i) Every natural number is a rational number but the vice-versa is not always true.

(ii) Every whole number and integer is a rational number but vice-versa is not always true.

Example 1.5: Which of the following are rational numbers and which are not?

$$-2, \frac{5}{3}, -17, \frac{15}{7}, \frac{18}{5}, -\frac{7}{6}$$

Solution:

(i) –2 can be written as $\frac{-2}{1}$, which is of the form $\frac{p}{q}$, $q \neq 0$. Therefore, –2 is a rational number.

(ii) $\frac{5}{3}$ is a rational number, as it is of the form $\frac{p}{q}$, $q \neq 0$

(iii) –17 is also a rational number as it is of the form $\frac{-17}{1}$

(iv) Similarly, $\frac{15}{7}, \frac{18}{5}$ and $\frac{-7}{6}$ are all rational numbers according to the same argument

Example 1.6: Write the following rational numbers in their lowest terms:

(i) $\frac{-24}{192}$ (ii) $\frac{12}{168}$ (iii) $\frac{-21}{49}$

Solution:

(i) $\frac{-24}{192} = \frac{-3 \times 8}{3 \times 8 \times 8} = \frac{-1}{8}$

$-\frac{1}{8}$ is the lowest form of the rational number $\frac{-24}{192}$

(ii) $\frac{12}{168} = \frac{12}{12 \times 14} = \frac{1}{14}$

$\therefore \frac{1}{14}$ is the lowest form of the rational number $\frac{12}{168}$

(iii) $\frac{-21}{49} = \frac{-3 \times 7}{7 \times 7} = \frac{-3}{7}$

$\therefore \frac{-3}{7}$ is the lowest form of the rational number $\frac{-21}{49}$

Notes

1.4 EQUIVALENT FORMS OF A RATIONAL NUMBER

A rational number can be written in an equivalent form by multiplying/dividing the numerator and denominator of the given rational number by the same number.

For example

$$\frac{2}{3}=\frac{2\times 2}{3\times 2}=\frac{4}{6},\quad \frac{2}{3}=\frac{2\times 4}{3\times 4}=\frac{8}{12},\qquad \frac{2\times 8}{3\times 8}=\frac{16}{24}$$

$$\therefore \frac{4}{6},\quad \frac{8}{12},\quad \frac{16}{24} \text{ etc. are equivalent forms of the rational number } \frac{2}{3}$$

Similarly

$$\frac{3}{8}=\frac{6}{16}=\frac{21}{56}=\frac{27}{72}=\ldots$$

and $$\frac{4}{7}=\frac{8}{14}=\frac{12}{21}=\frac{28}{49}=\ldots$$

are equivalent forms of $\frac{3}{8}$ and $\frac{4}{7}$ respectively.

Example 1.7: Write five equivalent forms of the following rational numbers:

(i) $\frac{3}{17}$ (ii) $\frac{-5}{9}$

Solution:

(i) $$\frac{3}{17}=\frac{3\times 2}{17\times 2}=\frac{6}{34},\ \frac{3}{17}=\frac{3\times 4}{17\times 4}=\frac{12}{68},\ \frac{3\times(-3)}{17\times(-3)}=\frac{-9}{-51}$$

$$\frac{3\times 8}{17\times 8}=\frac{24}{136},\ \frac{3}{17}\times\frac{7}{7}=\frac{21}{119}$$

$\therefore$ Five equivalent forms of $\frac{3}{17}$ are

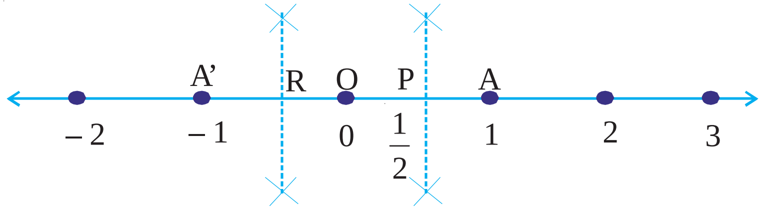
$$\frac{6}{34},\frac{12}{68},\frac{-9}{-51},\frac{24}{136},\frac{21}{119}$$

(ii) As in part (i), five equivalent forms of $\frac{-5}{9}$ are

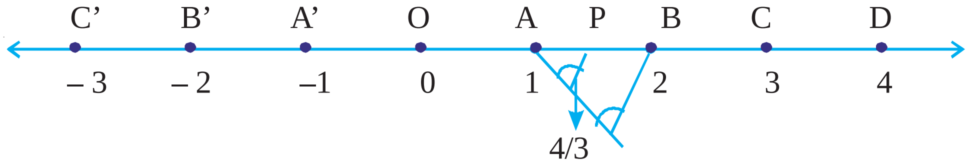
$$\frac{-10}{18}, \frac{-15}{27}, \frac{-20}{36}, \frac{-60}{108}, \frac{-35}{63}$$

1.5 RATIONAL NUMBERS ON THE NUMBER LINE

We know how to represent integers on the number line. Let us try to represent $\frac{1}{2}$ on the number line. The rational number $\frac{1}{2}$ is positive and will be represented to the right of zero. As $0 < \frac{1}{2} < 1$, $\frac{1}{2}$ lies between 0 and 1. Divide the distance OA in two equal parts. This can be done by bisecting OA at P. Let P represent $\frac{1}{2}$. Similarly R, the mid-point of OA', represents the rational number $-\frac{1}{2}$.



Similarly, $\frac{4}{3}$ can be represented on the number line as below:



As $1 < \frac{4}{3} < 2$, therefore $\frac{4}{3}$ lies between 1 and 2. Divide the distance AB in three equal parts. Let one of this part be AP

Now $\frac{4}{3} = 1 + \frac{1}{3} = \text{OA} + \text{AP} = \text{OP}$

The point P represents $\frac{4}{3}$ on the number line.

1.6 COMPARISON OF RATIONAL NUMBERS

In order to compare two rational numbers, we follow any of the following methods:

(i) If two rational numbers, to be compared, have the same denominator, compare their numerators. The number having the greater numerator is the greater rational number.

Thus for the two rational numbers $\frac{5}{17}$ and $\frac{9}{17}$, with the same positive denominator

$$17, \frac{9}{17} > \frac{5}{17} \text{ as } 9 > 5$$

$$\therefore \frac{9}{17} > \frac{5}{17}$$

(ii) If two rational numbers are having different denominators, make their denominators equal by taking their equivalent form and then compare the numerators of the resulting rational numbers. The number having a greater numerator is greater rational number.

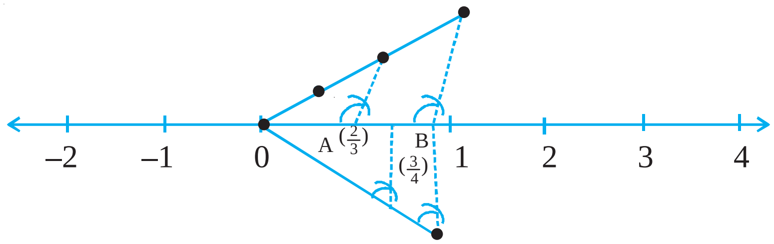
For example, to compare two rational numbers $\frac{3}{7}$ and $\frac{6}{11}$, we first make their denominators same in the following manner:

$$\frac{3\times11}{7\times11} = \frac{33}{77} \text{ and } \frac{9\times7}{11\times7} = \frac{42}{77}$$

As $42 > 33$, $\frac{42}{77} > \frac{33}{77}$ or $\frac{6}{11} > \frac{3}{7}$

(iii) By plotting two given rational numbers on the number line we see that the rational number to the right of the other rational number is greater.

For example, take $\frac{2}{3}$ and $\frac{3}{4}$, we plot these numbers on the number line as below:



$0<\frac{2}{3}<1$ and $0<\frac{3}{4}<1$. It means $\frac{2}{3}$ and $\frac{3}{4}$ both lie between 0 and 1. By the method of dividing a line into equal number of parts, A represents $\frac{2}{3}$ and B represents $\frac{3}{4}$

As B is to the right of A, $\frac{3}{4}>\frac{2}{3}$ or $\frac{2}{3}<\frac{3}{4}$

$\therefore$ Out of $\frac{2}{3}$ and $\frac{3}{4}$, $\frac{3}{4}$ is the greater number.

CHECK YOUR PROGRESS 1.1

1. Identify rational numbers and integers from the following:

   $4, \frac{-3}{4}, \frac{5}{6}, -36, \frac{12}{7}, \frac{3}{-8}, \frac{15}{7}, -6$

2. From the following identify those which are not :
   - (i) natural numbers
   - (ii) whole numbers
   - (iii) integers
   - (iv) rational numbers

   $-\frac{7}{4}, 16, \frac{-3}{7}, -15, 0, \frac{5}{17}, \frac{3}{-4}, -\frac{4}{3}$

3. By making the following rational numbers with same denominator, simplify the following and specify whether the result in each case is a natural number, whole number, integer or a rational number:

   (i) $3+\frac{7}{3}$ (ii) $-3+\frac{10}{4}$ (iii) $-8-13$ (iv) $12-12$

   (v) $\frac{9}{2}-\frac{1}{2}$ (vi) $2\times\frac{5}{7}$ (vii) $8\div 3$

4. Use the number line to add the following:-

   (i) 9 + (–7) (ii) (–5) + (–3) (iii) (–3) + (4)

5. Which of the following are rational numbers in lowest term?

$$\frac{8}{12}, \frac{5}{7}, \frac{-3}{12}, \frac{-6}{7}, \frac{2\sqrt{3}}{\sqrt{27}}, \frac{15}{24}$$

6. Which of the following rational numbers are integers?

$$-10, \frac{15}{5}, \frac{-5}{15}, \frac{13}{5}, \frac{27}{9}, \frac{7\times 3}{14}, \frac{-6}{-2}$$

7. Write 3 rational numbers equivalent to given rational numbers:

$$\frac{2}{5}, \frac{-5}{6}, \frac{17}{3}$$

8. Represent the following rational numbers on the number line.

$$\frac{2}{5}, \frac{3}{4}, \frac{1}{2}$$

9. Compare the following rational numbers by (i) changing them to rational numbers in equivalent forms (ii) using number line:

(a) $\frac{2}{3}$ and $\frac{3}{4}$ (b) $\frac{3}{5}$ and $\frac{7}{9}$ (c) $\frac{-2}{3}$ and $\frac{-1}{2}$

(d) $\frac{3}{7}$ and $\frac{5}{11}$ (e) $\frac{-7}{6}$ and $\frac{3}{2}$

1.7 FOUR FUNDAMENTAL OPERATIONS ON RATIONAL NUMBERS

1.7.1 Addition of Rational Numbers

(a) Consider the addition of rational numbers $\frac{p}{q}, \frac{r}{q}$

$$\frac{p}{q} + \frac{r}{q} = \frac{p+r}{q}$$

For example

(i) $\frac{2}{3} + \frac{5}{3} = \frac{2+5}{3} = \frac{7}{3}$

(ii) $\frac{3}{17} + \frac{9}{17} = \frac{3+9}{17} = \frac{12}{17}$

and (iii) $\frac{14}{3} + \left(\frac{-5}{3}\right) = \frac{14-5}{3} = \frac{9}{3} = 3$

(b) Consider the two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$.

$$\frac{p}{q}+\frac{r}{s}=\frac{ps}{qs}+\frac{rq}{sq}=\frac{ps+rq}{qs}$$

For example,

$$\text{(i)}\ \frac{3}{4}+\frac{2}{3}\quad=\frac{3\times3+4\times2}{4\times3}=\frac{9+8}{12}=\frac{17}{12}$$

$$\text{(ii)}\ -\frac{4}{5}+\frac{7}{8}\ =\frac{-4\times8+5\times7}{5\times8}=\frac{35-32}{40}=\frac{3}{40}$$

From the above two cases, we generalise the following rule:

(a) The addition of two rational numbers with common denominator is the rational number with common denominator and numerator as the sum of the numerators of the two rational numbers.

(b) The sum of two rational numbers with different denominators is a rational number with the denominator equal to the product of the denominators of two rational numbers and the numerator equal to sum of the product of the numerator of first rational number with the denominator of second and the product of numerator of second rational number and the denominator of the first rational number.

Let us take sone examples:

Example 1.8: Add the following rational numbers:

(i) $\frac{2}{7}$ and $\frac{6}{7}$ (ii) $\frac{4}{17}$ and $\frac{-3}{17}$ (iii) $-\frac{5}{11}$ and $\frac{-3}{11}$

Solution: (i) $\frac{2}{7}+\frac{6}{7}=\frac{2+6}{7}=\frac{8}{7}$

$$\therefore \frac{2}{7}+\frac{6}{7}=\frac{8}{7}$$

(ii) $\frac{4}{17}+\frac{(-3)}{17}=\frac{4+(-3)}{17}=\frac{4-3}{17}=\frac{1}{17}$

$$\therefore \frac{4}{17}+\frac{(-3)}{17}=\frac{1}{17}$$

Notes

(iii) $\left(-\frac{5}{11}\right)+\left(\frac{-3}{11}\right)=\frac{(-5)+(-3)}{11}=\frac{-5-3}{11}=\frac{-8}{11}$

$$\therefore \left(-\frac{5}{11}\right)+\left(-\frac{3}{11}\right)=-\frac{8}{11}$$

Example 1.9: Add each of the following rational numbers:

(i) $\frac{3}{4}$ and $\frac{1}{7}$ (ii) $\frac{2}{7}$ and $\frac{3}{5}$ (iii) $\frac{5}{9}$ and $-\frac{4}{15}$

Solution: (i) We have $\frac{3}{4}+\frac{1}{7}$

$$=\frac{3\times7}{4\times7}+\frac{1\times4}{7\times4}$$

$$=\frac{21}{28}+\frac{4}{28}=\frac{21+4}{28}$$

$$=\frac{25}{28}$$

$$\therefore \frac{3}{4}+\frac{1}{7}=\frac{25}{28} \text{ or} \left[\frac{3\times7+4\times1}{4\times7}=\frac{21+4}{28}=\frac{25}{28}\right]$$

(ii) $\frac{2}{7}+\frac{3}{5}$

$$=\frac{2\times5}{7\times5}+\frac{3\times7}{5\times7}$$

$$=\frac{10}{35}+\frac{21}{35}$$

$$=\frac{10+21}{35}=\frac{31}{35}$$

$$\therefore \frac{2}{7}+\frac{3}{5}=\frac{31}{35} \text{ or} \left[\frac{2\times5+3\times7}{35}=\frac{10+21}{35}=\frac{31}{35}\right]$$

(iii) $\frac{5}{9}+\frac{(-4)}{15}$

$$=\frac{5\times15}{9\times15}+\frac{(-4)\times9}{15\times9}$$

$$=\frac{75}{135}+\frac{(-36)}{135}$$

$$=\frac{75-36}{135}=\frac{39}{135}=\frac{3\times13}{3\times45}=\frac{13}{45}$$

$$\therefore\ \frac{5}{9}+\frac{(-4)}{15}=\frac{13}{45}\text{ or }\left[\frac{5\times15+9\times(-4)}{9\times15}=\frac{75-36}{135}=\frac{39}{135}=\frac{13}{45}\right]$$

1.7.2 Subtraction of Rational Numbers

(a) $\frac{p}{q}-\frac{r}{q}=\frac{p-r}{q}$

(b) $\frac{p}{q}-\frac{r}{s}=\frac{ps-qr}{qs}$

Example 1.10: Simplify the following:

(i) $\frac{7}{4}-\frac{1}{4}$ (ii) $\frac{3}{5}-\frac{2}{12}$

Solution: (i) $\frac{7}{4}-\frac{1}{4}=\frac{7-1}{4}=\frac{6}{4}=\frac{2\times3}{2\times2}=\frac{3}{2}$

(ii) $\frac{3}{5}-\frac{2}{12}=\frac{3\times12}{5\times12}-\frac{2\times5}{12\times5}$

$$=\frac{36}{60}-\frac{10}{60}=\frac{36-10}{60}$$

$$=\frac{26}{60}=\frac{13\times2}{30\times2}=\frac{13}{30}$$

1.7.3 Multiplication and Division of Rational Numbers

(i) Multiplication of two rational number $\left(\frac{p}{q}\right)$ and $\left(\frac{r}{s}\right)$, q ≠ 0, s ≠ 0 is the rational number $\frac{pr}{ps}$ where q$s\neq0$

$$=\frac{\text{product of numerators}}{\text{product of denominators}}$$

(ii) Division of two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$, such that q ≠ 0, s ≠ 0, is the rational number $\frac{ps}{qr}$, where qr ≠ 0

In other words $\left(\frac{p}{q}\right) \div \left(\frac{r}{s}\right) = \frac{p}{q} \times \left(\frac{s}{r}\right)$

Or (First rational number) × (Reciprocal of the second rational number)

Let us consider some examples.

Example 1.11: Multiply the following rational numbers:

(i) $\frac{3}{7}$ and $\frac{2}{9}$ (ii) $\frac{5}{6}$ and $\left(\frac{-2}{19}\right)$ (iii) $\frac{7}{13}$ and $\left(\frac{-2}{-5}\right)$

Solution: (i) $\frac{3}{7} \times \frac{2}{9} = \frac{3 \times 2}{7 \times 9} = \frac{3 \times 2}{7 \times 3 \times 3} = \frac{2}{21}$

$$\therefore \left(\frac{3}{7}\right) \times \left(\frac{2}{9}\right) = \frac{2}{21}$$

(ii) $\frac{5}{6} \times \left(\frac{-2}{19}\right) = \frac{5 \times (-2)}{6 \times 19}$

$$= -\frac{2 \times 5}{2 \times 3 \times 19} = -\frac{5}{57}$$

$$\therefore \left(\frac{5}{6}\right) \times \left(-\frac{2}{19}\right) = -\frac{5}{57}$$

(iii) $\frac{7}{13} \times \left(\frac{-2}{-5}\right) = \left(\frac{7}{13}\right)\left(\frac{-(-2)}{5}\right)$

$$= \frac{7}{13} \times \frac{2}{5} = \frac{7 \times 2}{13 \times 5} = \frac{14}{65}$$

$$\therefore \left(\frac{7}{13}\right) \times \left(\frac{-2}{-5}\right) = \frac{14}{65}$$

Example 1.12: Simply the following:

(i) $\left(\frac{3}{4}\right) \div \left(\frac{7}{12}\right)$ (ii) $\frac{9}{16} \div \left(-\frac{105}{12}\right)$ (iii) $\left(\frac{87}{27}\right) \div \left(\frac{29}{18}\right)$

Solution: (i) $\left(\frac{3}{4}\right) \div \left(\frac{7}{12}\right)$

$$= \left(\frac{3}{4}\right) \times \left(\frac{12}{7}\right) \qquad \left[\text{Reciprocal of } \frac{7}{12} \text{ is } \frac{12}{7}\right]$$

$$= \frac{3\times12}{4\times7} = \frac{3\times3\times4}{7\times4} = \frac{9}{7}$$

$$\therefore \left(\frac{3}{4}\right) \div \left(\frac{7}{12}\right) = \frac{9}{7}$$

(ii) $\left(\frac{9}{16}\right) \div \left(\frac{-105}{2}\right)$

$$\left(\frac{9}{16}\right) \times \left(\frac{2}{-105}\right) \qquad \left[\text{Reciprocal of } \frac{-105}{2} \text{ is } \frac{2}{-105}\right]$$

$$= -\frac{9\times2}{2\times8\times3\times35} = -\frac{3\times3\times2}{2\times8\times3\times35}$$

$$= \frac{-3}{8\times35} = \frac{-3}{280}$$

$$\therefore \left(\frac{9}{16}\right) \div \left(\frac{-105}{2}\right) = \frac{-3}{280}$$

(iii) $\left(\frac{87}{27}\right) \div \left(\frac{29}{18}\right)$

$$= \left(\frac{87}{27}\right) \times \left(\frac{18}{29}\right) = \frac{87}{27} \times \frac{18}{29} = \frac{29\times3\times2\times9}{9\times3\times29} = \frac{2}{1}$$

$$\therefore \left(\frac{87}{27}\right) \div \left(\frac{29}{18}\right) = \frac{2}{1}$$

CHECK YOUR PROGRESS 1.2

1. Add the following rational numbers:

   (i) $\frac{3}{7}, \frac{6}{7}$ (ii) $\frac{2}{15}, -\frac{6}{15}$ (iii) $\frac{3}{20}, \frac{-7}{-20}$ (iv) $\frac{1}{8}, \frac{3}{8}$

2. Add the following rational numbers:

   (i) $\frac{3}{2}, \frac{5}{3}$ (ii) $\frac{17}{7}, \frac{5}{9}$ (iii) $\frac{2}{5}, \frac{-5}{7}$

3. Perform the indicated operations:

   (i) $\left(-\frac{7}{8}+\frac{-5}{12}\right)+\frac{3}{16}$ (ii) $\left(\frac{7}{3}+\frac{3}{4}\right)+\left(-\frac{3}{5}\right)$

4. Subtract:-

   (i) $\frac{7}{15}$ from $\frac{13}{15}$ (ii) $\frac{7}{3}$ from $-\frac{5}{3}$ (iii) $\frac{3}{7}$ from $\frac{9}{24}$

5. Simplify:-

   (i) $\left(3\frac{1}{5}+\frac{7}{5}-2\frac{1}{6}\right)$ (ii) $\frac{5}{2}+\frac{13}{4}-6\frac{3}{4}$

6. Multiply:-

   (i) $\frac{2}{11}$ by $\frac{5}{6}$ (ii) $-\frac{3}{11}$ by $\frac{-33}{35}$ (iii) $\frac{-11}{3}$ by $\frac{-27}{77}$

7. Divide:

   (i) $\frac{1}{2}$ by $\frac{1}{4}$ (ii) $\frac{-7}{4}$ by $\frac{-4}{5}$ (iii) $\frac{35}{33}$ by $\frac{-7}{22}$

8. Simplify the following:

   (i) $\left(\frac{2}{3}+\frac{7}{8}\right)\times\frac{8}{25}\div\frac{37}{15}$ (ii) $\left[\left(\frac{3}{4}-\frac{2}{3}\right)\div\frac{1}{4}\right]\times 21$

9. Divide the sum of $\frac{16}{7}$ and $\frac{-3}{14}$ by their difference.

10. A number when multiplied by $\frac{13}{3}$ gives $\frac{39}{12}$. Find the number.

1.8 DECIMAL REPRESENTATION OF A RATIONAL NUMBER

You are familiar with the division of an integer by another integer and expressing the result as a decimal number. The process of expressing a rational number into decimal form is to carryout the process of long division using decimal notation.

Let us consider some examples.

Example 1.13: Represent each one of the following into a decimal number:

(i) $\frac{12}{5}$ (ii) $\frac{-27}{25}$ (iii) $\frac{13}{16}$

Solution: i) Using long division, we get

$$\begin{array}{r} 2.4 \\ 5\overline{)12.0} \\ \underline{10} \\ 2.0 \\ \underline{2.0} \\ \times \end{array}$$

Hence, $\frac{12}{5} = 2.4$

ii)

$$\begin{array}{r} 25\overline{)-27}\,(-1.08 \\ \underline{25} \\ 200 \\ \underline{200} \\ \times \end{array}$$

Hence, $\frac{-27}{25} = -1.08$

iii)

$$\begin{array}{r} 0.8125 \\ 16\overline{)13.0000} \\ \underline{128} \\ 20 \\ \underline{16} \\ 40 \\ \underline{32} \\ 80 \\ \underline{80} \\ \times \end{array}$$

Hence, $\frac{13}{16} = 0.8125$

From the above examples, it can be seen that the division process stops after a finite number of steps, when the remainder becomes zero and the resulting decimal number has a finite number of decimal places. Such decimals are known as **terminating decimals**.

Note: Note that in the above division, the denominators of the rational numbers had only 2 or 5 or both as the only prime factors.

Alternatively, we could have written $\frac{12}{5}$ as $\frac{12\times2}{5\times2} = \frac{24}{10} = 2.4$ and similarly for the others

Let us consider another example.

Example 14: Write the decimal representation of each of the following:

(a) $\frac{7}{3}$ (b) $\frac{2}{7}$ (c) $\frac{5}{11}$

Solution: (a)

$$\begin{array}{r} 2.33 \\ 3\overline{)7.00} \\ \underline{6} \\ 1.0 \\ \underline{9} \\ 1.0 \\ \underline{9} \\ 1.00 \end{array}$$

Here the remainder 1 repeats.

$\therefore$ The decimal is not a terminating decimal

$\frac{7}{3} = 2.333\ldots$ or $2.\bar{3}$

(b)

$$\begin{array}{r} 0.28571428 \\ 7\overline{)2.000} \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 4 \end{array}$$

Here when the remainder is 3, the digit after that start repeating

$\frac{2}{7} = 0.\overline{285714}$

Note: A bar over a digit or a group of digits implies that digit or that group of digits starts repeating itself indefinitely.

(c)

$$\begin{array}{r} 0.454 \\ 11\overline{)5.00} \\ \underline{44} \\ 60 \\ \underline{55} \\ 50 \\ \underline{44} \\ 50\ldots \end{array}$$

Here again when the remainder is 5, the digits after 5 start repeating

$\therefore \frac{5}{11} = 0.\overline{45}$

From the above, it is clear that in cases where the denominator has factors other than 2 or 5, the decimal representation starts repeating. Such decimals are called non-terminating repeating decimals.

Thus, we see from examples 1.13 and 1.14 that the decimal representation of a rational number is

(i) either a terminating decimal (and the remainder is zero after a finite number of steps)

(ii) or a non-terminating repeating decimal (where the division will never end)

∴ Thus, a rational number is either a terminating decimal or a non-terminating repeating decimal

1.8 EXPRESSING DECIMAL EXPANSION OF A RATIONAL NUMBER IN p/q FORM

Let us explain it through examples

Example 1.15: Express (i) 0.48 and (ii) 0.1357 in $\frac{p}{q}$ form

Solution: (i) $0.48 = \frac{48}{100} = \frac{12}{25}$

(ii) $0.1375 = \frac{1375}{10000} = \frac{55}{400} = \frac{11}{80}$

Example 1.16: Express (i) 0.666... (ii) 0.374374... in $\frac{p}{q}$ form

Solution: (i) Let $x = 0.666...$ (A)

$\therefore\ 10\,x = 6.666...$ (B)

(B) – (A) gives $9\,x = 6$ or $x = \frac{2}{3}$

(ii) Let $x = 0.374374374....$ (A)

$1000\,x = 374.374374374....$ (B)

(B) – (A) gives $999\,x = 374$

or $x = \frac{374}{999}$

$$\therefore 0.374374374... = \frac{374}{999}$$

The above example illustrates that:

A terminating decimal or a non-terminating recurring decimal represents a rational number

Note: The non- terminating recurring decimals like 0.374374374... are written as $0.\overline{374}$. The bar on the group of digits 374 indicate that the group of digits repeats again and again.

CHECK YOUR PROGRESS 1.3

1. Represent the following rational numbers in the decimal form:

   (i) $\frac{31}{80}$ (ii) $\frac{12}{25}$ (iii) $\frac{12}{8}$ (iv) $\frac{75}{12}$ (v) $\frac{91}{63}$

2. Represent the following rational numbers in the decimal form:

   (i) $\frac{2}{3}$ (ii) $\frac{5}{7}$ (iii) $\frac{25}{11}$

3. Represent the following decimals in the form $\frac{p}{q}$.

   (a) (i) 2.3 (ii) – 3.12 (iii) –0.715 (iv) 8.146

   (b) (i) $0.\overline{333}$ (ii) $3.\overline{42}$ (iii) – 0.315315315...

1.9 RATIONAL NUMBERS BETWEEN TWO RATIONAL NUMBERS

Is it possible to find a rational number between two given rational numbers. To explore this, consider the following examples.

Example 1.17: Find a rational number between $\frac{3}{4}$ and $\frac{6}{5}$

Solution: Let us try to find the number $\frac{1}{2}\left(\frac{3}{4}+\frac{6}{5}\right)$

$$=\frac{1}{2}\left(\frac{15+24}{20}\right)=\frac{39}{40}$$

Now $\frac{3}{4}=\frac{3\times10}{4\times10}=\frac{30}{40}$

and $\frac{6}{5}=\frac{6\times8}{5\times8}=\frac{48}{40}$

Obviously $\frac{30}{40}<\frac{39}{40}<\frac{48}{40}$

i.e. $\frac{39}{40}$ is a rational number between the rational numbers $\frac{3}{4}$ and $\frac{6}{5}$.

Note: $\frac{3}{4}=0.75$, $\frac{39}{40}=0.975$ and $\frac{6}{5}=1.2$

$\therefore 0.75<0.975<1.2$

or $\frac{3}{4}<\frac{39}{40}<\frac{6}{5}$

$\therefore$ This can be done by either way:

(i) reducing each of the given rational number with a common base and then taking their average

or (ii) by finding the decimal expansions of the two given rational numbers and then taking their average.

The question now arises, "How many rationals can be found between two given rationals? Consider the following examples.

Example 1.18: Find 3 rational numbers between $\frac{1}{2}$ and $\frac{3}{4}$.

Solution: $\frac{1}{2}=\frac{1\times8}{2\times8}=\frac{8}{16}$

and $\frac{3}{4}=\frac{3\times4}{4\times4}=\frac{12}{16}$

As $\frac{8}{16}<\frac{9}{16}<\frac{10}{16}<\frac{11}{16}<\frac{12}{16}$

$\therefore$ We have been able to find 3 rational numbers

$$\frac{9}{16}, \frac{10}{16} \text{ and } \frac{11}{16} \text{ between } \frac{1}{2} \text{ and } \frac{3}{4}$$

In fact, we can find any number of rationals between two given numbers.

Again $\quad \dfrac{1}{2} = \dfrac{50}{2\times 50} = \dfrac{50}{100}$

$$\frac{3}{4} = \frac{3\times 25}{4\times 25} = \frac{75}{100}$$

As $\quad \dfrac{50}{100} < \dfrac{51}{100} < \dfrac{52}{100} < \dfrac{53}{100} < \ldots\ldots < \dfrac{72}{100} < \dfrac{73}{100} < \dfrac{74}{100} < \dfrac{75}{100} < \ldots.\ \text{(i)}$

$\therefore$ we have been able to find 24 rational numbers between $\dfrac{1}{2}$ and $\dfrac{3}{4}$ as given in (i) above.

We can continue in this way further.

Note: From the above it is clear that between any two rationals an infinite number of rationals can be found.

CHECK YOUR PROGRESS 1.4

1. Find a rational number between the following rational numbers:

   (i) $\dfrac{3}{4}$ and $\dfrac{4}{3}$ (ii) 5 and 6 (iii) $-\dfrac{3}{4}$ and $\dfrac{1}{3}$

2. Find two rational numbers between the following rational numbers:

   (i) $-\dfrac{2}{3}$ and $\dfrac{1}{2}$ (ii) $-\dfrac{2}{3}$ and $-\dfrac{1}{4}$

3. Find 5 rational numbers between the following rational numbers:

   (i) 0.27 and 0.30 (ii) 7.31 and 7.35

   (iii) 20.75 and 26.80 (iv) 1.001 and 1.002

1.10 IRRATIONAL NUMBERS

We have seen that the decimal expansion of a rational number is either terminating or is a non-terminating and repeating deimal.

Are there decimals which are neither terminating nor non-terminating but repeating decimals? Consider the following decimal:

0.10 100 1000 10000 1....... (i)

You can see that this decimal has a definite pattern and it can be written indefinitely, and there is no block of digits which is repeating. Thus, it is an example of a non-terminating and non-repeating decimal. A similar decimal is given as under:

0.1 2 3 4 5 6 7 8 9 10 11 12 13..... (ii)

Can you write the next group of digits in (i) and (ii)? The next six digits in (i) are 000001... and in (ii) they are 14 15 16 ...

Such decimals as in (i) and (ii) represent irrational numbers.

Thus, *a decimal expansion which is neither terminating nor is repeating represents an irrational number.*

1.11 INADEQUACY OF RATIONAL NUMBERS

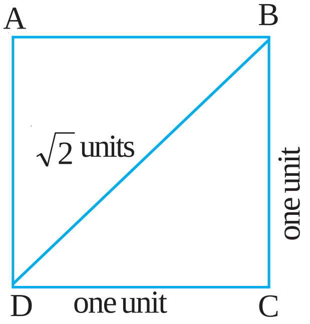
Can we measure all the lengths in terms of rational numbers? Can we measure all weights in terms of rational numbers?

Let us examine the following situation:

Consider a square ABCD, each of whose sides is 1 unit.

Naturally the diagonal BD is of length $\sqrt{2}$ units.

It can be proved that $\sqrt{2}$ is not a rational number, as there is no rational, whose square is 2, [Proof is beyond the scope of this lesson].

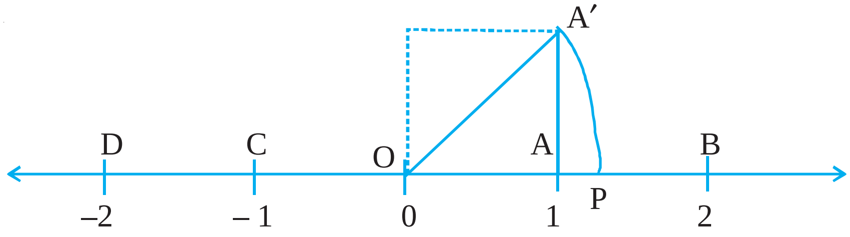


We conclude that we can not exactly measure the lengths of all line-segments using rationals, in terms of a given unit of length. Thus, the rational numbers are inadequate to measure all lengths in terms of a given unit. This inadequacy necessitates the extension of rational numbers to irrationals (which are not rational)

We have also read that corresponding to every rational number, there corresponds a point on the number line. Consider the converse of this statement:

Given a point on the number line, will it always correspond to a rational number? The answer to this question is also "No". For clarifying this, we take the following example.

On the number line take points O, A, B, C and D representing rational 0, 1, 2, –1 and –2 respectively. At A draw AA′⊥ to OA such that AA′ = 1 unit



$\therefore OA' = \sqrt{1^2+1^2} = \sqrt{2}$ units. Taking O as centre and radius OA′, if we draw an arc, we reach the point P, which represents the number $\sqrt{2}$.

As $\sqrt{2}$ is irrational, we conclude that there are points on the number line (like P) which are not represented by a rational number. Similarly, we can show that we can have points like $\sqrt{3},\ 2\sqrt{3},\ 5\sqrt{2}$ etc, which are not represented by rationals.

∴ The number line, consisting of points corresponding to rational numbers, has gaps on it. Therefore, the number line consists of points corresponding to rational numbers and irrational numbers both.

We have thus extended the system of rational numbers to include irrational numbers also. The system containing rationals and irrationals both is called the Real Number System.

The system of numbers consisting of all rational and irrational numbers is called the system of real numbers.

CHECK YOUR PROGRESS 1.5

1. Write the first three digits of the decimal representation of the following:

   $\sqrt{2},\ \sqrt{3},\ \sqrt{5}$

2. Represent the following numbers on the real number line:

   (i) $\frac{\sqrt{2}}{2}$ (ii) $1+\sqrt{2}$ (iii) $\frac{\sqrt{3}}{2}$

1.12 FINDING IRRATIONAL NUMBER BETWEEN TWO GIVEN NUMBERS

Let us illustrate the process of finding an irrational number between two given numbers with the help of examples.

Example 1.19: Find an irrational number between 2 and 3.

Solution: Consider the number $\sqrt{2\times 3}$

We know that $\sqrt{6}$ approximately equals 2.45.

∴ It lies between 2 and 3 and it is an irrational number.

Example 1.20: Find an irrational number lying between $\sqrt{3}$ and 2.

Solution: Consider the number $\frac{\sqrt{3}+2}{2}$

$$=1+\frac{\sqrt{3}}{2}\approx 1+\frac{1.732}{2}=1.866$$

$$\therefore \quad \frac{\sqrt{3}+2}{2} \approx 1.866 \text{ lies betwen } \sqrt{3} \ (\approx 1.732) \text{ and } 2$$

∴ The required irrational number is $\frac{\sqrt{3}+2}{2}$

CHECK YOUR PROGRESS 1.6

1. Find an irrational number between the following pairs of numbers

   (i) 2 and 4 (ii) $\sqrt{3}$ and 3 (iii) $\sqrt{2}$ and $\sqrt{3}$

2. Can you state the number of irrationals between 1 and 2? Illustrate with three examples.

1.13 ROUNDING OFF NUMBERS TO A GIVEN NUMBER OF DECIMAL PLACES

It is sometimes convenient to write the approximate value of a real number upto a desired number of decimal places. Let us illustrate it by examples.

Example 1.21: Express 2.71832 approximately by rounding it off to two places of decimals.

Solution: We look up at the third place after the decimal point. In this case it 8, which is more than 5. So the approximate value of 2.71832, upto two places of decimal is 2.72.

Example 1.22: Find the approximate value of 12.78962 correct upto 3 places of decimals.

Solution: The fourth place of decimals is 6 (more than 5) so we add 1 to the third place to get the approximate value of 12.78962 correct upto three places of decimals as 12.790.

Thus, we observe that to round off a number to some given number of places, we observe the next digit in the decimal part of the number and proceed as below

(i) If the digit is less than 5, we ignore it and state the answer without it.

(ii) If the digit is 5 or more than 5, we add 1 to the preceeding digit to get the required number upto desired number of decimal places.

CHECK YOUR PROGRESS 1.7

1. Write the approximate value of the following correct upto 3 place of decimals.

(i) 0.77777 (ii) 7.3259 (iii) 1.0118

(iv) 3.1428 (v) 1.1413

LET US SUM UP

- Recall of natural numbers, whole numbers, integers with four fundamental operations is done.
- Representation of above on the number line.
- Extension of integers to rational numbers - A rational number is a number which can be put in the form p/q, where p and q are integers and $q \neq 0$.
- When q is made positive and p and q have no other common factor, then a rational number is said to be in standard form or lowest form.
- Two rational numbers are said to be the equivalent form of the number if standard forms of the two are same.
- The rational numbers can be represented on the number line.
- Corresponding to a rational number, there exists a unique point on the number line.
- The rational numbers can be compared by
    - reducing them with the same denominator and comparing their numerators.
    - when represented on the number line, the greater rational number lies to the right of the other.

- As in integers, four fundamental operations can be performed on rational numbers also.
- The decimal representation of a rational number is either terminating or non-terminating repeating.
- There exist infinitely many rational numbers between two rational numbers.
- There are points other than those representing rationals on the number line. That shows inadequacy of system of rational numbers.
- The sytem of rational numbers is extended to real numbers.
- Rationals and irrationals together constitute the system of real numbers.
- We can always find an irrational number between two given numbers.
- The decimal representation of an irrational number is non-terminating non repeating.
- We can find the approximate value of a rational or an irrational number upto a given number of decimals.

TERMINAL EXERCISE

1. From the following pick out:

   (i) natural numbers

   (ii) integers which are not natural numbers

   (iii) rationals which are not natural numbers

   (iv) irrational numbers

   $$-3, 17, \frac{6}{7}, \frac{-3}{8}, 0, -32, \frac{3}{14}, \frac{11}{6}, \sqrt{2}, 2+\sqrt{3}$$

2. Write the following integers as rational numbers:

   (i) – 14 (ii) 13 (iii) 0 (iv) 2

   (v) 1 (vi) –1 (vii) –25

3. Express the following rationals in lowest terms:

   $$\frac{6}{8}, \frac{14}{21}, \frac{-17}{153}, \frac{13}{273}$$

4. Express the following rationals in decimal form:

   (i) $\frac{11}{80}$ (ii) $\frac{8}{25}$ (iii) $\frac{14}{8}$ (iv) $\frac{15}{6}$ (v) $\frac{98}{35}$

(vi) $\frac{15}{7}$ (vii) $-\frac{7}{6}$ (viii) $\frac{115}{11}$ (ix) $-\frac{17}{13}$ (x) $\frac{126}{36}$

5. Represent the following decimals in $\frac{p}{q}$ form:

(i) 2.4 (ii) – 0.32 (iii) 8.14 (iv) $3.\overline{24}$

(v) 0.415415415...

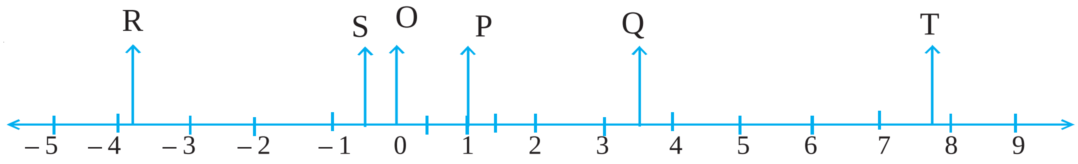
6. Find a rational number betwen the following rational numbers:

(i) $\frac{3}{4}$ and $\frac{7}{8}$ (ii) – 2 and – 3 (iii) $-\frac{4}{5}$ and $\frac{1}{3}$

7. Find three rational numbers between the following rational numbers:

(i) $\frac{3}{4}$ and $\frac{-3}{4}$ (ii) 0.27 and 0.28 (iii) 1.32 and 1.34

8. Write the rational numbers corresponding to the points O, P, Q, R, S and T on the number line in the following figure:



9. Find the sum of the following rational numbers:

(i) $\frac{3}{5}, \frac{-7}{5}$ (ii) $-\frac{7}{9}, \frac{5}{9}$ (iii) $\frac{3}{5}, \frac{7}{3}$ (iv) $\frac{9}{5}, \frac{2}{3}$ (v) $\frac{18}{7}, -\frac{7}{6}$

10. Find the product of the following rationals:

(i) $\frac{3}{5}, \frac{7}{3}$ (ii) $\frac{19}{5}, \frac{2}{3}$ (iii) $\frac{15}{7}, \frac{-14}{5}$

11. Write an irrational number between the following pairs of numbers:

(i) 1 and 3 (ii) $\sqrt{3}$ and 3 (iii) $\sqrt{2}$ and $\sqrt{5}$ (iv) $-\sqrt{2}$ and $\sqrt{2}$

12. How many rational numbers and irrational numbers lie between the numbers 2 and 7?

13. Find the approximate value of the following numbers correct to 2 places of decimals:

(i) 0.338 (ii) 3.924 (iii) 3.14159 (iv) 3.1428

14. Write the value of following correct upto 3 places of decimals:

(i) $\frac{3}{4}$ (ii) $2+\sqrt{2}$ (iii) 1.7326 (iv) 0.9999...

15. Simplify the following as irrational numbers. The first one is done for you.

(i) $12\sqrt{3}+5\sqrt{3}-7\sqrt{3}=\sqrt{3}[12+5-7]=10\sqrt{3}$

(ii) $3\sqrt{2}-2\sqrt{8}+7\sqrt{2}$

(iii) $3\sqrt{2}\times 2\sqrt{3}\times 5\sqrt{6}$

(iv) $[(\sqrt{8}\times 3\sqrt{2})\times 6\sqrt{2}]\div 36\sqrt{2}$

ANSWERS TO CHECK YOUR PROGRESS

1.1

1. Integers: 4, – 36, – 6

Rational Numbers: $4, \frac{-3}{4}, \frac{5}{6}, -36, \frac{12}{7}, \frac{-3}{8}, \frac{15}{7}, -6$

2. (i) $-\frac{7}{4}, -\frac{3}{7}, -15, 0.\frac{5}{17}, -\frac{3}{4}, -\frac{4}{3}$

(ii) $-\frac{7}{4}, -\frac{3}{7}, -15, \frac{5}{17}, -\frac{3}{4}, -\frac{4}{3}$

(iii) $-\frac{7}{4}, -\frac{3}{7}, \frac{5}{17}, -\frac{3}{4}, -\frac{4}{3}$

(iv) All are rational numbers.

3. (i) $\frac{16}{3}$, rational (ii) $-\frac{1}{2}$, rational (iii) –21, integer and rational

(iv) zero, whole number, integer and rational (v) 4, All

(vi) $\frac{10}{7}$, rational (vii) $\frac{8}{3}$, rational

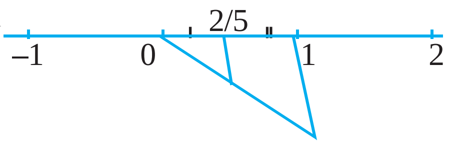
4. (i) 2 (ii) – 8 (iii) 1

5. $\frac{5}{7}, \frac{-6}{7}, \frac{2\sqrt{3}}{\sqrt{27}}$

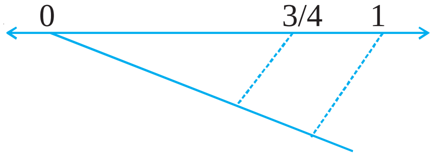
6. $-10, \frac{15}{5}, \frac{27}{9}, \frac{-6}{-2}$

7. (i) $\frac{2}{5}=\frac{4}{10}=\frac{6}{15}=\frac{8}{20}$ (ii) $-\frac{5}{6}=-\frac{10}{12}=-\frac{15}{18}=-\frac{20}{24}$ (iii) $\frac{17}{3}=\frac{34}{6}=\frac{51}{9}=\frac{68}{12}$

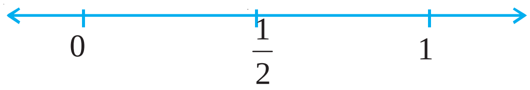
8. (i)



(ii)



(iii)



9. (a) $\frac{3}{4}>\frac{2}{3}$ (b) $\frac{7}{9}>\frac{3}{5}$ (c) $\frac{-1}{2}>\frac{-2}{3}$ (d) $\frac{5}{11}>\frac{3}{7}$

(e) $\frac{3}{2}>-\frac{7}{6}$

1.2

1. (i) $\frac{9}{7}$ (ii) $-\frac{4}{15}$ (iii) $\frac{1}{2}$ (iv) $\frac{1}{2}$

2. (i) $\frac{19}{6}$ (ii) $\frac{188}{63}$ (iii) $-\frac{11}{35}$

3. (i) $-\frac{53}{48}$ (ii) $\frac{149}{60}$

4. (i) $\frac{2}{5}$ (ii) -4 (iii) $\frac{-3}{56}$

5. (i) $\frac{73}{30}$ (ii) -1

6. (i) $\frac{5}{33}$ (ii) $\frac{9}{35}$ (iii) $\frac{9}{7}$

7. (i) 2 (ii) $\frac{35}{16}$ (iii) $-\frac{10}{3}$

8. (i) $\frac{1}{5}$ (ii) 7

9. $\frac{29}{35}$

10. $\frac{3}{4}$

1.3

1. (i) 0.3875 (ii) 0.48 (iii) 1.5 (iv) 6.25 (v) $1.\overline{4}$

2. (i) $0.\overline{6}$ (ii) $0.\overline{714285}$ (iii) $2.\overline{27}$

3. (a) (i) $\frac{23}{10}$ (ii) $-\frac{78}{25}$ (iii) $-\frac{143}{200}$ (iv) $\frac{4073}{500}$

(b) (i) $\frac{1}{3}$ (ii) $\frac{113}{33}$ (iii) $-\frac{35}{111}$

1.4

1. (i) $\frac{25}{24}$ (ii) 5.5 (iii) $\frac{-5}{24}$

2. (i) 0.2 and 0.3 (ii) – 0.30, – 0.35

3. (i) 0.271, 0.275, 0, 281, 0.285, 0.291

(ii) 7.315, 7.320 7.325, 7.330, 7.331

(iii) 21.75, 22.75, 23.75, 24.75, 25.75

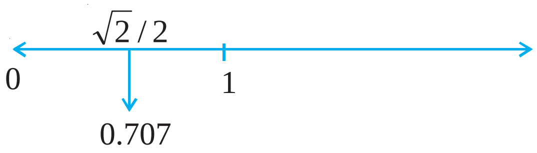
(iv) 1.0011, 1.0012, 1.0013, 1.0014, 1.0015

Note: *Can be other answers as well.*

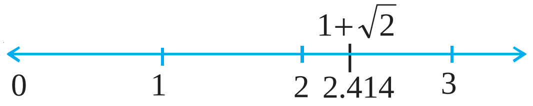
1.5

1. 1.414, 1.732, 2.236

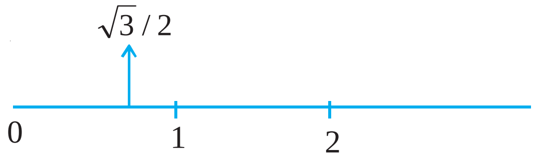
2. (i)



(ii)



(iii)



1.6

1. (i) $\sqrt{5}$ (ii) $\sqrt{3}+1$ (iii) $\dfrac{\sqrt{2}+\sqrt{3}}{2}$

2. Infinitely many:

1.0001, 1.0002, ..., 1.0010, 1.0011,....., 1.0020, 1.0021, ...

1.7

1. (i) 0.778 (ii) 7.326 (iii) 1.012 (iv) 3.143 (v) 1.141

ANASWERS TO TERMINAL EXERCISE

1. Natural: 17,

Integers but not natural numbers, –3, 0, – 32

Rationals but not natural numbers: $-3, \frac{6}{7}, \frac{-3}{8}, 0, -32, \frac{3}{14}, \frac{11}{6}$

Irrationals but not rationals: $\sqrt{2}$, $2+\sqrt{3}$

2. (i) $-\dfrac{14}{1}$ (ii) $\dfrac{13}{1}$ (iii) $\dfrac{0}{1}$ (iv) $\dfrac{2}{1}$

(v) $\dfrac{1}{1}$ (vi) $\dfrac{-1}{1}$ (vii) $\dfrac{-25}{1}$

3. $\dfrac{3}{4}, \dfrac{2}{3}, -\dfrac{1}{9}, \dfrac{1}{21}$

4. (i) 0.1375 (ii) 0.32 (iii) 1.75 (iv) 2.5 (v) 2.8
(vi) 2.142857 (vii) $-1.\overline{166}$ (viii) $10.\overline{45}$ (ix) $-1.\overline{307692}$ (x) 3.5

5. (i) $\frac{12}{5}$ (ii) $\frac{-8}{25}$ (iii) $\frac{407}{50}$ (iv) $\frac{107}{33}$ (v) $\frac{415}{999}$

6. (i) $\frac{13}{16}$ (ii) – 2.5 (iii) zero

7. (i) 0.50, 0.25, 0.00 (ii) 0.271, 0.274, 0.277 (iii) 1.325. 1.33, 1.335

8. (i) R: – 3.8 (ii) S: – 0.5 (iii) O: 0.00 (iv) S: $-0.\overline{33}$ (v) Q: 3.5
(vi) T: $7.\overline{66}$

9. (i) $-\frac{4}{5}$ (ii) $-\frac{2}{9}$ (iii) $\frac{44}{15}$ (iv) $\frac{37}{15}$ (v) $\frac{59}{42}$

10. (i) $\frac{7}{5}$ (ii) $\frac{38}{15}$ (iii) – 6

11. (i) $\sqrt{3}$ (ii) $1+\sqrt{3}$ (iii) $\sqrt{3}$ (iv) $\frac{\sqrt{2}}{2}$

12. Infinitely many

13. (i) 0.34 (ii) 3.92 (iii) 3.14 (iv) 3.14

14. (i) 0.75 (ii) 3.414 (iii) 1.733 (iv) 1.000

15. (ii) $6\sqrt{2}$ (iii) 180 (iv) 2

2

EXPONENTS AND RADICALS

We have learnt about multiplication of two or more real numbers in the earlier lesson. You can very easily write the following

$$4 \times 4 \times 4 = 64, 11 \times 11 \times 11 \times 11 = 14641 \text{ and}$$

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256$$

Think of the situation when 13 is to be multiplied 15 times. How difficult is it to write?

$$13 \times 13 \times 13 \times \text{.................}15 \text{ times?}$$

This difficulty can be overcome by the introduction of exponential notation. In this lesson, we shall explain the meaning of this notation, state and prove the laws of exponents and learn to apply these. We shall also learn to express real numbers as product of powers of prime numbers.

In the next part of this lesson, we shall give a meaning to the number $a^{1/q}$ as qth root of a. We shall introduce you to radicals, index, radicand etc. Again, we shall learn the laws of radicals and find the simplest form of a radical. We shall learn the meaning of the term "rationalising factor' and rationalise the denominators of given radicals.

OBJECTIVES

After studying this lesson, you will be able to

- *write a repeated multiplication in exponential notation and vice-versa;*
- *identify the base and exponent of a number written in exponential notation;*
- *express a natural number as a product of powers of prime numbers uniquely;*
- *state the laws of exponents;*
- *explain the meaning of* a^{0}, a^{-m} *and* $a^{\frac{p}{q}}$;
- *simplify expressions involving exponents, using laws of exponents;*

- *identify radicals from a given set of irrational numbers;*
- *identify index and radicand of a surd;*
- *state the laws of radicals (or surds);*
- *express a given surd in simplest form;*
- *classify similar and non-similar surds;*
- *reduce surds of different orders to those of the same order;*
- *perform the four fundamental operations on surds;*
- *arrange the given surds in ascending/descending order of magnitude;*
- *find a rationalising factor of a given surd;*
- *rationalise the denominator of a given surd of the form* $\frac{1}{a+b\sqrt{x}}$ and $\frac{1}{\sqrt{x}+\sqrt{y}}$, *where x and y are natural numbers and a and b are integers;*
- *simplify expressions involving surds.*

EXPECTED BACKGROUND KNOWLEDGE

- Prime numbers
- Four fundamental operations on numbers
- Rational numbers
- Order relation in numbers.

2.1 EXPONENTIAL NOTATION

Consider the following products:

(i) 7×7 (ii) $3 \times 3 \times 3$ (iii) $6 \times 6 \times 6 \times 6 \times 6$

In (i), 7 is multiplied twice and hence 7×7 is written as 7^2.

In (ii), 3 is multiplied three times and so $3 \times 3 \times 3$ is written as 3^3.

In (iii), 6 is multiplied five times, so $6 \times 6 \times 6 \times 6 \times 6$ is written as $6^{5.}$

7^2 is read as “7 raised to the power 2” or “second power of 7”. Here, 7 is called base and 2 is called exponent (or index)

Similarly, 3^3 is read as “3 raised to the power 3”or “third power of 3”. Here, 3 is called the base and 3 is called exponent.

Similarly, 6^5 is read as “6 raised to the power 5”or “Fifth power of 6”. Again 6 is base and 5 is the exponent (or index).

From the above, we say that

The notation for writing the product of a number by itself several times is called the Exponential Notation or Exponential Form.

Thus, 5 × 5 × .... 20 times = 5^{20} and (–7) × (–7) × .... 10 times = $(-7)^{10}$

In 5^{20}, 5 is the base and exponent is 20.

In $(-7)^{10}$, base is –7 and exponent is 10.

Similarly, exponential notation can be used to write precisely the product of a ratioinal number by itself a number of times.

Thus, $$\frac{3}{5}\times\frac{3}{5}\times.........16 \text{ times} = \left(\frac{3}{5}\right)^{16}$$

and $$\left(-\frac{1}{3}\right)\times\left(-\frac{1}{3}\right)\times..........10 \text{ times} = \left(-\frac{1}{3}\right)^{10}$$

In general, if a is a rational number, multiplied by itself m times, it is written as a^{m}.

Here again, a is called the base and m is called the exponent

Let us take some examples to illustrate the above discussion:

Example 2.1: Evaluate each of the following:

(i) $\left(\frac{2}{7}\right)^{3}$ (ii) $\left(-\frac{3}{5}\right)^{4}$

Solution: (i) $$\left(\frac{2}{7}\right)^{3} = \frac{2}{7}\times\frac{2}{7}\times\frac{2}{7} = \frac{(2)^3}{(7)^3} = \frac{8}{343}$$

(ii) $$\left(-\frac{3}{5}\right)^{4} = \left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right) = \frac{(-3)^4}{(5)^5} = \frac{81}{625}$$

Example 2.2: Write the following in exponential form:

(i) (–5) × (–5) × (–5) × (–5) × (–5) × (–5) × (–5)

(ii) $\left(\frac{3}{11}\right)\times\left(\frac{3}{11}\right)\times\left(\frac{3}{11}\right)\times\left(\frac{3}{11}\right)$

Solution: (i) (–5) × (–5) × (–5) × (–5) × (–5) × (–5) × (–5) = $(-5)^{7}$

(ii) $$\left(\frac{3}{11}\right)\times\left(\frac{3}{11}\right)\times\left(\frac{3}{11}\right)\times\left(\frac{3}{11}\right) = \left(\frac{3}{11}\right)^{4}$$

Example 2.3: Express each of the following in exponential notation and write the base and exponent in each case.

(i) 4096 (ii) $\frac{125}{729}$ (iii) – 512

Solution: (i) $4096 = 4 \times 4 \times 4 \times 4 \times 4 \times 4$

$= (4)^6$

Alternatively $4096 = (2)^{12}$

Base = 2, exponent =12

Here, base = 4 and exponent = 6

(ii) $\frac{125}{729} = \frac{5}{9}\times\frac{5}{9}\times\frac{5}{9} = \left(\frac{5}{9}\right)^3$

Here, base = $\left(\frac{5}{9}\right)$ and exponent = 3

(iii) $512 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^9$

Here, base = 2 and exponent = 9

Example 2.4: Simplify the following:

$$\left(\frac{3}{2}\right)^3 \times \left(\frac{4}{3}\right)^4$$

Solution:

$$\left(\frac{3}{2}\right)^3 = \frac{3}{2}\times\frac{3}{2}\times\frac{3}{2} = \frac{3^3}{2^3}$$

Similarly

$$\left(\frac{4}{3}\right)^4 = \frac{4^4}{3^4}$$

$$\left(\frac{3}{2}\right)^3 \times \left(\frac{4}{3}\right)^4 = \frac{3^3}{2^3}\times\frac{4^4}{3^4}$$

$$= \frac{3^3}{8}\times\frac{16\times16}{3^4} = \frac{32}{3}$$

Example 2.5: Write the reciprocal of each of the following and express them in exponential form:

(i) 3^5 (ii) $\left(\frac{3}{4}\right)^2$ (iii) $\left(-\frac{5}{6}\right)^9$

Solution: (i) $3^5 = 3 \times 3 \times 3 \times 3 \times 3$

$= 243$

$$\therefore \text{ Reciprocal of } 3^5 = \frac{1}{243} = \left(\frac{1}{3}\right)^5$$

(ii) $\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$

$$\therefore \text{ Reciprocal of } \left(\frac{3}{4}\right)^2 = \frac{4^2}{3^2} = \left(\frac{4}{3}\right)^2$$

(iii) $\left(-\frac{5}{6}\right)^9 = \frac{(-5)^9}{6^9}$

$$\therefore \text{ Reciprocal of } \left(-\frac{5}{6}\right)^9 = \frac{-6^9}{5^9} = \left(\frac{-6}{5}\right)^9$$

From the above example, we can say that if $\frac{p}{q}$ is any non-zero rational number and m is any positive integer, then the reciprocal of $\left(\frac{p}{q}\right)^m$ is $\left(\frac{q}{p}\right)^m$.

CHECK YOUR PROGRESS 2.1

1. Write the following in exponential form:

(i) (–7) × (–7) × (–7) × (–7)

(ii) $\left(\frac{3}{4}\right) \times \left(\frac{3}{4}\right) \times \ldots.$ 10 times

(iii) $\left(-\frac{5}{7}\right) \times \left(-\frac{5}{7}\right) \times \ldots.$ 20 times

2. Write the base and exponent in each of the following:

(i) $(-3)^5$ (ii) $(7)^4$ (iii) $\left(-\frac{2}{11}\right)^8$

3. Evaluate each of the following

(i) $\left(\frac{3}{7}\right)^4$ (ii) $\left(\frac{-2}{9}\right)^4$ (iii) $\left(-\frac{3}{4}\right)^3$

4. Simplify the following:

(i) $\left(\frac{7}{3}\right)^5 \times \left(\frac{3}{7}\right)^6$

(ii) $\left(-\frac{5}{6}\right)^2 \div \left(-\frac{3}{5}\right)^2$

5. Find the reciprocal of each of the following:

(i) 3^5 (ii) $(-7)^4$ (iii) $\left(-\frac{3}{5}\right)^4$

2.2 PRIME FACTORISATION

Recall that any composite number can be expressed as a product of prime numbers. Let us take the composite numbers 72, 760 and 7623.

(i) $72 = 2 \times 2 \times 2 \times 3 \times 3$

$= 2^3 \times 3^2$

(ii) $760 = 2 \times 2 \times 2 \times 5 \times 19$

$= 2^3 \times 5^1 \times 19^1$

(iii) $7623 = 3 \times 3 \times 7 \times 11 \times 11$

$= 3^2 \times 7^1 \times 11^2$

$$\begin{array}{r|l} 2 & 72 \\ \hline 2 & 36 \\ \hline 2 & 18 \\ \hline 3 & 9 \\ \hline & 3 \end{array} \qquad \begin{array}{r|l} 3 & 7623 \\ \hline 3 & 2541 \\ \hline 7 & 847 \\ \hline 11 & 121 \\ \hline & 11 \end{array} \qquad \begin{array}{r|l} 2 & 760 \\ \hline 2 & 380 \\ \hline 2 & 190 \\ \hline 5 & 95 \\ \hline & 19 \end{array}$$

We can see that any natural number, other than 1, can be expressed as a product of powers of prime numbers in a unique manner, apart from the order of occurrence of factors. Let us consider some examples

Example 2.6: Express 24300 in exponential form.

Solution: $24300 = 3 \times 3 \times 3 \times 3 \times 2 \times 2 \times 5 \times 5 \times 3$

$\therefore$ 24300 = $2^2 \times 3^5 \times 5^2$

Example 2.7: Express 98784 in exponential form.

Solution:

$$\begin{array}{r|l} 2 & 98784 \\ \hline 2 & 49392 \\ \hline 2 & 24696 \\ \hline 2 & 12348 \\ \hline 2 & 6174 \\ \hline 3 & 3087 \\ \hline 3 & 1029 \\ \hline 7 & 343 \\ \hline 7 & 49 \\ \hline & 7 \end{array}$$

$\therefore$ 98784 = $2^5 \times 3^2 \times 7^3$

CHECK YOUR PROGRESS 2.2

1. Express each of the following as a product of powers of primes, i.e, in exponential form:

   (i) 429 (ii) 648 (iii) 1512

2. Express each of the following in exponential form:

   (i) 729 (ii) 512 (iii) 2592

   (iv) $\frac{1331}{4096}$ (v) $-\frac{243}{32}$

2.3 LAWS OF EXPONENTS

Consider the following

(i) $3^2 \times 3^3$ $= (3 \times 3) \times (3 \times 3 \times 3) = (3 \times 3 \times 3 \times 3 \times 3)$

$= 3^5 = 3^{2+3}$

(ii) $(-7)^2 \times (-7)^4 = [(-7) \times (-7)] \times [(-7) \times (-7) \times (-7) \times (-7)]$

$= [(-7) \times (-7) \times (-7) \times (-7) \times (-7) \times (-7)]$

$= (-7)^6 = (-7)^{2+4}$

(iii) $\left(\frac{3}{4}\right)^3 \times \left(\frac{3}{4}\right)^4 = \left(\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}\right) \times \left(\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}\right)$

$$=\left(\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\right)$$

$$=\left(\frac{3}{4}\right)^{7}=\left(\frac{3}{4}\right)^{3+4}$$

(iv) $a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a) = a^7 = a^{3+4}$

From the above examples, we observe that

Law 1: If a is any non-zero rational number and m and n are two positive integers, then

$$a^m \times a^n = a^{m+n}$$

Example 2.8: Evaluate $\left(-\frac{3}{2}\right)^{3}\times\left(-\frac{3}{2}\right)^{5}$.

Solution: Here $a = -\frac{3}{2}$, $m = 3$ and $n = 5$.

$$\therefore\ \left(-\frac{3}{2}\right)^{3}\times\left(-\frac{3}{2}\right)^{5}=\left(-\frac{3}{2}\right)^{3+5}=\left(-\frac{3}{2}\right)^{8}=\frac{6561}{256}$$

Example 2.9: Find the value of

$$\left(\frac{7}{4}\right)^{2}\times\left(\frac{7}{4}\right)^{3}$$

Solution: As before,

$$\left(\frac{7}{4}\right)^{2}\times\left(\frac{7}{4}\right)^{3}=\left(\frac{7}{4}\right)^{2+3}=\left(\frac{7}{4}\right)^{5}=\frac{16807}{1024}$$

Now study the following:

(i) $7^5 \div 7^3 = \frac{7^5}{7^3} = \frac{7\times7\times7\times7\times7}{7\times7\times7} = 7\times7 = 7^2 = 7^{5-3}$

(ii) $(-3)^7 \div (-3)^4 = \frac{(-3)^7}{(-3)^4} = \frac{(-3)\times(-3)\times(-3)\times(-3)\times(-3)\times(-3)\times(-3)}{(-3)\times(-3)\times(-3)\times(-3)}$

$$=(-3)(-3)(-3)=(-3)^3=(-3)^{7-4}$$

From the above, we can see that

Law 2: If a is any non-zero rational number and m and n are positive integers ($m > n$), then

$$a^m \div a^n = a^{m-n}$$

Example 2.10: Find the value of $\left(\frac{35}{25}\right)^{16} \div \left(\frac{35}{25}\right)^{13}$.

Solution: $\left(\frac{35}{25}\right)^{16} \div \left(\frac{35}{25}\right)^{13}$

$$= \left(\frac{35}{25}\right)^{16-13} = \left(\frac{35}{25}\right)^{3} = \left(\frac{7}{5}\right)^{3} = \frac{343}{125}$$

In Law 2, $m < n \Rightarrow n > m$,

then $$a^m \div a^n = a^{-(n-m)} = \frac{1}{a^{m-n}}$$

Law 3: When $n > m$

$$a^m \div a^n = \frac{1}{a^{m-n}}$$

Example 2.11: Find the value of $\left(\frac{3}{7}\right)^{6} \div \left(\frac{3}{7}\right)^{9}$

Solution: Here $a = \frac{3}{7}$, $m = 6$ and $n = 9$.

$$\therefore \left(\frac{3}{7}\right)^{6} \div \left(\frac{3}{7}\right)^{9} = \left(\frac{3}{7}\right)^{\frac{1}{9-6}}$$

$$= \frac{7^3}{3^3} = \frac{343}{27}$$

Let us consider the following:

(i) $\left(3^3\right)^2 = 3^3 \times 3^3 = 3^{3+3} = 3^6 = 3^{3\times 2}$

(ii) $\left[\left(\frac{3}{7}\right)^2\right]^5 = \left(\frac{3}{7}\right)^2 \times \left(\frac{3}{7}\right)^2 \times \left(\frac{3}{7}\right)^2 \times \left(\frac{3}{7}\right)^2 \times \left(\frac{3}{7}\right)^2$

$$\left(\frac{3}{7}\right)^{2+2+2+2+2}=\left(\frac{3}{7}\right)^{10}=\left(\frac{3}{7}\right)^{2\times5}$$

From the above two cases, we can infer the following:

Law 4: If a is any non-zero rational number and m and n are two positive integers, then

$$\left(a^m\right)^n=a^{mn}$$

Let us consider an example.

Example 2.12: Find the value of $\left[\left(\frac{2}{5}\right)^2\right]^3$

Solution: $$\left[\left(\frac{2}{5}\right)^2\right]^3=\left[\frac{2}{5}\right]^{2\times3}=\left(\frac{2}{5}\right)^6=\frac{64}{15625}$$

2.3.1 Zero Exponent

Recall that $a^m\div a^n=a^{m-n}$, if $m>n$

$$=\frac{1}{a^{n-m}}, \text{ if } n>m$$

Let us consider the case, when $m=n$

$$\therefore\ a^m\div a^m=a^{m-m}$$

$$\Rightarrow\frac{a^m}{a^m}=a^0$$

$$\Rightarrow1=a^0$$

Thus, we have another important law of exponents,.

Law 5: If a is any rational number other than zero, then $a^o=1$.

Example 2.13: Find the value of

(i) $\left(\frac{2}{7}\right)^0$ (ii) $\left(\frac{-3}{4}\right)^0$

Solution: (i) Using $a^0=1$, we get $\left(\frac{2}{7}\right)^0=1$

(ii) Again using $a^0 = 1$, we get $\left(\frac{-3}{4}\right)^0 = 1.$

CHECK YOUR PROGRESS 2.3

1. Simplify and express the result in exponential form:

(i) $(7)^2 \times (7)^3$ (ii) $\left(\frac{3}{4}\right)^3 \times \left(\frac{3}{4}\right)^2$ (iii) $\left(-\frac{7}{8}\right)^1 \times \left(-\frac{7}{8}\right)^2 \times \left(-\frac{7}{8}\right)^3$

2. Simplify and express the result in exponential form:

(i) $(-7)^9 \div (-7)^7$ (ii) $\left(\frac{3}{4}\right)^8 \div \left(\frac{3}{4}\right)^2$ (iii) $\left(\frac{-7}{3}\right)^{18} \div \left(\frac{-7}{3}\right)^3$

3. Simplify and express the result in exponential form:

(i) $\left(2^6\right)^3$ (ii) $\left[\left(\frac{3}{4}\right)^3\right]^2$ (iii) $\left[\left(-\frac{5}{9}\right)^3\right]^5$

(iv) $\left(\frac{11}{3}\right)^5 \times \left(\frac{15}{7}\right)^0$ (v) $\left(-\frac{7}{11}\right)^0 \times \left(-\frac{7}{11}\right)^3$

4. Which of the following statements are true?

(i) $7^3 \times 7^3 = 7^6$ (ii) $\left(\frac{3}{11}\right)^5 \times \left(\frac{3}{11}\right)^2 = \left(\frac{3}{11}\right)^7$

(iii) $\left[\left(\frac{4}{9}\right)^5\right]^4 = \left(\frac{4}{9}\right)^9$ (iv) $\left[\left(\frac{3}{19}\right)^6\right]^2 = \left(\frac{3}{19}\right)^8$

(v) $\left(\frac{3}{11}\right)^0 = 0$ (vi) $\left(-\frac{3}{2}\right)^2 = -\frac{9}{4}$

(vii) $\left(\frac{8}{15}\right)^5 \times \left(\frac{7}{6}\right)^0 = \left(\frac{8}{15}\right)^5$

2.4 NEGATIVE INTEGERS AS EXPONENTS

i) We know that the reciprocal of 5 is $\frac{1}{5}$. We write it as 5^{-1} and read it as 5 raised to power –1.

ii) The reciprocal of (–7) is $-\frac{1}{7}$. We write it as $(-7)^{-1}$ and read it as (–7) raised to the power –1.

iii) The reciprocal of $5^2 = \frac{1}{5^2}$. We write it as 5^{-2} and read it as '5 raised to the power (–2)'.

From the above all, we get

If a is any non-zero rational number and m is any positive integer, then the reciprocal of a^m $\left(i.e. \frac{1}{a^m}\right)$ is written as a^{-m} and is read as 'a raised to the power (–m)'. Therefore,

$$\frac{1}{a^m} = a^{-m}$$

Let us consider an example.

Example 2.14: Rewrite each of the following with a positive exponent:

(i) $\left(\frac{3}{8}\right)^{-2}$ (ii) $\left(-\frac{4}{7}\right)^{-7}$

Solution:

$$\text{(i)} \left(\frac{3}{8}\right)^{-2} = \frac{1}{\left(\frac{3}{8}\right)^2} = \frac{1}{\frac{3^2}{8^2}} = \frac{8^2}{3^2} = \left(\frac{8}{3}\right)^2$$

$$\text{(ii)} \left(-\frac{4}{7}\right)^{-7} = \frac{1}{\left(-\frac{4}{7}\right)^7} = \frac{7^7}{(-4)^7} = \left(-\frac{7}{4}\right)^7$$

From the above example, we get the following result:

If $\frac{p}{q}$ is any non-zero rational number and m is any positive integer, then $\left(\frac{p}{q}\right)^{-m} = \frac{q^m}{p^m} = \left(\frac{q}{p}\right)^m$.

2.5 LAWS OF EXPONENTS FOR INTEGRAL EXPONENTS

After giving a meaning to negative integers as exponents of non-zero rational numbers, we can see that laws of exponents hold good for negative exponents also.

For example.

(i) $$\left(\frac{3}{5}\right)^{-4}\times\left(\frac{3}{5}\right)^{3}=\frac{1}{\left(\frac{3}{5}\right)^{4}}\times\left(\frac{3}{5}\right)^{3}=\frac{3^{3-4}}{5}$$

(ii) $$\left(-\frac{2}{3}\right)^{-2}\times\left(-\frac{2}{3}\right)^{-3}=\frac{1}{\left(-\frac{2}{3}\right)^{2}}\times\frac{1}{\left(-\frac{2}{3}\right)^{3}}=\frac{1}{\left(-\frac{2}{3}\right)^{2+3}}=\left(-\frac{2}{3}\right)^{-2-3}$$

(iii) $$\left(-\frac{3}{4}\right)^{-3}\div\left(-\frac{3}{4}\right)^{-7}=\frac{1}{\left(-\frac{3}{4}\right)^{3}}\div\frac{1}{\left(-\frac{3}{4}\right)^{7}}=\frac{1}{\left(-\frac{3}{4}\right)^{3}}\times\left(-\frac{3}{4}\right)^{7}=\left(-\frac{3}{4}\right)^{7-3}$$

(iv) $$\left(\left(\frac{2}{7}\right)^{-2}\right)^{3}=\left[\left(\frac{7}{2}\right)^{2}\right]^{3}=\left(\frac{7}{2}\right)^{6}=\left(\frac{2}{7}\right)^{-6}=\left(\frac{2}{7}\right)^{-2\times 3}$$

Thus, from the above results, we find that laws 1 to 5 hold good for negative exponents also.

∴ For any non-zero rational numbers a and b and any integers m and n,

1. $a^m \times a^n \quad = a^{m+n}$
2. $a^m \div a^n \quad = a^{m-n}$ if $m > n$

   $= a^{n-m}$ if $n > m$
3. $(a^m)^n = a^{mn}$
4. $(a \times b)^m = a^m \times b^m$

CHECK YOUR PROGRESS 2.4

1. Express $\left(\frac{-3}{7}\right)^{-2}$ as a rational number of the form $\frac{p}{q}$:

2. Express as a power of rational number with positive exponent:

(i) $\left(\frac{3}{7}\right)^{-4}$ (ii) $12^5 \times 12^{-3}$ (iii) $\left[\left(\frac{3}{13}\right)^{-3}\right]^4$

3. Express as a power of a rational number with negative index:

(i) $\left(\frac{3}{7}\right)^{4}$ (ii) $\left[(7)^2\right]^5$ (iii) $\left[\left(-\frac{3}{4}\right)^{2}\right]^5$

4. Simplify:

(i) $\left(\frac{3}{2}\right)^{-3} \times \left(\frac{3}{2}\right)^{7}$ (ii) $\left(-\frac{2}{3}\right)^{-3} \times \left(-\frac{2}{3}\right)^{4}$ (iii) $\left(-\frac{7}{5}\right)^{-4} \div \left(-\frac{7}{5}\right)^{-7}$

5. Which of the following statements are true?

(i) $a^{-m} \times a^n = a^{-m-n}$

(ii) $(a^{-m})^n = a^{-mn}$

(iii) $a^m \times b^m = (ab)^m$

(iv) $a^m \div b^m = \left(\frac{a}{b}\right)^m$

(v) $a^{-m} \times a^{o} = a^m$

2.6 MEANING OF $a^{p/q}$

You have seen that for all integral values of *m* and *n*,

$$a^m \times a^n = a^{m+n}$$

What is the method of defining $a^{1/q}$, if *a* is positive rational number and q is a natural number.

Consider the multiplication

$$\underbrace{a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times a^{\frac{1}{q}} \ldots\ldots\ldots \times a^{\frac{1}{q}}}_{q \text{ times}} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\ldots\ldots q \text{ times}}$$

$$= a^{\frac{q}{q}} = a$$

In other words, the qth power of $a^{\frac{1}{q}} = a$ or

in other words $a^{\frac{1}{q}}$ is the qth root of a and is written as $\sqrt[q]{a}$.

For example,

$$7^{\frac{1}{4}} \times 7^{\frac{1}{4}} \times 7^{\frac{1}{4}} \times 7^{\frac{1}{4}} = 7^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = 7^{\frac{4}{4}} = 7^1 = 7$$

or $7^{\frac{1}{4}}$ is the fourth root of 7 and is written as $\sqrt[4]{7}$,

Let us now define rational powers of a

If a is a positive real number, p is an integer and q is a natural number, then

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

We can see that

$$\underbrace{a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \ldots\ldots\ldots \times a^{\frac{p}{q}}}_{q \text{ times}} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\ldots\ldots q \text{ times}} = a^{\frac{p}{q}.q} = a^p$$

$$\therefore a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

$\therefore$ $a^{p/q}$ is the qth root of a^p

Consequently, $7^{\frac{2}{3}}$ is the cube root of 7^2.

Let us now write the laws of exponents for rational exponents:

(i) $a^m \times a^n = a^{m+n}$

(ii) $a^m \div a^n = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$

(iv) $(ab)^m = a^m b^m$

(v) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Let us consider some examples to verify the above laws:

Example 2.15: Find the value of

(i) $(625)^{\frac{1}{4}}$ (ii) $(243)^{\frac{2}{5}}$ (iii) $\left(\frac{16}{81}\right)^{-3/4}$

Solution:

(i) $(625)^{\frac{1}{4}} = (5\times5\times5\times5)^{\frac{1}{4}} = (5^4)^{\frac{1}{4}} = 5^{4\times\frac{1}{4}} = 5$

(ii) $(243)^{\frac{2}{5}} = (3\times3\times3\times3\times3)^{\frac{2}{5}} = (3^5)^{\frac{2}{5}} = 3^{5\times\frac{2}{5}} = 3^2 = 9$

(iii) $\left(\frac{16}{81}\right)^{\frac{-3}{4}} = \left(\frac{2\times2\times2\times2}{3\times3\times3\times3}\right)^{\frac{-3}{4}}$

$$= \left[\left(\frac{2}{3}\right)^4\right]^{\frac{-3}{4}} = \left(\frac{2}{3}\right)^{4\times\left(\frac{-3}{4}\right)} = \left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

CHECK YOUR PROGRESS 2.5

1. Simplify each of the following:

   (i) $(16)^{3/4}$

   (ii) $\left(\frac{27}{125}\right)^{-\frac{2}{3}}$

2. Simplify each of the following:

   (i) $(625)^{-\frac{1}{4}} \div (25)^{-\frac{1}{2}}$

   (ii) $\left(\frac{7}{8}\right)^{-\frac{1}{4}} \times \left(\frac{7}{8}\right)^{\frac{1}{2}} \times \left(\frac{7}{8}\right)^{\frac{3}{4}}$

   (iii) $\left(\frac{13}{16}\right)^{-\frac{3}{4}} \times \left(\frac{13}{16}\right)^{\frac{1}{4}} \times \left(\frac{13}{16}\right)^{\frac{3}{2}}$

2.7 SURDS

We have read in first lesson that numbers of the type $\sqrt{2}, \sqrt{3}$ and $\sqrt{5}$ are all irrational numbers. We shall now study irrational numbers of a particular type called **radicals** or **surds**.

A surd is defined as a positive irrational number of the type $\sqrt[n]{x}$, where it is not possible to find exactly the nth root of x, where x is a positive rational number.

The number $\sqrt[n]{x}$ is a surd if and only if

(i) it is an irrational number

(ii) it is a root of the positive rational number

2.7.1 Some Terminology

In the surd $\sqrt[n]{x}$, the symbol $\sqrt{}$ is called a **radical sign**. The index 'n' is called the **order** of the surd and *x* is called the **radicand**.

Note: i) When order of the surd is not mentioned, it is taken as 2. For example, order of $\sqrt{7}\ \left(=\sqrt[2]{7}\right)$ is 2.

ii) $\sqrt[3]{8}$ is not a surd as its value can be determined as 2 which is a rational.

iii) $\sqrt{2+\sqrt{2}}$, although an irrational number, is not a surd because it is the square root of an irrational number.

2.8 PURE AND MIXED SURD

i) A surd, with rational factor is 1 only, other factor being rrational is called a **pure surd**.

For example, $\sqrt[5]{16}$ and $\sqrt[3]{50}$ are pure surds.

ii) A surd, having rational factor other than 1 alongwith the irrational factor, is called a mixed surd.

For example, $2\sqrt{3}$ and $3\sqrt[3]{7}$ are mixed surds.

2.9 ORDER OF A SURD

In the surd $5\sqrt[3]{4}$, 5 is called the **co-efficient** of the surd, 3 is the **order** of the surd and 4 is the **radicand**. Let us consider some examples:

Example 2.16: State which of the following are surds?

(i) $\sqrt{49}$ (ii) $\sqrt{96}$ (iii) $\sqrt[3]{81}$ (iv) $\sqrt[3]{256}$

Solution: (i) $\sqrt{49} = 7$, which is a rational number.

$\therefore\ \sqrt{49}$ is not a surd.

(ii) $\sqrt{96} = \sqrt{4\times4\times6} = 4\sqrt{6}$

$\therefore\ \sqrt{96}$ is an irrational number.

$\Rightarrow \sqrt{96}$ is a surd.

(iii) $\sqrt[3]{81} = \sqrt[3]{3\times3\times3\times3} = 3\sqrt[3]{3}$, which is irrational

$\therefore\ \sqrt[3]{81}$ is a surd.

(iv) $\sqrt[3]{256} = \sqrt[3]{4\times4\times4\times4} = 4\sqrt[3]{4}$

$\therefore\ \sqrt[3]{256}$ is irrational.

$\Rightarrow\ \sqrt[3]{256}$ is a surd

$\therefore$ (ii), (iii) and (iv) are surds.

Example 2.17: Find "index" and "radicand" in each of the following:

(i) $\sqrt[5]{117}$ (ii) $\sqrt{162}$ (iii) $\sqrt[4]{213}$ (iv) $\sqrt[4]{214}$

Solution: (i) index is 5 and radicand is 117.

(ii) index is 2 and radicand is 162.

(iii) index is 4 and radicand is 213.

(iv) index is 4 and radicand is 214.

Example 2.18: Identify "pure" and "mixed" surds from the following:

(i) $\sqrt{42}$ (ii) $4\sqrt[3]{18}$ (iii) $2\sqrt[4]{98}$

Solution: (i) $\sqrt{42}$ is a pure surd.

(ii) $4\sqrt[3]{18}$ is a mixed surd.

(iii) $2\sqrt[4]{98}$ is a mixed surd.

2.10 LAWS OF RADICALS

Given below are Laws of Radicals: (without proof):

(i) $\left[\sqrt[n]{a}\right]^n = a$

(ii) $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$

(iii) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

where *a* and *b* are positive rational numbers and *n* is a positive integer.

Let us take some examples to illustrate.

Example 2.19: Which of the following are surds and which are not? Use laws of radicals to ascertain.

(i) $\sqrt{5} \times \sqrt{80}$ (ii) $2\sqrt{15} \div 4\sqrt{10}$

(iii) $\sqrt[3]{4} \times \sqrt[3]{16}$ (iv) $\sqrt{32} \div \sqrt{27}$

Solution: (i) $\sqrt{5} \times \sqrt{80} = \sqrt{5 \times 80} = \sqrt{400} = 20\cdot$

which is a rational number.

$\therefore\ \sqrt{5} \times \sqrt{80}$ is not a surd.

(ii) $2\sqrt{15} \div 4\sqrt{10} = \dfrac{2\sqrt{15}}{4\sqrt{10}} = \dfrac{\sqrt{15}}{2\sqrt{10}}$

$= \dfrac{\sqrt{15}}{\sqrt{2 \times 2 \times 10}} = \dfrac{\sqrt{15}}{\sqrt{40}} = \sqrt{\dfrac{3}{8}}$, which is irrational.

$\therefore\ 2\sqrt{15} \div 4\sqrt{10}$ is a surd.

(iii) $\sqrt[3]{4} \times \sqrt[3]{16} = \sqrt[3]{64} = 4 \Rightarrow$ It is not a surd.

(iv) $\sqrt{32} \div \sqrt{27} = \dfrac{\sqrt{32}}{\sqrt{27}} = \sqrt{\dfrac{32}{27}}$, which is irrational

$\therefore\ \sqrt{32} \div \sqrt{27}$ is a surd.

CHECK YOUR PROGRESS 2.6

1. For each of the following, write index and the radicand:

(i) $\sqrt[4]{64}$ (ii) $\sqrt[6]{343}$ (iii) $\sqrt{119}$

2. State which of the following are surds:

(i) $\sqrt[3]{64}$ (ii) $\sqrt[4]{625}$ (iii) $\sqrt[6]{216}$

(iv) $\sqrt{5}\times\sqrt{45}$ (v) $3\sqrt{2}\times 5\sqrt{6}$

3. Identify pure and mixed surds out of the following:

(i) $\sqrt{32}$ (ii) $2\sqrt[3]{12}$ (iii) $13\sqrt[3]{91}$ (iv) $\sqrt{35}$

2.11 LAWS OF SURDS

Recall that the surds can be expressed as numbers with fractional exponents. Therefore, laws of indices studied in this lesson before, are applicable to them also. Let us recall them here:

(i) $\sqrt[n]{x}\,.\,\sqrt[n]{y} = \sqrt[n]{xy}$ or $x^{\frac{1}{n}}.y^{\frac{1}{n}} = (xy)^{\frac{1}{n}}$

(ii) $\dfrac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\dfrac{x}{y}}$ or $\dfrac{x^{\frac{1}{n}}}{y^{\frac{1}{n}}} = \left(\dfrac{x}{y}\right)^{\frac{1}{n}}$

(iii) $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x} = \sqrt[n]{\sqrt[m]{x}}$ or $\left(x^{\frac{1}{n}}\right)^{\frac{1}{m}} = x^{\frac{1}{mn}} = \left(x^{\frac{1}{m}}\right)^{\frac{1}{n}}$

(iv) $\sqrt[n]{x^m} = x^{\frac{m}{n}}$ or $\left(x^m\right)^{\frac{1}{n}} = x^{\frac{m}{n}}$

(v) $\sqrt[n]{x^p} = \sqrt[mn]{x^{pn}}$ or $\left(x^p\right)^{\frac{1}{m}} = x^{\frac{p}{m}} = x^{\frac{pn}{mn}} = \left(x^{pn}\right)^{\frac{1}{mn}}$

Here, *x* and *y* are positive rational numbers and m, n and p are positive integers.

Let us illustrate these laws by examples:

(i) $\sqrt[3]{3}\ \sqrt[3]{8} = 3^{\frac{1}{3}}\times 8^{\frac{1}{3}} = (24)^{\frac{1}{3}} = \sqrt[3]{24} = \sqrt[3]{3\times 8}$

(ii) $\dfrac{(5)^{\frac{1}{3}}}{(9)^{\frac{1}{3}}} = \left(\dfrac{5}{9}\right)^{\frac{1}{3}} = \sqrt[3]{\dfrac{5}{9}}$

(iii) $\sqrt[3]{\sqrt[2]{7}} = \sqrt[3]{7^{\frac{1}{2}}} = \left(7^{\frac{1}{2}}\right)^{\frac{1}{3}} = 7^{\frac{1}{6}} = \sqrt[6]{7} = \sqrt[2\times3]{7} = \sqrt[2]{\sqrt[3]{7}}$

(iv) $\sqrt[5]{4^3} = \left(4^3\right)^{\frac{1}{5}} = 4^{\frac{3}{5}} = 4^{\frac{9}{15}} = \sqrt[15]{4^9} = \sqrt[3\times5]{4^{3\times3}}$

Thus, we see that the above laws of surds are verified.

An important point: The order of a surd can be changed by multiplying the index of the surd and index of the radicand by the same positive number.

For example $\sqrt[3]{2} = \sqrt[6]{2^2} = \sqrt[6]{4}$

and $\sqrt[4]{3} = \sqrt[8]{3^2} = \sqrt[8]{9}$

2.12 SIMILAR (OR LIKE) SURDS

Two surds are said to be **similar**, if they can be reduced to the same irrational factor, without consideration for co-efficient.

For example, $3\sqrt{5}$ and $7\sqrt{5}$ are similar surds. Again consider $\sqrt{75} = 5\sqrt{3}$ and $\sqrt{12} = 2\sqrt{3}$. Now $\sqrt{75}$ and $\sqrt{12}$ are expressed as $5\sqrt{3}$ and $2\sqrt{3}$. Thus, they are similar surds.

2.13 SIMPLEST (LOWEST) FORM OF A SURD

A surd is said to be in its simplest form, if it has

a) smallest possible index of the sign

b) no fraction under radical sign

c) no factor of the form a^n, where a is a positive integer, under the radical sign of index n.

For example, $\sqrt[3]{\dfrac{125}{18}} = \sqrt[3]{\dfrac{125\times12}{18\times12}} = \dfrac{5}{6}\sqrt[3]{12}$

Let us take some examples.

Example 2.20: Express each of the following as pure surd in the simplest form:

(i) $2\sqrt{7}$ (ii) $4\sqrt[4]{7}$ (iii) $\dfrac{3}{4}\sqrt{32}$

Solution:

(i) $2\sqrt{7} = \sqrt{2^2 \times 7} = \sqrt{4 \times 7} = \sqrt{28}$, which is a pure surd.

(ii) $4\sqrt[4]{7} = \sqrt[4]{4^4 \times 7} = \sqrt[4]{256 \times 7} = \sqrt[4]{1792}$, which is a pure surd.

(iii) $\frac{3}{4}\sqrt{32} = \sqrt{32 \times \frac{9}{16}} = \sqrt{18}$, which is a pure surd.

Example 2.21: Express as a mixed surd in the simplest form:

(i) $\sqrt{128}$ (ii) $\sqrt[6]{320}$ (iii) $\sqrt[3]{250}$

Solution:

(i) $\sqrt{128} = \sqrt{64 \times 2} = 8\sqrt{2}$,

which is a mixed surd.

(ii) $\sqrt[6]{320} = 6\sqrt{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 5}$

$= \sqrt[6]{2^6 \times 5} = 2\sqrt[6]{5}$, which is a mixed surd.

(iii) $\sqrt[3]{250} = \sqrt[3]{5 \times 5 \times 5 \times 2} = 5\sqrt[3]{2}$, which is a mixed surd.

CHECK YOUR PROGRESS 2.7

1. State which of the following are pairs of similar surds:

   (i) $\sqrt{8}, \sqrt{32}$ (ii) $5\sqrt{3}, 6\sqrt{18}$ (iii) $\sqrt{20}, \sqrt{125}$

2. Express as a pure surd:

   (i) $7\sqrt{3}$ (ii) $3\sqrt[3]{16}$ (iii) $\frac{5}{8}\sqrt{24}$

3. Express as a mixed surd in the simplest form:

   (i) $\sqrt[3]{250}$ (ii) $\sqrt[3]{243}$ (iii) $\sqrt[4]{512}$

2.14 FOUR FUNDAMENTAL OPERATIONS ON SURDS

2.14.1 Addition and Subtraction of Surds

As in rational numbers, surds are added and subtracted in the same way.

For example, $5\sqrt{3}+17\sqrt{3}=(5+17)\sqrt{3}=22\sqrt{3}$

and $12\sqrt{5}-7\sqrt{5}=[12-7]\sqrt{5}=5\sqrt{5}$

For adding and subtracting surds, we first change them to similar surds and then perform the operations.

For example i) $\sqrt{50}+\sqrt{288}$

$= \sqrt{5\times5\times2}+\sqrt{12\times12\times2}$

$= 5\sqrt{2}+12\sqrt{2}=\sqrt{2}(5+12)=17\sqrt{2}$

ii) $\sqrt{98}-\sqrt{18}$

$= \sqrt{7\times7\times2}-\sqrt{3\times3\times2}$

$= 7\sqrt{2}-3\sqrt{2}=(7-3)\sqrt{2}=4\sqrt{2}$

Example 2.22: Simplify each of the following:

(i) $4\sqrt{6}+2\sqrt{54}$

(ii) $45\sqrt{6}-3\sqrt{216}$

Solution: (i) $4\sqrt{6}+2\sqrt{54}$

$= 4\sqrt{6}+2\sqrt{3\times3\times6}$

$= 4\sqrt{6}+6\sqrt{6}=10\sqrt{6}$

(ii) $45\sqrt{6}-3\sqrt{216}$

$= 45\sqrt{6}-3\sqrt{6\times6\times6}$

$= 45\sqrt{6}-18\sqrt{6}$

$= 27\sqrt{6}$

Example 2.23: Show that

$$24\sqrt{45}-16\sqrt{20}+\sqrt{245}-47\sqrt{5}=0$$

Solution: $24\sqrt{45}-16\sqrt{20}+\sqrt{245}-47\sqrt{5}$

$$= 24\sqrt{3\times3\times5} - 16\sqrt{2\times2\times5} + \sqrt{7\times7\times5} - 47\sqrt{5}$$

$$= 72\sqrt{5} - 32\sqrt{5} + 7\sqrt{5} - 47\sqrt{5}$$

$$= \sqrt{5}[72 - 32 + 7 - 47]$$

$$= \sqrt{5}\times0 = 0 = \text{RHS}$$

Example 2.24: Simplify: $2\sqrt[3]{16000} + 8\sqrt[3]{128} - 3\sqrt[3]{54} + \sqrt[4]{32}$

Solution: $2\sqrt[3]{16000} = 2\sqrt[3]{10\times10\times10\times8\times2} = 2\times10\times2\sqrt[3]{2} = 40\sqrt[3]{2}$

$$8\sqrt[3]{128} = 8\sqrt[3]{4\times4\times4\times2} = 32\sqrt[3]{2}$$

$$3\sqrt[3]{54} = 3\sqrt[3]{3\times3\times3\times2} = 9\sqrt[3]{2}$$

$$\sqrt[4]{32} = 2\sqrt[4]{2}$$

∴ Required expression

$$= 40\sqrt[3]{2} + 32\sqrt[3]{2} - 9\sqrt[3]{2} + 2\sqrt[4]{2}$$
$$= (40 + 32 - 9)\sqrt[3]{2} + 2\sqrt[4]{2}$$
$$= 63\sqrt[3]{2} + 2\sqrt[4]{2}$$

CHECK YOUR PROGRESS 2.8

Simplify each of the following:

1. $\sqrt{175} + \sqrt{112}$
2. $\sqrt{32} + \sqrt{200} + \sqrt{128}$
3. $3\sqrt{50} + 4\sqrt{18}$
4. $\sqrt{108} - \sqrt{75}$
5. $\sqrt[3]{24} + \sqrt[3]{81} - 8\sqrt[3]{3}$
6. $6\sqrt[3]{54} - 2\sqrt[3]{16} + 4\sqrt[3]{128}$
7. $12\sqrt{18} + 6\sqrt{20} - 6\sqrt{147} + 3\sqrt{50} + 8\sqrt{45}$

2.14.2 Multiplication and Division in Surds

Two surds can be multiplied or divided if they are of the same order. We have read that the order of a surd can be changed by multiplying or dividing the index of the surd and index of the radicand by the same positive number. Before multiplying or dividing, we change them to the surds of the same order.

Let us take some examples:

$$\sqrt{3}\times\sqrt{2}=\sqrt{3\times2}=\sqrt{6} \qquad \left[\sqrt{3} \text{ and } \sqrt{2} \text{ are of same order}\right]$$

$$\sqrt{12}\div\sqrt{2}=\frac{\sqrt{12}}{\sqrt{2}}=\sqrt{6}$$

Let us multiply $\sqrt{3}$ and $\sqrt[3]{2}$

$$\sqrt{3}=\sqrt[6]{3^3}=\sqrt[6]{27}$$

$$\sqrt[3]{2}=\sqrt[6]{4}$$

$$\therefore \sqrt{3}\times\sqrt[3]{2}=\sqrt[6]{27}\times\sqrt[6]{4}=\sqrt[6]{108}$$

$$\text{and } \frac{\sqrt{3}}{\sqrt[3]{2}}=\frac{\sqrt[6]{27}}{\sqrt[6]{4}}=\sqrt[6]{\frac{27}{4}}$$

Let us consider an example:

Example 2.25: (i) Multiply $5\sqrt[3]{16}$ and $11\sqrt[3]{40}$.

(ii) Divide $15\sqrt[3]{13}$ by $6\sqrt[6]{5}$.

Solution: (i) $5\sqrt[3]{16} \times 11\sqrt[3]{40}$

$$= 5\times11\times\sqrt[3]{2\times2\times2\times2}\times\sqrt[3]{2\times2\times2\times5}$$

$$= 55\times2\times2\sqrt[3]{2}\ \sqrt[3]{5}$$

$$= 220\sqrt[3]{10}$$

(ii) $$\frac{15\sqrt[3]{13}}{6\sqrt[6]{5}}=\frac{5}{2}.\frac{\sqrt[6]{13^2}}{\sqrt[6]{5}}=\frac{5}{2}\sqrt[6]{\frac{169}{5}}$$

Example 2.26: Simplify and express the result in simplest form:

$$2\sqrt{50}\times\sqrt{32}\times2\sqrt{72}$$

Solution: $2\sqrt{50} = 2\sqrt{5\times5\times2} = 10\sqrt{2}$

$\sqrt{32} = \sqrt{2\times2\times2\times2\times2} = 4\sqrt{2}$

$2\sqrt{72} = 2\times6\sqrt{2} = 12\sqrt{2}$

$\therefore$ Given expression

$= 10\sqrt{2}\times4\sqrt{2}\times12\sqrt{2}$

$= 960\sqrt{2}$

2.15 COMPARISON OF SURDS

To compare two surds, we first change them to surds of the same order and then compare their radicands along with their co-efficients. Let us take some examples:

Example 2.27: Which is greater $\sqrt{\frac{1}{4}}$ or $\sqrt[3]{\frac{1}{3}}$?

Solution: $\sqrt{\frac{1}{4}} = \sqrt[6]{\left(\frac{1}{4}\right)^3} = \sqrt[6]{\frac{1}{64}}$

$\sqrt[3]{\frac{1}{3}} = \sqrt[6]{\frac{1}{9}}$

$\frac{1}{9} > \frac{1}{64} \Rightarrow \sqrt[6]{\frac{1}{9}} > \sqrt[6]{\frac{1}{64}} \Rightarrow \sqrt[3]{\frac{1}{3}} > \sqrt{\frac{1}{4}}$

Example 2.28: Arrange in ascending order: $\sqrt[3]{2}$, $\sqrt{3}$ and $\sqrt[6]{5}$.

Solution: LCM of 2, 3, and 6 is 6.

$\therefore \sqrt[3]{2} = \sqrt[6]{2^2} = \sqrt[6]{4}$

$\sqrt{3} = \sqrt[6]{3^3} = \sqrt[6]{27}$

$\sqrt[6]{5} = \sqrt[6]{5}$

Now $\sqrt[6]{4} < \sqrt[6]{5} < \sqrt[6]{27}$

$\Rightarrow \sqrt[3]{2} < \sqrt[6]{5} < \sqrt{3}$

CHECK YOUR PROGRESS 2.9

1. Multipliy $\sqrt[3]{32}$ and $5\sqrt[3]{4}$.
2. Multipliy $\sqrt{3}$ and $\sqrt[3]{5}$.
3. Divide $\sqrt[3]{135}$ by $\sqrt[3]{5}$.
4. Divide $2\sqrt{24}$ by $\sqrt[3]{320}$.
5. Which is greater $\sqrt[4]{5}$ or $\sqrt[3]{4}$?
6. Which in smaller: $\sqrt[5]{10}$ or $\sqrt[4]{9}$?
7. Arrange in ascending order:

   $\sqrt[3]{2},\ \sqrt[6]{3},\ \sqrt[3]{4}$
8. Arrange in descending order:

   $\sqrt[3]{2},\ \sqrt[4]{3},\ \sqrt[3]{4}$

2.16 RATIONALISATION OF SURDS

Consider the products:

(i) $3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3$

(ii) $5^{\frac{7}{11}} \times 5^{\frac{4}{11}} = 5$

(iii) $7^{\frac{1}{4}} \times 7^{\frac{3}{4}} = 7$

In each of the above three multiplications, we see that on multiplying two surds, we get the result as rational number. In such cases, each surd is called the rationalising factor of the other surd.

(i) $\sqrt{3}$ is a rationalising factor of $\sqrt{3}$ and vice-versa.

(ii) $\sqrt[11]{5^4}$ is a rationalising factor of $\sqrt[11]{5^7}$ and vice-versa.

(iii) $\sqrt[4]{7}$ is a rationalising factor of $\sqrt[4]{7^3}$ and vice-versa.

In other words, the process of converting surds to rational numbers is called rationalisation and two numbers which on multiplication give the rational number is called the rationalisation factor of the other.

For example, the rationalising factor of $\sqrt{x}$ is $\sqrt{x}$, of $\sqrt{3}+\sqrt{2}$ is $\sqrt{3}-\sqrt{2}$.

Note:

(i) The quantities $x-\sqrt{y}$ and $x+\sqrt{y}$ are called conjugate surds. Their sum and product are always rational.

(ii) Rationalisation is usually done of the denominator of an expression involving irrational surds.

Let us consider some examples.

Example 2.29: Find the rationalising factors of $\sqrt{18}$ and $\sqrt{12}$.

Solution: $\sqrt{18} = \sqrt{3\times3\times2} = 3\sqrt{2}$

$\therefore$ Rationalising factor is $\sqrt{2}$.

$\sqrt{12} = \sqrt{2\times2\times3} = 2\sqrt{3}$.

$\therefore$ Rationalising factor is $\sqrt{3}$.

Example 2.30: Rationalise the denominator of $\dfrac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}}$.

Solution:

$$\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}} = \frac{\left(\sqrt{2}+\sqrt{5}\right)\left(\sqrt{2}+\sqrt{5}\right)}{\left(\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+\sqrt{5}\right)} = \frac{\left(\sqrt{2}+\sqrt{5}\right)^2}{-3}$$

$$= -\frac{7+2\sqrt{10}}{3} = -\frac{7}{3}-\frac{2}{3}\sqrt{10}$$

Example 2.31: Rationalise the denominator of $\dfrac{4+3\sqrt{5}}{4-3\sqrt{5}}$.

Solution:

$$\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = \frac{\left(4+3\sqrt{5}\right)\left(4+3\sqrt{5}\right)}{\left(4-3\sqrt{5}\right)\left(4+3\sqrt{5}\right)}$$

$$= \frac{16+45+24\sqrt{5}}{16-45} = -\frac{61}{29}-\frac{24}{29}\sqrt{5}$$

Example 2.32: Rationalise the denominator of $\frac{1}{\sqrt{3}-\sqrt{2}+1}$.

Solution:
$$\frac{1}{\sqrt{3}-\sqrt{2}+1} = \frac{\left(\sqrt{3}-\sqrt{2}\right)-1}{\left[\left(\sqrt{3}-\sqrt{2}\right)+1\right]\left[\left(\sqrt{3}-\sqrt{2}\right)-1\right]}$$

$$= \frac{\sqrt{3}-\sqrt{2}-1}{\left(\sqrt{3}-\sqrt{2}\right)^2-1} = \frac{\sqrt{3}-\sqrt{2}-1}{4-2\sqrt{6}}$$

$$= \frac{\sqrt{3}-\sqrt{2}-1}{4-2\sqrt{6}} \times \frac{4+2\sqrt{6}}{4+2\sqrt{6}}$$

$$= \frac{4\sqrt{3}-4\sqrt{2}-4+6\sqrt{2}-4\sqrt{3}-2\sqrt{6}}{16-24}$$

$$= -\frac{\sqrt{2}-2-\sqrt{6}}{4} = \frac{\sqrt{6}-\sqrt{2}+2}{4}$$

Example 2.33: If $\frac{3+2\sqrt{2}}{3-\sqrt{2}} = a+b\sqrt{2},$ find the values of *a* and *b*.

Solution:
$$\frac{3+2\sqrt{2}}{3-\sqrt{2}} = \frac{3+2\sqrt{2}}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{9+4+9\sqrt{2}}{9-2}$$

$$= \frac{13+9\sqrt{2}}{7} = \frac{13}{7}+\frac{9}{7}\sqrt{2} = \mathrm{a}+\mathrm{b}\sqrt{2}$$

$$\Rightarrow \mathrm{a} = \frac{13}{7}, \qquad \mathrm{b} = \frac{9}{7}$$

CHECK YOUR PROGRESS 2.10

1. Find the rationalising factor of each of the following:

   (i) $\sqrt[3]{49}$ (ii) $\sqrt{2}+1$ (iii) $\sqrt[3]{x^2}+\sqrt[3]{y^2}+\sqrt[3]{xy}$

2. Simplify by rationalising the denominator of each of the following:

   (i) $\frac{12}{\sqrt{5}}$ (ii) $\frac{2\sqrt{3}}{\sqrt{17}}$ (iii) $\frac{\sqrt{11}-\sqrt{5}}{\sqrt{11}+\sqrt{5}}$ (iv) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

3. Simplify: $\frac{2+\sqrt{3}}{2-\sqrt{3}}+\frac{2-\sqrt{3}}{2+\sqrt{3}}$

4. Rationalise the denominator of $\frac{1}{\sqrt{3}-\sqrt{2}-1}$

5. If $a=3+2\sqrt{2}$. Find $a+\frac{1}{a}$.

6. If $\frac{2+5\sqrt{7}}{2-5\sqrt{7}}=x+\sqrt{7}y$, find x and y.

LET US SUM UP

- $a \times a \times a \times \ldots.. m$ times $= a^m$ is the exponential form, where a is the base and m is the exponent.
- Laws of exponent are:

  (i) $a^m \times a^n = a^{m+n}$ (ii) $a^m \div a^n = a^{m-n}$ (iii) $(ab)^m = a^m b^m$ (iv) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

  (v) $\left(a^m\right)^n = a^{mn}$ (vi) $a^o = 1$ (vii) $a^{-m} = \frac{1}{a^m}$

- $a^{\frac{p}{q}} = \sqrt[q]{a^p}$
- An irrational number $\sqrt[n]{x}$ is called a surd, if x is a rational number and nth root of x is not a rational number.
- In $\sqrt[n]{x}$, n is called index and x is called radicand.
- A surd with rational co-efficient (other than 1) is called a mixed surd.
- The order of the surd is the number that indicates the root.
- The order of $\sqrt[n]{x}$ is n
- Laws of radicals (a > 0, b > 0)

  (i) $\left[\sqrt[n]{a}\right]^n = a$ (ii) $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ (iii) $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

- Operations on surds

$$x^{\frac{1}{n}} \times y^{\frac{1}{n}} = (xy)^{\frac{1}{n}}; \left(x^{\frac{1}{n}}\right)^{\frac{1}{m}} = x^{\frac{1}{mn}} = \left(x^{\frac{1}{m}}\right)^{\frac{1}{n}}; \frac{x^{\frac{1}{n}}}{y^{\frac{1}{n}}} = \left(\frac{x}{y}\right)^{\frac{1}{n}}$$

$$\left(x^{m}\right)^{\frac{1}{n}} = x^{\frac{m}{n}}; \sqrt[m]{x^{a}} = \sqrt[mn]{x^{an}} \text{ or } \left(x^{a}\right)^{\frac{1}{m}} = x^{\frac{a}{m}} = x^{\frac{an}{mn}} = \left(x^{an}\right)^{\frac{1}{mn}}$$

- Surds are similar if they have the same irrational factor.
- Similar surds can be added and subtracted.
- Orders of surds can be changed by multiplying index of the surds and index of the radicand by the same positive number.
- Surds of the same order are multiplied and divided.
- To compare surds, we change surds to surds of the same order. Then they are compared by their radicands alongwith co-efficients.
- If the product of two surds is rational, each is called the rationalising factor of the other.
- $\mathrm{x}+\sqrt{\mathrm{y}}$ is called rationalising factor of $\mathrm{x}-\sqrt{\mathrm{y}}$ and vice-versa.

TERMINAL EXERCISE

1. Express the following in exponential form:

   (i) 5 × 3 × 5 × 3 × 7 × 7 × 7 × 9 × 9

   (ii) $\left(\frac{-7}{9}\right) \times \left(\frac{-7}{9}\right) \times \left(\frac{-7}{9}\right) \times \left(\frac{-7}{9}\right)$

2. Simplify the following:

   (i) $\left(-\frac{5}{6}\right)^{3} \times \left(\frac{7}{5}\right)^{2} \times \left(\frac{3}{7}\right)^{3}$

   (ii) $\left(\frac{3}{7}\right)^{2} \times \frac{35}{27} \times \left(-\frac{1}{5}\right)^{2}$

3. Simplify and express the result in exponential form:

   (i) $(10)^{2} \times (6)^{2} \times (5)^{2}$

(ii) $\left(-\frac{37}{19}\right)^{20} \div \left(-\frac{37}{19}\right)^{20}$

(iii) $\left[\left(\frac{3}{13}\right)^{3}\right]^{5}$

4. Simplify each of the following:

   (i) $3^{o}+7^{o}+37^{o}-3$ (ii) $(7^{o}+3^{o})\,(7^{o}-3^{o})$

5. Simplify the following:

   (i) $(32)^{12} \div (32)^{-6}$ (ii) $(111)^{6} \times (111)^{-5}$ (iii) $\left(-\frac{2}{9}\right)^{-3} \times \left(-\frac{2}{9}\right)^{5}$

6. Find x so that $\left(\frac{3}{7}\right)^{-3} \times \left(\frac{3}{7}\right)^{11} = \left(\frac{3}{7}\right)^{x}$

7. Find x so that $\left(\frac{3}{13}\right)^{-2} \times \left(\frac{3}{13}\right)^{-9} = \left(\frac{3}{13}\right)^{2x+1}$

8. Express as a product of primes and write the answers of each of the following in exponential form:

   (i) 6480000 (ii) 172872 (iii) 11863800

9. The star sirus is about 8.1×10^{13} km from the earth. Assuming that the light travels at 3.0×10^{5} km per second, find how long light from sirus takes to reach earth.

10. State which of the following are surds:

    (i) $\sqrt{\frac{36}{289}}$ (ii) $\sqrt[9]{729}$ (iii) $\sqrt[3]{\sqrt{5}+1}$ (iv) $\sqrt[4]{3125}$

11. Express as a pure surd:

    (i) $3\sqrt[2]{3}$ (ii) $5\sqrt[3]{4}$ (iii) $5\sqrt[5]{2}$

12. Express as a mixed surd in simplest form:

    (i) $\sqrt[4]{405}$ (ii) $\sqrt[5]{320}$ (iii) $\sqrt[3]{128}$

13. Which of the following are pairs of similar surds?

    (i) $\sqrt{112}, \sqrt{343}$ (ii) $\sqrt[3]{625}, \sqrt[3]{3125 \times 25}$ (iii) $\sqrt[6]{216}, \sqrt{250}$

14. Simplify each of the following:

(i) $4\sqrt{48}-\frac{5}{2}\sqrt{\frac{1}{3}}+6\sqrt{3}$

(ii) $\sqrt{63}+\sqrt{28}-\sqrt{175}$

(iii) $\sqrt{8}+\sqrt{128}-\sqrt{50}$

15. Which is greater?

(i) $\sqrt{2}$ or $\sqrt[3]{3}$ (ii) $\sqrt[3]{6}$ or $\sqrt[4]{8}$

16. Arrange in descending order:

(i) $\sqrt{3}, \sqrt[3]{4}, \sqrt[4]{5}$ (ii) $\sqrt{2}, \sqrt{3}, \sqrt[3]{4}$

17. Arrange in ascending order:

$\sqrt[3]{16}, \sqrt{12}, \sqrt[6]{320}$

18. Simplify by rationalising the denominator:

(i) $\frac{3}{\sqrt{6}-\sqrt{7}}$ (ii) $\frac{12}{\sqrt{7}-\sqrt{3}}$ (iii) $\frac{\sqrt{5}-2}{\sqrt{5}+2}$

19. Simplify each of the following by rationalising the denominator:

(i) $\frac{1}{1+\sqrt{2}-\sqrt{3}}$ (ii) $\frac{1}{\sqrt{7}+\sqrt{5}-\sqrt{12}}$

20. If $\frac{5+2\sqrt{3}}{7+4\sqrt{3}}=a+b\sqrt{3},$ find the values of a and b, where a and b are rational numbers.

21. If $x=7+4\sqrt{3}$, find the value of $x+\frac{1}{x}$.

ANSWERS TO CHECK YOUR PROGRESS

2.1

1. (i) $(-7)^4$ (ii) $\left(\frac{3}{4}\right)^{10}$ (iii) $\left(\frac{-5}{7}\right)^{20}$

2.

	Base	Exponent
(i)	-3	5
(ii)	7	4
(iii)	$-\frac{2}{11}$	8

3. (i) $\frac{81}{2401}$ (ii) $\frac{16}{6561}$ (iii) $-\frac{27}{64}$

4. (i) $\frac{3}{7}$ (ii) $\frac{625}{324}$

5. (i) $\left(\frac{1}{3}\right)^5$ (ii) $\left(-\frac{1}{7}\right)^4$ (iii) $\left(-\frac{5}{3}\right)^4$

2.2

1. (i) $3^1 \times 11^1 \times 13^1$ (ii) $2^3 \times 3^4$ (iii) $2^3 \times 3^3 \times 7^1$

2. (i) 3^6 (ii) 2^9 (iii) $2^5 \times 3^4$

(iv) $\frac{11^3}{2^{12}}$ (v) $\frac{(-7)^3}{2^5}$

2.3

1. (i) $(7)^5$ (ii) $\left(\frac{3}{4}\right)^5$ (iii) $\left(-\frac{7}{8}\right)^6$

2. (i) $(-7)^2$ (ii) $\left(\frac{3}{4}\right)^6$ (iii) $\left(-\frac{7}{8}\right)^{15}$

3. (i) 2^{18} (ii) $\left(\frac{3}{4}\right)^6$ (iii) $\left(-\frac{5}{9}\right)^{15}$

(iv) $\left(\frac{11}{3}\right)^5$ (v) $\left(-\frac{7}{11}\right)^3$

4. True: (i), (ii), (vii)

False: (iii), (iv), (v), (vi)

2.4

1. $\frac{49}{9}$

2. (i) $\left(\frac{7}{3}\right)^4$ (ii) 12^2 (iii) $\left(\frac{13}{3}\right)^{12}$

3. (i) $\left(\frac{7}{3}\right)^{-4}$ (ii) $\left(\frac{1}{7}\right)^{-10}$ (iii) $\left(-\frac{4}{3}\right)^{-10}$

4. (i) $\frac{81}{16}$ (ii) $-\frac{2}{3}$ (iii) $-\frac{343}{125}$

5. True: (ii), (iii), (iv)

2.5

1. (i) 8 (ii) $\frac{25}{9}$

2. (i) 1 (ii) $\frac{7}{8}$ (iii) $\frac{13}{16}$

2.6

1. (i) 4, 64 (ii) 6, 343 (iii) 2, 119

2. (iii), (iv)

3. Pure: (i), (iv)

   Mixed: (ii), (iii)

2.7

1. (i), (iii)

2. (i) $\sqrt{147}$ (ii) $\sqrt[3]{432}$ (iii) $\sqrt{\frac{75}{8}}$

3. (i) $5\sqrt[3]{2}$ (ii) $3\sqrt[3]{9}$ (iii) $4\sqrt[4]{2}$

2.8

1. $9\sqrt{7}$ 2. $22\sqrt{2}$ 3. $27\sqrt{2}$ 4. $\sqrt{3}$

5. $-3\sqrt{3}$ 6. $30\sqrt[3]{2}$ 7. $51\sqrt{2}+36\sqrt{5}-42\sqrt{3}$

2.9

1. $20\sqrt[3]{2}$ 2. $3\sqrt[3]{5}$ 3. 3 4. $\sqrt[6]{\frac{216}{25}}$

5. $\sqrt[3]{4}$ 6. $\sqrt[4]{9}$ 7. $\sqrt[6]{3}, \sqrt[3]{2}, \sqrt[3]{4}$ 8. $\sqrt[3]{4}, \sqrt[4]{3}, \sqrt[3]{2}$

2.10

1. (i) $\sqrt[3]{7}$ (ii) $\sqrt{2}-1$ (iii) $\sqrt[3]{x}-\sqrt[3]{y}$

2. (i) $\frac{12}{5}\sqrt{5}$ (ii) $\frac{2\sqrt{51}}{17}$ (iii) $\frac{8}{3}-\frac{\sqrt{55}}{3}$ (iv) $2+\sqrt{3}$

3. 14

4. $-\frac{1}{4}\left[2+\sqrt{6}+\sqrt{2}\right]$

5. 6

6. $-\frac{179}{171}-\frac{20\sqrt{7}}{171}$

ANSWERS TO TERMINAL EXERCISE

1. (i) $5^2 \times 3^2 \times 7^3 \times 9^2$ (ii) $\left(-\frac{7}{9}\right)^4$

2. (i) $-\frac{5}{56}$ (ii) $\frac{1}{105}$

3. (i) $2^4 \times 3^2 \times 5^4$ (ii) 1 (iii) $\left(\frac{3}{13}\right)^{15}$

4. (i) zero (ii) zero

5. (i) $(32)^{18}$ (ii) 111 (iii) $\left(\frac{2}{9}\right)^2$

6. $x = 8$

7. $x = -6$

8. $2^7 \times 3^4 \times 5^4$

9. $3^3 \times 10^7$ seconds

10. (ii), (iii), (iv)

11. (i) $\sqrt[2]{27}$ (ii) $\sqrt[3]{500}$ (iii) $\sqrt[5]{6250}$

12. (i) $3\sqrt[4]{5}$ (ii) $2\sqrt[5]{10}$ (iii) $4\sqrt[3]{2}$

13. (i), (ii)

14. (i) $\frac{127}{6}\sqrt{3}$ (ii) zero (iii) $5\sqrt{2}$

15. (i) $\sqrt[3]{3}$ (ii) $\sqrt[3]{6}$

16. (i) $\sqrt{3}, \sqrt[3]{4}, \sqrt[4]{5}$ (ii) $\sqrt{3}, \sqrt[3]{4}, \sqrt{2}$

17. $\sqrt[3]{16}, \sqrt[6]{320}, \sqrt{12}$

18. (i) $-3\left(\sqrt{6}+\sqrt{7}\right)$ (ii) $3\left(\sqrt{7}+\sqrt{3}\right)$ (ii) $9-4\sqrt{5}$

19. (i) $\frac{2+\sqrt{2}+\sqrt{6}}{4}$ (ii) $\frac{7\sqrt{5}+5\sqrt{7}+2\sqrt{105}}{70}$

20. $a = 11$, $b = -6$

21. 14

3

ALGEBRAIC EXPRESSIONS AND POLYNOMIALS

So far, you had been using arithmetical numbers, which included natural numbers, whole numbers, fractional numbers, etc. and fundamental operations on those numbers. In this lesson, we shall introduce algebraic numbers and some other basic concepts of algebra like constants, variables, algebraic expressions, special algebraic expressions, called polynomials and four fundamental operations on them.

OBJECTIVES

After studying this lesson, you will be able to

- *identify variables and constants in an expression;*
- *cite examples of algebraic expressions and their terms;*
- *understand and identify a polynomial as a special case of an algebraic expression;*
- *cite examples of types of polynomials in one and two variables;*
- *identify like and unlike terms of polynomials;*
- *determine degree of a polynomial;*
- *find the value of a polynomial for given value(s) of variable(s), including zeros of a polynomial;*
- *perform four fundamental operations on polynomials.*

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of number systems and four fundamental operations.
- Knowledge of other elementary concepts of mathematics at primary and upper primary levels.

3.1 INTRODUCTION TO ALGEBRA

You are already familiar with numbers 0, 1, 2, 3, ..., $\frac{1}{2}, \frac{3}{4}, ...\sqrt{2}, ...$ etc. and operations of addition (+), subtraction (–), multiplication (×) and division (÷) on these numbers. Sometimes, letters called **literal numbers**, are also used as symbols to represent numbers. Suppose we want to say "The cost of one book is twenty rupees".

In arithmetic, we write : The cost of one book = ₹ 20

In algebra, we put it as: the cost of one book in rupees is x. Thus x stands for a number.

Similarly, a, b, c, x, y, z, etc. can stand for number of chairs, tables, monkeys, dogs, cows, trees, etc. The use of letters help us to think in more general terms.

Let us consider an example, you know that if the side of a square is 3 units, its perimeter is 4 × 3 units. In algebra, we may express this as

$$p = 4\ s$$

where p stands for the number of units of perimeter and s those of a side of the square.

On comparing the language of arithmetic and the language of algebra we find that the language of algebra is

(a) more precise than that of arithmetic.

(b) more general than that of arithmetic.

(c) easier to understand and makes solutions of problems easier.

A few more examples in comparative form would confirm our conclusions drawn above:

Verbal statement	**Algebraic statement**
(i) A number increased by 3 gives 8	$a + 3 = 8$
(ii) A number increased by itself gives 12	$x + x = 12$, written as $2x = 12$
(iii) Distance = speed × time	$d = s \times t$, written as $d = st$
(iv) A number, when multiplied by itself and added to 5 gives 9	$b \times b + 5 = 9$, written as $b^2 + 5 = 9$
(v) The product of two successive natural numbers is 30	$y \times (y + 1) = 30$, wrtten as $y\ (y + 1) = 30$, where y is a natural number.

Since literal numbers are used to represent numbers of arithmetic, symbols of operation +, –, × and ÷ have the same meaning in algebra as in arithmetic. Multiplication symbols in algebra are often omitted. Thus for 5 × a we write 5a and for a × b we write ab.

3.2 VARIABLES AND CONSTANTS

Consider the months — January, February, March, ..., December of the year 2009. If we represent 'the year 2009' by a and 'a month' by x we find that in this situation 'a' (year 2009) is a fixed entity whereas x can be any one of January, February, March, ..., December. Thus, x is not fixed. It varies. We say that in this case 'a' is a **constant** and 'x' is a **variable**.

Similarly, when we consider students of class X and represent class X by, say, b and a student by, say, y; we find that in this case b (class X) is fixed and so b is a constant and y (a student) is a variable as it can be any one student of class X.

Let us consider another situation. If a student stays in a hostel, he will have to pay fixed room rent, say, ₹ 1000. The cost of food, say ₹ 100 per day, depends on the number of days he takes food there. In this case room rent is constant and the number of days, he takes food there, is variable.

Now think of the numbers.

$$4, -14, \sqrt{2}, \frac{\sqrt{3}}{2}, -\frac{4}{15}, 3x, \frac{21}{8}y, \sqrt{2}z$$

You know that $4, -14, \sqrt{2}, \frac{\sqrt{3}}{2}$, and $-\frac{4}{15}$ are real numbers, each of which has a fixed value while $3x, \frac{21}{8}y$ and $\sqrt{2}z$ contain unknown x, y and z respectively and therefore do not have fixed values like 4, –14, etc. Their values depend on x, y and z respectively. Therefore, x, y and z are variables.

Thus, *a variable is literal number which can have different values whereas a constant has a fixed value.*

In algebra, we usually denote constants by a, b, c and variables x, y, z. However, the context will make it clear whether a literal number has denoted a constant or a variable.

3.3 ALBEGRAIC EXPRESSIONS AND POLYNOMIALS

Expressions, involving arithmetical numbers, variables and symbols of operations are called algebraic expressions. Thus, $3+8, 8x+4, 5y, 7x-2y+6, \frac{1}{\sqrt{2x}}, \frac{x}{\sqrt{y-2}}, \frac{ax+by+cz}{x+y+z}$ are all algebraic expressions. You may note that 3 + 8 is both an arithmetic as well as algebraic expression.

An algebraic expression is a combination of numbers, variables and arithmetical operations.

One or more signs + or – separates an algebraic expression into several parts. Each part along with its sign is called a **term** of the expression. Often, the plus sign of the first term is omitted in writing an algebraic expression. For example, we write x – 5y + 4 instead of writing + x – 5y + 4. Here x, – 5y and 4 are the three terms of the expression.

In $\frac{1}{3}xy$, $\frac{1}{3}$ is called the numerical coefficient of the term and also of xy. **coefficient** of x is $\frac{1}{3}y$ and that of y is $\frac{1}{3}x$. When the numerical coefficient of a term is +1 or –1, the '1' is usually omitted in writing. Thus, numerical coefficent of a term, say, x^2y is +1 and that of $-x^2y$ is –1.

An algebraic expression, in which variable(s) does (do) not occur in the denominator, exponents of variable(s) are whole numbers and numerical coefficients of various terms are real numbers, is called a polynomial.

In other words,

(i) No term of a polynomial has a variable in the denominator;

(ii) In each term of a polynomial, the exponents of the variable(s) are non-negative integers; and

(iii) Numerical coefficient of each term is a real number.

Thus, for example, 5, 3x –y , $\frac{1}{3}a - b + \frac{7}{2}$ and $\frac{1}{4}x^3 - 2y^2 + xy - 8$ are all polynomials whereas $x^3 - \frac{1}{x}, \sqrt{x+y}$ and $x^{\frac{2}{3}} + 5$ are not polynomials.

$x^2 + 8$ is a polynomial in one variable x and $2x^2 + y^3$ is a polynomial in two variables x and y. In this lesson, we shall restrict our discussion of polynomials including two variables only.

General form of a polynomial in one variable x is:

$$a_0 + a_1x + a_2x^2 + ...+a_nx^n$$

where coefficients a_0, a_1, a_2, ...a_n are real numbers, x is a variable and n is a whole number. a_0, a_1x, a_2x^2, ..., a_nx^n are (n + 1) terms of the polynomial.

An algebraic expression or a polynomial, consisting of only one term, is called a **monomial**.

Thus, –2, 3y, $-5x^2$, xy, $\frac{1}{2}x^2y^3$ are all monomials.

An algebraic expression or a polynomial, consisting of only two terms, is called a **binomial**. Thus, 5 + x, $y^2 - 8x$, $x^3 - 1$ are all bionomials.

An algebraic expression or a polynomial, consisting of only three terms, is called a **trinomial**. Thus $x + y + 1$, $x^2 + 3x + 2$, $x^2 + 2xy + y^2$ are all trinomials.

The terms of a polynomial, having the same variable(s) and the same exponents of the variable(s), are called like terms.

For example, in the expression

$$3xy + 9x + 8xy - 7x + 2x^2$$

the terms 3xy and 8 xy are like terms; also 9x and –7x are **like terms** whereas 9x and $2x^2$ are not like terms. Terms that are not like, are called **unlike terms**. In the above expression 3xy and –7x are also unlike terms.

Note that arithmetical numbers are like terms. For example, in the polynomials $x^2 + 2x + 3$ and $x^3 - 5$, the terms 3 and – 5 are regrded as like terms since $3 = 3x^0$ and $-5 = -5x^0$.

The terms of the expression

$$2x^2 - 3xy + 9y^2 - 7y + 8$$

are all unlike, i.e., there are no two like terms in this expression.

Example 3.1: Write the variables and constants in $2x^2y + 5$.

Solution: Variables : x and y

Constants: 2 and 5

Example 3.2: In $8x^2y^3$, write the coefficient of

(i) x^2y^3 (ii) x^2 (iii) y^3

Solution: (i) $8x^2y^3 = 8 \times (x^2y^3)$

$\therefore$ Coefficient of x^2y^3 is 8

(ii) $8x^2y^3 = 8y^3 \times (x^2)$

$\therefore$ Coefficient of x^2 is $8y^3$.

(iii) $8x^2y^3 = 8x^2 \times (y^3)$

$\therefore$ Coefficient of y^3 is $8x^2$.

Example 3.3: Write the terms of expression

$$3x^2y - \frac{5}{2}x - \frac{1}{3}y + 2$$

Solution: The terms of the given expression are

$$3x^2y,\ -\frac{5}{2}x,\ -\frac{1}{3}y,\ 2$$

Example 3.4: Which of the following algebraic expressions are polynomials?

(i) $\frac{1}{2}+x^3-2x^2+\sqrt{6}x$ (ii) $x+\frac{1}{x}$

(iii) $2x^2+3x-5\sqrt{x}+6$ (iv) $5-x-x^2-x^3$

Solution: (i) and (iv) are polynomials.

In (ii), second term is $\frac{1}{x}=x^{-1}$. Since second term contains negative exponent of the variable, the expression is not a polynomial.

In (iii), third term is $-5\sqrt{x}=-5x^{\frac{1}{2}}$. Since third term contains fractional exponent of the variable, the expression is not a polynomial.

Example 3.5: Write like terms, if any, in each of the following expressions:

(i) $x+y+2$ (ii) $x^2-2y-\frac{1}{2}x^2+\sqrt{3}y-8$

(iii) $1-2xy+2x^2y-2xy^2+5x^2y^2$ (iv) $\frac{2}{\sqrt{3}}y-\frac{1}{3}z+\frac{\sqrt{5}}{3}y+\frac{1}{3}$

Solution: (i) There are no like terms in the expression.

(ii) x^2 and $-\frac{1}{2}x^2$ are like terms, also $-2y$ and $\sqrt{3}\,y$ are like terms

(iii) There are no like terms in the expression.

(iv) $\frac{2}{\sqrt{3}}y$ and $\frac{\sqrt{5}}{3}y$ are like terms

CHECK YOUR PROGRESS 3.1

1. Write the variables and constants in each of the following:

(i) $1+y$ (ii) $\frac{2}{3}x+\frac{1}{3}y+7$ (iii) $\frac{4}{5}x^2y^3$

(iv) $\frac{2}{5}xy^5+\frac{1}{2}$ (v) $2x^2+y^2-8$ (vi) $x+\frac{1}{x}$

2. In $2x^2y$, write the coefficient of

   (i) x^2y (ii) x^2 (iii) y

3. Using variables and operation symbols, express each of the following verbal statements as algebraic statements:

   (i) three less than a number equals fifteen.

   (ii) A number increased by five gives twenty-two.

4. Write the terms of each of the following expressions:

   (i) $2 + abc$ (ii) $a + b + c + 2$ (iii) $x^2y - 2xy^2 - \frac{1}{2}$

   (iv) $\frac{1}{8}x^3y^2$

5. Identify like terms, if any, in each of the following expressions:

   (i) $-xy^2 + x^2y + y^2 + \frac{1}{3}y^2x$ (ii) $6a + 6b - 3ab + \frac{1}{4}a^2b + ab$

   (iii) $ax^2 + by^2 + 2c - a^2x - b^2y - \frac{1}{3}c^2$

6. Which of the following algebraic expressions are polynomials?

   (i) $\frac{1}{3}x^3 + 1$ (ii) $5^2 - y^2 - 2$ (iii) $4x^{-3} + 3y$

   (iv) $5\sqrt{x+y} + 6$ (v) $3x^2 - \sqrt{2}\,y^2$ (vi) $y^2 - \frac{1}{y^2} + 4$

7. Identify each of the following as a monomial, binomial or a trinomial:

   (i) $x^3 + 3$ (ii) $\frac{1}{3}x^3y^3$ (iii) $2y^2 + 3yz + z^2$

   (iv) $5 - xy - 3x^2y^2$ (v) $7 - 4x^2y^2$ (vi) $-8x^3y^3$

3.4 DEGREE OF A POLYNOMIAL

The sum of the exponents of the variables in a term is called the **degree** of that term. For example, the degree of $\frac{1}{2}x^2y$ is 3 since the sum of the exponents of x and y is 2 + 1, i.e., 3. Similarly, the degree of the term $2x^5$ is 5. The degree of a non-zero constant, say, 3 is 0 since it can be written as $3 = 3 \times 1 = 3 \times x^0$, as $x^0 = 1$.

A polynomial has a number of terms separated by the signs + or –. **The degree of a polynomial is the same as the degree of its term or terms having the highest degree and non-zero coefficient.**

For example, consider the polynomial

$$3x^4y^3 + 7\,xy^5 - 5x^3y^2 + 6xy$$

It has terms of degrees 7, 6, 5, and 2 respectively, of which 7 is the highest. Hence, the degree of this polynomial is 7.

A polynomial of degree 2 is also called a **quadratic polynomial**. For example, $3 - 5x + 4x^2$ and $x^2 + xy + y^2$ are quadratic polynomials.

Note that **the degree of a non-zero constant polynomial is taken as zero.**

When all the coefficients of variable(s) in the terms of a polynomial are zeros, the polynomial is called a **zero polynomial. The degree of a zero polynomial is not defined.**

3.5 EVALUATION OF POLYNOMIALS

We can evaluate a polynomial for given value of the variable occuring in it. Let us understand the steps involved in evaluation of the polynomial $3x^2 - x + 2$ for $x = 2$. Note that we restrict ourselves to polynomials in one variable.

Step 1: Substitute given value(s) in place of the variable(s).

Here, when $x = 2$, we get $3 \times (2)^2 - 2 + 2$

Step 2: Simplify the numerical expression obtained in Step 1.

$$3 \times (2)^2 - 2 + 2 = 3 \times 4 = 12$$

Therefore, when $x = 2$, we get $3x^2 - x + 2 = 12$

Let us consider another example.

Example 3.6: Evaluate

(i) $1 - x^5 + 2x^6 + 7\,x$ for $x = \frac{1}{2}$

(ii) $5x^3 + 3x^2 - 4x - 4$ for $x = 1$

Solution: (i) For $x = \frac{1}{2}$, the value of the given polynomial is:

$$= 1 - \left(\frac{1}{2}\right)^5 + 2\left(\frac{1}{2}\right)^6 + 7 \times \frac{1}{2}$$

$$= 1 - \frac{1}{32} + \frac{1}{32} + \frac{7}{2}$$

$$= \frac{9}{2} = 4\frac{1}{2}$$

(ii) For x = 1, the value of the given polynomial is:

$5 \times (1)^3 + 3 \times (1)^2 - 4 \times 1 - 4$

$= 5 + 3 - 4 - 4 = 0$

3.6 ZERO OF A POLYNOMIAL

The value(s) of the variable for which the value of a polynomial in one variable is zero is (are) called **zero(s) of the polynomial**. In Example 3.6(ii) above, the value of the polynomial $5x^3 + 3x^2 - 4x - 4$ for x = 1 is zero. Therefore, we say that x = 1 is a zero of the polynomial $5x^3 + 3x^2 - 4x - 4$.

Let us consider another example.

Example 3.7: Determine whether given value is a zero of the given polynomial:

(i) $x^3 + 3x^2 + 3x + 2;\ x = -1$

(ii) $x^4 - 4x^3 + 6x^2 - 4x + 1;\ x = 1$

Solution: (i) For x = – 1, the value of the given polynomial is

$(-1)^3 + 3 \times (-1)^2 + 3 \times (-1) + 2$

$= -1 + 3 - 3 + 2$

$= 1 \ (\neq 0)$

Hence, x = – 1 is not a zero of the given polynomial.

(ii) For x = 1, the value of the given polynomial is

$(1)^4 - 4 \times (1)^3 + 6 \times (1)^2 - 4 \times 1 + 1$

$= 1 - 4 + 6 - 4 + 1$

$= 0$

Hence, x = 1 is a zero of the given polynomial.

CHECK YOUR PROGRESS 3.2

1. Write the degree of each of the following monomials:

(i) $\frac{18}{5}x^7$ (ii) $\frac{7}{8}y^3$ (iii) 10 x (iv) 27

2. Rewrite the following monomials in increasing order of their degrees:

$$-3x^6,\ \frac{2}{9}x^2,\ 9x,\ -25x^3,\ 2.5$$

3. Determine the degree of each of the following polynomials:

(i) $5x^6y^4 + 1$ (ii) $10^5 + xy^3$ (iii) $x^2 + y^2$ (iv) $x^2y + xy^2 - 3xy + 4$

4. Evaluate each of the following polynomials for the indicated value of the variable:

(i) $x^2 - 25$ for $x = 5$ (ii) $x^2 + 3x - 5$ for $x = -2$

(iii) $\frac{2}{3}x^3 + \frac{4}{5}x^2 - \frac{7}{5}$ for $x = -1$ (iv) $2x^3 - 3x^2 - 3x + 12$ for $x = -2$

5. Verify that each of $x = 2$ and $x = 3$ is a zero of the polynomial $x^2 - 5x + 6$.

3.7 ADDITION AND SUBTRACTION OF POLYNOMIALS

You are now familiar that polynomials may consist of like and unlike terms. In adding polynomials, we add their like terms together. Similarly, in subtracting a polynomial from another polynomial, we subtract a term from a like term. The question, now, arises 'how do we add or subtract like terms?' Let us take an example.

Suppose we want to add like terms 2x and 3x. The procedure, that we follow in arithmetic, we follow in algebra too. You know that

$$5 \times 6 + 5 \times 7 = 5 \times (6 + 7)$$

$$6 \times 5 + 7 \times 5 = (6 + 7) \times 5$$

Therefore, $2x + 3x = 2 \times x + 3 \times x$

$$= (2 + 3) \times x$$

$$= 5 \times x$$

$$= 5x$$

Similarly, $2xy + 4\,xy = (2 + 4)\,xy = 6xy$

$$3x^2y + 8x^2y = (3 + 8)x^2y = 11x^2y$$

In the same way, since

$$7 \times 5 - 6 \times 5 = (7 - 6) \times 5 = 1 \times 5$$

$$\therefore\ 5y - 2y = (5 - 2) \times y = 3y$$

and $9x^2y^2 - 5x^2y^2 = (9 - 5)x^2y^2 = 4x^2y^2$

In view of the above, we conclude:

1. *The sum of two (or more) like terms is a like term whose numerical coefficient is the sum of the numerical coefficients of the like terms.*
2. *The difference of two like terms is a like term whose numerical coefficient is the difference of the numerical coefficients of the like terms.*

Therefore, to add two or more polynomials, we take the following steps:

Step 1: Group the like terms of the given polynomials together.

Step 2: Add the like terms together to get the sum of the given polynomials.

Example 3.8: Add $-3x + 4$ and $2x^2 - 7x - 2$

Solution:

$$(-3x + 4) + (2x^2 - 7x - 2)$$
$$= 2x^2 + (-3x - 7x) + (4 - 2)$$
$$= 2x^2 + (-3 - 7)x + 2$$
$$= 2x^2 + (-10)x + 2$$
$$= 2x^2 - 10x + 2$$
$$\therefore\ (-3x + 4) + (2x^2 - 7x - 2) = 2x^2 - 10x + 2$$

Polynomials can be added more conveniently if

(i) the given polynomials are so arranged that their like terms are in one column, and

(ii) the coefficients of each column (i.e. of the group of like terms) are added

Thus, Example 3.8 can also be solved as follows:

$$\begin{array}{lll} & -3x & +4 \\ 2x^2 & -7x & -2 \\ \hline 2x^2 + & (-7 - 3)x & + (4 - 2) \end{array}$$

$$\therefore\ (-3x + 4) + (2x^2 - 7x - 2) = 2x^2 - 10x + 2$$

Example 3.9: Add $5x + 3y - \frac{3}{4}$ and $-2x + y + \frac{7}{4}$

Solution:

$$\begin{array}{l} 5x + 3y - \frac{3}{4} \\ -2x + y + \frac{7}{4} \\ \hline 3x + 4y + \left(\frac{7}{4} - \frac{3}{4}\right) \\ \hline = 3x + 4y + 1 \end{array}$$

$$\therefore \left(5x + 3y - \frac{3}{4}\right) + \left(-2x + y + \frac{7}{4}\right) = 3x + 4y + 1$$

Example 3.10: Add $\frac{3}{2}x^3 + x^2 + x + 1$ and $x^4 - \frac{x^3}{2} - 3x + 1$

Solution: $\frac{3}{2}x^2 + x^2 + x + 1$

$$+x^4 - \frac{1}{2}x^3 \qquad -3x + 1$$

$$x^4 + \left(\frac{3}{2} - \frac{1}{2}\right)x^3 + x^2 + (1-3)x + (1+1)$$

$$= x^4 + x^3 + x^2 - 2x + 2$$

$$\therefore \left(\frac{3}{2}x^2 + x^2 + x + 1\right) + \left(x^4 - \frac{x^3}{2} - 3x + 1\right) = x^4 + x^3 + x^2 - 2x + 2$$

In order to subtract one polynomial from another polynomial, we go through the following three steps:

Step 1: Arrange the given polynomials in columns so that like terms are in one column.

Step 2: Change the sign (from + to – and from – to +) of each term of the polynomial to be subtracted.

Step 3: Add the like terms of each column separately.

Let us understand the procedure by means of some examples.

Example 3.11: Subtract $-4x^2 + 3x + \frac{2}{3}$ from $9x^2 - 3x - \frac{2}{7}$.

Solution: $9x^2 - 3x - \frac{2}{7}$

$$-4x^2 + 3x + \frac{2}{3}$$

$$+ \qquad - \qquad -$$

$$(9+4)x^2 + (-3-3)x + \left(-\frac{2}{7} - \frac{2}{3}\right)$$

$$= 13x^2 - 6x - \frac{20}{21}$$

$$\therefore \left(9x^2-3x-\frac{2}{7}\right)-\left(-4x^2+3x+\frac{2}{3}\right)=13x^2-6x-\frac{20}{21}$$

Example 3.12: Subtract $3x - 5x^2 + 7 + 3x^3$ from $2x^2 - 5 + 11x - x^3$.

Solution:

$$\begin{array}{l} -x^3 + 2x^2 + 11x - 5 \\ 3x^3 \;\; - 5x^2 + 3x \;\; + 7 \\ - \quad + \quad - \qquad - \\ \hline (-1-3)x^3 + (2+5)x^2 + (11-3)x + (-5-7) \\ \hline = -4x^3 + 7x^2 + 8x - 12 \end{array}$$

$$\therefore (2x^2 - 5 + 11x - x^3) - (3x - 5x^2 + 7 + 3x^3) = -4x^3 + 7x^2 + 8x - 12$$

Example 3.13: Subtract $12xy - 5y^2 - 9x^2$ from $15xy + 6y^2 + 7x^2$.

Solution:

$$\begin{array}{l} 15xy + 6y^2 + 7x^2 \\ 12xy - 5y^2 - 9x^2 \\ - \quad\; + \quad\; + \\ \hline 3xy + 11y^2 + 16x^2 \\ \hline \end{array}$$

Thus, $(15xy + 6y^2 + 7x^2) - (12xy - 5y^2 - 9x^2) = 3xy + 11y^2 + 16x^2$

We can also directly subtract without arranging expressions in columns as follows:

$$(15xy + 6y^2 + 7x^2) - (12xy - 5y^2 - 9x^2)$$

$$= 15xy + 6y^2 + 7x^2 - 12xy + 5y^2 + 9x^2$$

$$= 3xy + 11y^2 + 16x^2$$

In the same manner, we can add more than two polynomials.

Example 3.14: Add polynomials $3x + 4y - 5x^2$, $5y + 9x$ and $4x - 17y - 5x^2$.

Solution:

$$\begin{array}{l} 3x + 4y - 5x^2 \\ 9x + 5y \\ 4x - 17y - 5x^2 \\ \hline 16x - 8y - 10x^2 \\ \hline \end{array}$$

$$\therefore (3x + 4y - 5x^2) + (5y + 9x) + (4x - 17y - 5x^2) = 16x - 8y - 10x^2$$

Example 3.15: Subtract $x^2 - x - 1$ from the sum of $3x^2 - 8x + 11$, $-2x^2 + 12x$ and $-4x^2 + 17$.

Solution: Firstly we find the sum of $3x^2 - 8x + 11, - 2x^2 + 12x$ and $- 4x^2 + 17$.

$$\begin{array}{rrr}
3x^2 & - 8x & + 11 \\
- 2x^2 & + 12x & \\
- 4x^2 & & + 17 \\
\hline
- 3x^2 & + 4x & + 28 \\
\hline
\end{array}$$

Now, we subtract $x^2 - x - 1$ from this sum.

$$\begin{array}{rrr}
- 3x^2 & + 4x & + 28 \\
x^2 & - x & - 1 \\
- & + & + \\
\hline
- 4x^2 & + 5x & + 29 \\
\hline
\end{array}$$

Hence, the required result is $- 4x^2 + 5x + 29$.

CHECK YOUR PROGRESS 3.3

1. Add the following pairs of polynomials:

   (i) $\frac{2}{3}x^2 + x + 1; \quad \frac{3}{7}x^2 + \frac{1}{4}x + 5$

   (ii) $\frac{7}{5}x^3 - x^2 + 1; \quad 2x^2 + x - 3$

   (iii) $7x^2 - 3x + 4y; \quad 3x^3 + 5x^2 - 4x + \frac{7}{3}y$

   (iv) $2x^3 + 7x^2y - 5xy + 7; \ - 2x^2y + 7x^3 - 3xy - 7$

2. Add:

   (i) $x^2 - 3x + 5, \ 5 + 7x - 3x^2$ and $x^2 + 7$

   (ii) $\frac{1}{3}x^2 + \frac{7}{8}x - 5, \ \frac{2}{3}x^2 + 5 + \frac{1}{8}x$ and $- x^2 - x$

   (iii) $a^2 - b^2 + ab, \ b^2 - c^2 + bc$ and $c^2 - a^2 + ca$

   (iv) $2a^2 + 3b^2, 5a^2 - 2b^2 + ab$ and $- 6a^2 - 5ab + b^2$

3. Subtract:

   (i) $7x^3 - 3x^2 + 2$ from $x^2 - 5x + 2$

   (ii) $3y - 5y^2 + 7 + 3y^3$ from $2y^2 - 5 + 11y - y^3$

(iii) $2z^3 + 7z - 5z^2 + 2$ from $5z + 7 - 3z^2 + 5z^3$

(iv) $12x^3 - 3x^2 + 11x + 13$ from $5x^3 + 7x^2 + 2x - 4$

4. Subtract $4a - b - ab + 3$ from the sum of $3a - 5b + 3ab$ and $2a + 4b - 5ab$.

3.8 MULTIPLICATION OF POLYNOMIALS

To multiply a monomial by another monomial, we make use of laws of exponents and the rule of signs. For example,

$$3a \times a^2b^2c^2 = (3 \times 1)\, a^{2+1}\, b^2\, c^2 = 3a^3b^2c^2$$

$$-5x \times 2\, xy^3 = (-5 \times 2)\, x^{1+1}\, y^3 = -10\, x^2y^3$$

$$-\frac{1}{2}y^2z \times \left(-\frac{1}{3}\right)yz = \left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)y^{2+1}z^{1+1} = \frac{1}{6}y^3z^2$$

To multiply a polynomial by a monomial, we multiply each term of the polynomial by the monomial. For example

$$x^2y \times (-y^2 + 2xy + 1) = x^2y \times (-y^2) + (x^2y) \times 2xy + (x^2y) \times 1$$

$$= -x^2y^3 + 2x^3y^2 + x^2y$$

To multiply a polynomial by another polynomial, we multiply each term of one polynomial by each term of the other polynomial and simplify the result by combining the terms. It is advisable to arrange both the polynomials in increasing or decreasing powers of the variable. For example,

$$(2n + 3)(n^2 - 3n + 4) = 2n \times n^2 + 2n \times (-3n) + 2n \times 4 + 3 \times n^2 + 3 \times (-3n) + 3 \times 4$$

$$= 2n^3 - 6n^2 + 8n + 3n^2 - 9n + 12$$

$$= 2n^3 - 3n^2 - n + 12$$

Let us take some more examples.

Example 3.16: Find the product of $(0.2x^2 + 0.7\,x + 3)$ and $(0.5\,x^2 - 3x)$

Solution: $(0.2x^2 + 0.7\,x + 3) \times (0.5\,x^2 - 3x)$

$$= 0.2x^2 \times 0.5\,x^2 + 0.2x^2 \times (-3x) + 0.7\,x \times 0.5\,x^2 + 0.7\,x \times (-3x) + 3 \times 0.5x^2 + 3\times (-3x)\,)$$

$$= 0.1x^4 - 0.60x^3 + 0.35x^3 - 2.1x^2 + 1.5\,x^2 - 9x$$

$$= 0.1\,x^4 - 0.25\,x^3 - 0.6x^2 - 9x$$

Example 3.17: Multiply $2x - 3 + x^2$ by $1 - x$.

Solution: Arranging polynomials in decreasing powers of x, we get

$$\begin{aligned}(x^2 + 2x - 3) \times (-x + 1) &= x^2 \times (-x) + x^2 \times (1) + 2x \times (-x) + 2x \times 1 - 3 \times (-x) \\ &\quad - 3 \times 1 \\ &= -x^3 + x^2 - 2x^2 + 2x + 3x - 3 \\ &= -x^3 - x^2 + 5x - 3\end{aligned}$$

Alternative method:

$$\begin{array}{rrrrl} & x^2 & +2x & -3 & \leftarrow \text{one polynomial} \\ \hline & -x & +1 & & \leftarrow \text{other polynomial} \\ -x^3 & -2x^2 & +3x & & \left.\right] \\ & +x^2 & +2x & -3 & \left.\right] \leftarrow \text{Partial products} \\ \hline -x^3 & -x^2 & +5x & -3 & \leftarrow \text{Product} \\ \hline \end{array}$$

3.9 DIVISION OF POLYNOMIALS

To divide a monomial by another monomial, we find the quotient of numerical coefficients and variable(s) separately using laws of exponents and then multiply these quotients. For example,

(i) $$25x^3y^3 \div 5x^2y = \frac{25x^3y^3}{5x^2y} = \frac{25}{5} \times \frac{x^3}{x^2} \times \frac{y^3}{y}$$

$$= 5 \times x^1 \times y^2$$

$$= 5xy^2$$

(ii) $$-12ax^2 \div 4x = -\frac{12ax^2}{4x} = \frac{-12}{4} \times \frac{a}{1} \times \frac{x^2}{x}$$

$$= -3ax$$

To divide a polynomial by a monomial, we divide each term of the polynomial by the monomial. For example,

(i) $$\left(15x^3 - 3x^2 + 18x\right) \div 3x = \frac{15x^3}{3x} - \frac{3x^2}{3x} + \frac{18x}{3x}$$

$$= 5x^2 - x + 6$$

(ii) $$\left(-8x^2 + 10x\right) \div (-2x) = \frac{-8x^2}{-2x} + \frac{10x}{-2x}$$

$$= \left(\frac{-8}{-2}\right)\left(\frac{x^2}{x}\right) + \frac{10}{(-2)} \times \frac{x}{x}$$

$$= 4x - 5$$

The process of division of a polynomial by another polynomial is done on similar lines as in arithmetic. Try to recall the process when you divided 20 by 3.

$$\text{Divisor} \longrightarrow \begin{array}{r} 6 \\ 3\overline{)20} \\ 18 \\ \hline 2 \\ \hline \end{array} \begin{array}{l} \longleftarrow \text{Quotient} \\ \longleftarrow \text{Dividend} \\ \\ \longleftarrow \text{Remainder} \end{array}$$

The steps involved in the process of division of a polynomial by another polynomial are explained below with the help of an example.

Let us divide $2x^2 + 5x + 3$ by $2x + 3$.

Step 1: Arrange the terms of both the polynomials in decreasing powers of the variable common to both the polynomials.

$$2x+3\overline{)2x^2+5x+3}$$

Step 2: Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.

$$\begin{array}{r} x \\ 2x+3\overline{)2x^2+5x+3} \end{array}$$

Step 3: Multiply all the terms of the divisor by the first term of the quotient and subtract the result from the dividend, to obtain a remainder (as next dividend)

$$\begin{array}{rl} & x \\ 2x+3) & \overline{2x^2+5x+3} \\ & 2x^2 + 3x \\ & \underline{- \quad\ -} \\ & 2x+3 \end{array}$$

Step 4: Divide the first term of the resulting dividend by the first term of the divisor and write the result as the second term of the quotient.

Step 5: Multiply all the terms of the divisor by the second term of the quotient and subtract the result from the resulting dividend of Step 4.

$$\begin{array}{rl} & x+1 \\ 2x+3) & \overline{2x^2+5x+3} \\ & 2x^2 + 3x \\ & \underline{- \quad\ -} \\ & 2x+3 \\ & 2x+3 \\ & \underline{- \quad -} \\ & \underline{0} \end{array}$$

Step 6: Repeat the process of Steps 4 and 5, till you get either the remainder zero or a polynomial having the highest exponent of the variable lower than that of the divisor.

In the above example, we got the quotient $x + 1$ and remainder 0.

Let us now consider some more examples.

Example 3.18 : Divide $x^3 - 1$ by $x - 1$.

Solution:

$$\begin{array}{rl} & x^2+x+1 \\ x-1) & \overline{x^3-1} \\ & x^3 \quad -x^2 \\ & \underline{- \qquad +} \\ & x^2 \quad -1 \\ & x^2 \quad -x \\ & \underline{- \qquad +} \\ & x \ -\ 1 \end{array}$$

$$\begin{array}{r}
x \;-\; 1 \\
- \quad + \\
\hline
0 \\
\hline
\end{array}$$

We get quotient $x^2 + x + 1$ and remainder 0.

Example 3.19: Divide $5x - 11 - 12x^2 + 2x^3$ by $2x - 5$.

Solution: Arranging the dividend in decreasing powers of x, we get it as

$2x^3 - 12x^2 + 5x - 11$

So,

$$\begin{array}{r|l}
 & x^2 - \frac{7}{2}x - \frac{25}{4} \\
\cline{2-2}
2x - 5 & 2x^3 - 12x^2 + 5x - 11 \\
 & 2x^3 - 5x^2 \\
 & - \qquad + \\
\cline{2-2}
 & -7x^2 + 5x - 11 \\
 & -7x^2 + \frac{35}{2}x \\
 & + \qquad - \\
\cline{2-2}
 & -\frac{25}{2}x - 11 \\
 & -\frac{25}{2}x + \frac{125}{4} \\
 & + \qquad - \\
\cline{2-2}
 & -\frac{169}{4} \\
\cline{2-2}
\end{array}$$

We get quotient $x^2 - \frac{7}{2}x - \frac{25}{4}$ and remainder $-\frac{169}{4}$.

CHECK YOUR PROGRESS 3.4

1. Multiply:

   (i) $9b^2c^2$ by $3b$ (ii) $5x^3y^5$ by $-2xy$

   (iii) $2xy + y^2$ by $-5x$ (iv) $x + 5y$ by $x - 3y$

2. Write the quotient:

   (i) $x^5y^3 \div x^2y^2$ (ii) $-28y^7z^2 \div \left(-4y^3z^2\right)$

   (iii) $\left(a^4 + a^3b^5\right) \div a^2$ (iv) $-15b^5c^6 \div 3b^2c^4$

3. Divide and write the quotient and the remainder:

   (i) $x^2 - 1$ by $x + 1$ (ii) $x^2 - x + 1$ by $x + 1$

   (iii) $6x^2 - 5x + 1$ by $2x - 1$ (iv) $2x^3 + 4x^2 + 3x + 1$ by $x + 1$

LET US SUM UP

- A literal number (unknown quantity), which can have various values, is called a variable.
- A constant has a fixed value.
- An algebraic expression is a combination of numbers, variables and arithmetical operations. It has one or more terms joined by the signs + or –.
- Numerical coefficient of a term, say, 2xy is 2. Coefficient of x is 2y and that of y is 2x.
- Numerical coefficient of non-negative x is + 1 and that of – x is – 1.
- An algebraic expression, in which variable(s) does (do) not occur in the denominator, exponents of variables are whole numbers and numerical coefficients of various terms are real numbers, is called a polynomial.
- The standard form of a polynomial in one variable x is:

  $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ (or written in reverse order) where $a_0, a_1, a_2, \dots\ a_n$ are real numbers and n, n–1, n–2, ..., 3, 2, 1 are whole numbers.
- An algebraic expression or a polynomial having one term is called a monomial, that having two terms a bionomial and the one having three terms a trinomial.
- The terms of an algebraic expression or a polynomial having the same variable(s) and same exponent(s) of variable(s) are called like terms. The terms, which are not like, are called unlike terms.
- The sum of the exponents of variables in a term is called the degree of that term.
- The degree of a polynomial is the same as the degree of its term or terms having the highest degree and non-zero numerical coefficient.
- The degree of a non-zero constant polynomial is zero.
- The process of substituting a numerical value for the variable(s) in an algebraic expression (or a polynomial) is called evaluation of the algebraic expression (or polynomial).
- The value(s) of variable(s), for which the value of a polynomial is zero, is (are) called zero(s) of the polynomial.
- The sum of two like terms is a like term whose numerical coefficient is the sum of the numerical coefficients of the two like terms.

- The difference of two like terms is a like term whose numerical coefficient is the difference of the numerical coefficients of the two like terms.
- To multiply or divide a polynomial by a monomial, we multiply or divide each term of the polynomial separately using laws of exponents and the rule of signs.
- To multiply a polynomial by a polynomial, we multiply each term of one polynomial by each term of the other polynomial and simplify the result by combining like terms.
- To divide a polynomial by a polynomial, we usually arrange the terms of both the polynomials in decreasing powers of the variable common to both of them and take steps of division on similar lines as in arithmetic in case of numbers.

TERMINAL EXERCISE

1. Mark a tick (✓) against the correct alternative:

(i) The coefficient of x^4 in $6x^4y^2$ is

(A) 6 (B) y^2 (C) $6y^2$ (D) 4

(ii) Numerical coefficient of the monomial $-x^2y^4$ is

(A) 2 (B) 6 (C) 1 (D) –1

(iii) Which of the following algebraic expressions is a polynomial?

(A) $\frac{1}{\sqrt{2}}x^2 - \sqrt{8} + 3.7x$ (B) $2x + \frac{1}{2x} - 4$

(C) $(x^2 - 2y^2) \div (x^2 + y^2)$ (D) $6 + \sqrt{x} - x - 15x^2$

(iv) How many terms does the expression $1 - \sqrt{2a^2b^3} - (7a)(2b) + \sqrt{3b^2}$ contain?

(A) 5 (B) 4 (C) 3 (D) 2

(v) Which of the following expressions is a binomial?

(A) $2x^2y^2$ (B) $x^2 + y^2 - 2xy$

(C) $2 + x^2 + y^2 + 2x^2y^2$ (D) $1 - 3xy^3$

(vi) Which of the following pairs of terms is a pair of like terms?

(A) 2a, 2b (B) $2xy^3, 2x^3y$

(C) $3x^2y, \frac{1}{\sqrt{2}}yx^2$ (D) 8, 16 a

(vii) A zero of the polynomial $x^2 - 2x - 15$ is

(A) $x = -5$ (B) $x = -3$

(C) $x = 0$ (D) $x = 3$

(viii) The degree of the polynomial $x^3y^4 + 9x^6 - 8y^5 + 17$ is

(A) 7 (B) 17

(C) 5 (D) 6

2. Using variables and operation symbols, express each of the following verbal statements as algebraic statement:

(i) A number added to itself gives six.

(ii) Four subtracted from three times a number is eleven.

(iii) The product of two successive odd numbers is thirty-five.

(iv) One-third of a number exceeds one-fifth of the number by two.

3. Determine the degree of each of the following polynomials:

(i) 3^{27} (ii) $x + 7x^2y^2 - 6xy^5 - 18$ (iii) $a^4x + bx^3$ where a and b are constants

(iv) $c^6 - a^3x^2y^2 - b^2x^3y$ Where a, b and c are constants.

4. Determine whether given value is a zero of the polynomial:

(i) $x^2 + 3x - 40;$ $x = 8$

(ii) $x^6 - 1;$ $x = -1$

5. Evaluate each of the following polynomials for the indicated value of the variable:

(i) $2x - \frac{3}{2}x^2 + \frac{4}{5}x^5 + 7x^3$ at $x = \frac{1}{2}$

(ii) $\frac{4}{5}y^3 + \frac{1}{5}y^2 - 6y - 65$ at $y = -5$

6. Find the value of $\frac{1}{2}n^2 + \frac{1}{2}n$ for $n = 10$ and verify that the result is equal to the sum of first 10 natural numbers.

7. Add:

(i) $\frac{7}{3}x^3 + \frac{2}{5}x^2 - 3x + \frac{7}{5}$ and $\frac{2}{3}x^3 + \frac{3}{5}x^2 - 3x + \frac{3}{5}$

(ii) $x^2 + y^2 + 4xy$ and $2y^2 - 4xy$

(iii) $x^3 + 6x^2 + 4xy$ and $7x^2 + 8x^3 + y^2 + y^3$

(iv) $2x^5 + 3x + \frac{2}{3}$ and $-3x^5 + \frac{2}{5}x - 3$

8. Subtract

   (i) $-x^2 + y^2 - xy$ from 0

   (ii) $a + b - c$ from $a - b + c$

   (iii) $x^2 - y^2x + y$ from $y^2x - x^2 - y$

   (iv) $-m^2 + 3mn$ from $3m^2 - 3mn + 8$

9. What should be added to $x^2 + xy + y^2$ to obtain $2x^2 + 3xy$?

10. What should be subtracted from $-13x + 5y - 8$ to obtain $11x - 16y + 7$?

11. The sum of two polynomials is $x^2 - y^2 - 2xy + y - 7$. If one of them is $2x^2 + 3y^2 - 7y + 1$, find the other.

12. If $A = 3x^2 - 7x + 8$, $B = x^2 + 8x - 3$ and $C = -5x^2 - 3x + 2$, find $B + C - A$.

13. Subtract $3x - y - xy$ from the sum of $3x - y + 2xy$ and $-y - xy$. What is the coefficient of x in the result?

14. Multiply

   (i) $a^2 + 5a - 6$ by $2a + 1$

   (ii) $4x^2 + 16x + 15$ by $x - 3$

   (iii) $a^2 - 2a + 1$ by $a - 1$

   (iv) $a^2 + 2ab + b^2$ by $a - b$

   (v) $x^2 - 1$ by $2x^2 + 1$

   (vi) $x^2 - x + 1$ by $x + 1$

   (vii) $x^2 + \frac{2}{3}x + \frac{5}{6}$ by $x - \frac{7}{4}$

   (viii) $\frac{2}{3}x^2 + \frac{5}{4}x - 3$ by $3x^2 + 4x + 1$

15. Subtract the product of $(x^2 - xy + y^2)$ and $(x + y)$ from the product of $(x^2 + xy + y^2)$ and $(x - y)$.

16. Divide

   (i) $8x^3 + y^3$ by $2x + y$

   (ii) $7x^3 + 18x^2 + 18x - 5$ by $3x + 5$

   (iii) $20x^2 - 15x^3y^6$ by $5x^2$

   (iv) $35a^3 - 21a^4b$ by $(-7a^3)$

   (v) $x^3 - 3x^2 + 5x - 8$ by $x - 2$

   (vi) $8y^2 + 38y + 35$ by $2y + 7$

   In each case, write the quotient and remainder.

ANSWERS TO CHECK YOUR PROGRESS

3.1

1. (i) y; 1 (ii) x, y; $\frac{2}{3}, \frac{1}{3}, 7$ (iii) x, y; $\frac{4}{5}$

(iv) x, y; $\frac{2}{5}, \frac{1}{2}$ (v) x, y; 2, –8 (vi) x; None

2. (i) 2 (ii) 2y (iii) $2x^2$

3. (i) $x - 3 = 15$ (ii) $x + 5 = 22$

4. (i) 2, abc (ii) a,b,c, 2 (iii) $x^2y, -2xy^2, -\frac{1}{2}$

   (iv) $\frac{1}{8}x^3y^2$

5. (i) $-xy^2, +\frac{1}{3}y^2x$ (ii) –3ab, + ab (iii) No like terms

6. (i) , (ii) and (v) 7. Monomials (ii) and (vi);

   Binomials: (i) and (v); Trinomials : (iii) and (iv)

3.2

1. (i) 7 (ii) 3 (iii) 1 (iv) 0

2. 2.5, 9x, $\frac{2}{9}x^2, -25x^3, -3x^6$

3. (i) 10 (ii) 4 (iii) 2 (iv) 3

4. (i) 0 (ii) –7 (iii) $-\frac{19}{15}$ (iv) 6

3.3

1. (i) $\frac{23}{11}x^2 + \frac{5}{4}x + 6$ (ii) $\frac{7}{5}x^3 + x^2 + x - 2$

   (iii) $3x^3 + 12x^2 - 7x + \frac{19}{3}y$ (iv) $9x^3 + 5x^2y - 8xy$

2. (i) $-x^2 + 4x + 17$ (ii) 0

   (iii) ab + bc +ca (iv) $a^2 + 2b^2 - 4ab$

3. (i) $-7x^3 + 4x^2 - 5x$ (ii) $-4y^3 + 7y^2 + 8y - 12$

   (iii) $3z^3 + 2z^2 - 2z + 5$ (iv) $-7x^3 + 10x^2 - 9x - 17$

4. a – ab – 3

3.4

1. (i) $27b^3c^2$ (ii) $-10x^4y^6$
   (iii) $-10x^2y - 5xy^2$ (iv) $x^2 + 2xy - 15y^2$
2. (i) x^3y (ii) $7y^4$ (iii) $a^2 + ab^5$ (iv) $-5b^3c^2$
3. (i) $x - 1; 0$ (ii) $x - 2; 3$ (iii) $3x - 1; 0$ (iv) $2x^2 + 2x + 1; 0$

ANSWERS TO TERMINAL EXERCISE

1. (i) C (ii) D (iii) A (iv) B (v) D (vi) C (vii) B (viii) A
2. (i) $y + y = 6$ (ii) $3y - 4 = 11$ (iii) $z(z + 2) = 35$ (iv) $\frac{x}{3} - \frac{x}{5} = 2$
3. (i) 0 (ii) 6 (iii) 3 (iv) 4
4. (i) No (ii) Yes
5. (i) $\frac{37}{24}$ (ii) 0
6. 55
7. (i) $3x^3 + x^2 - 6x + 2$ (ii) $x^2 + 3y^2$
   (iii) $9x^3 + 13x^2 + 4xy + y^2 + y^3$ (iv) $-x^5 + \frac{17}{5}x - \frac{7}{3}$
8. (i) $x^2 - y^2 + xy$ (ii) $2c - 2b$
   (iii) $2y^2x - 2x^2 - 2y$ (iv) $4m^2 - 6mn + 8$
9. $x^2 + 2xy - y^2$
10. $-24x + 21y - 15$
11. $-x^2 - 4y^2 - 2xy + 8y - 8$
12. $-7x^2 + 12x - 9$
13. $2xy - y; 2y$
14. (i) $2a^3 + 11a^2 - 7a - 6$ (ii) $4x^3 + 4x^2 - 33x - 45$
    (iii) $a^3 - 3a^2 + 3a - 1$ (iv) $a^3 + a^2b - ab^2 - b^3$
    (v) $2x^4 - x^2 - 1$ (vi) $x^3 + 1$
    (vii) $x^3 - \frac{13}{12}x^2 - \frac{x}{3} - \frac{35}{24}$ (viii) $2x^4 + \frac{77}{12}x^3 - \frac{10}{3}x^2 - \frac{43}{4}x - 3$
15. $-2y^3$
16. (i) $4x^2 - 2xy + y^2; 0$ (ii) $9x^2 - 9x + 21; -110$
    (iii) $4 - 3xy^6; 0$ (iv) $-5 + 3ab; 0$
    (iv) $x^2 - x + 3; -2$ (v) $4y + 5; 0$

4

SPECIAL PRODUCTS AND FACTORIZATION

In an earlier lesson you have learnt multiplication of algebraic expressions, particularly polynomials. In the study of algebra, we come across certain products which occur very frequently. By becoming familiar with them, a lot of time and labour can be saved as in those products, multiplication is performed without actually writing down all the steps. For example, products, such as 108 × 108, 97 × 97, 104 × 96, 99 × 99 × 99, can be easily calculated if you know the products $(a + b)^2$, $(a - b)^2$, $(a + b)(a - b)$, $(a - b)^3$ respectively. Such products are called **special products**.

Factorization is a process of finding the factors of certain given products such as $a^2 - b^2$, $a^3 + 8b^3$, etc. We will consider factoring only those polynomials in which coefficients are integers.

In this lesson, you will learn about certain special products and factorization of certain polynomials. Besides, you will learn about finding HCF and LCM of polynomials by factorization. In the end you will be made familiar with rational algebraic expressions and to perform fundamental operations on rational expressions.

OBJECTIVES

After studying this lesson, you will be able to

- *write formulae for special products* $(a \pm b)^2$, $(a + b)(a - b)$, $(x + a)(x + b)$, $(a + b)(a^2 - ab + b^2)$, $(a - b)(a^2 + ab + b^2)$, $(a \pm b)^3$ *and* $(ax + b)(cx + d)$;
- *calculate squares and cubes of numbers using formulae;*
- *factorise given polynomials including expressions of the forms* $a^2 - b^2$, $a^3 \pm b^3$;
- *factorise polynomials of the form* $ax^2 + bx + c$ $(a \neq 0)$ *by splitting the middle term;*
- *determine HCF and LCM of polynomials by factorization;*

- *cite examples of rational expressions in one and two variables;*
- *perform four fundamental operations on rational expressions.*

EXPECTED BACKGROUND KNOWLEDGE

- Number system and four fundamental operations
- Laws of exponents
- Algebraic expressions
- Four fundamental operations on polynomials
- HCF and LCM of numbers
- Elementary concepts of geometry and mensuration learnt at primary and upper primary levels.

4.1 SPECIAL PRODUCTS

Here, we consider some speical products which occur very frequently in algebra.

(1) Let us find $(a + b)^2$

$(a + b)^2 \quad = (a + b)\,(a + b)$

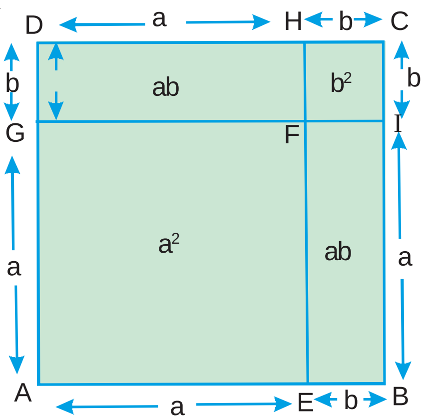
$= a(a + b) + b\,(a + b)$ [Distributive law]

$= a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$

Geometrical verification

Concentrate on the figure, given here, on the right



(i) $(a + b)^2$ = Area of square ABCD

= Area of square AEFG +

area of rectangle EBIF +

area of rectangle DGFH +

area of square CHFI

$= a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$

Thus, $\mathbf{(a + b)^2 = a^2 + 2ab + b^2}$

(2) Let us find $(a-b)^2$

$(a-b)^2 = (a-b)(a-b)$ [Distributive law]

$= a(a-b) - b(a-b)$

$= a^2 - ab - ab + b^2$

$= a^2 - 2ab + b^2$

Method 2: Using $(a+b)^2$

We know that $a - b = a + (-b)$

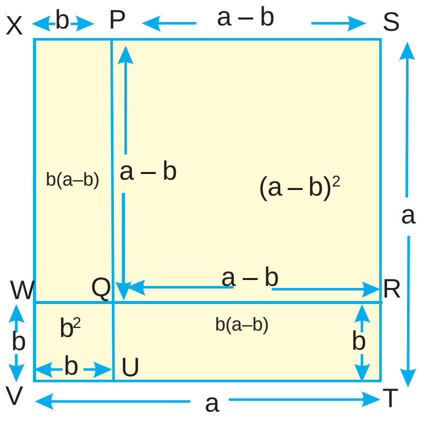
$\therefore (a-b)^2 = [a + (-b)]^2$

$= a^2 + 2(a)(-b) + (-b)^2$

$= a^2 - 2ab + b^2$

Geometrical verification

Concentrate on the figure, given here, on the right



$(a-b)^2$ = Area of square PQRS

= Area of square STVX –

[area of rectangle RTVW +

area of rectangle PUVX –

area of square QUVW]

$= a^2 - (ab + ab - b^2)$

$= a^2 - ab - ab + b^2$

$= a^2 - 2ab + b^2$

Thus, $\mathbf{(a-b)^2 = a^2 - 2ab + b^2}$

Deductions: We have

$(a+b)^2 = a^2 + 2ab + b^2$ .....(1)

$(a-b)^2 = a^2 - 2ab + b^2$ .....(2)

(1) + (2) gives

$\mathbf{(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)}$

(1) – (2) gives

$\mathbf{(a+b)^2 - (a-b)^2 = 4ab}$

(3) Now we find the product $(a + b)(a - b)$

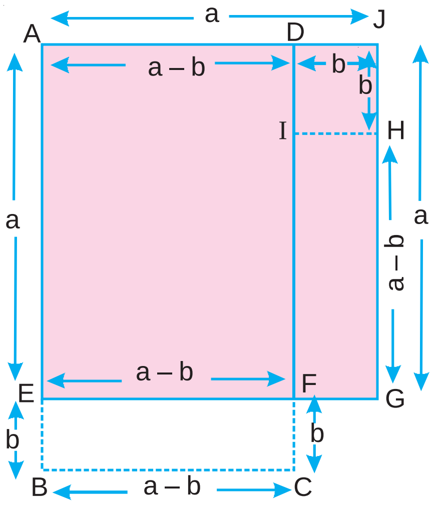
$(a + b)(a - b) = a(a - b) + b(a - b)$ [Distributive law]

$= a^2 - ab + ab - b^2$

$= a^2 - b^2$

Geometrical verification

Observe the figure, given here, on the right



$(a + b)(a - b)$ = Area of Rectangle ABCD

= Area of Rectangle AEFD + area of rectangle EBCF

= Area of Rectangle AEFD + Area of Rectangle FGHI

= [Area of Rectangle AEFD + Area of rectangle FGHI + Area of square DIHJ] – Area of square DIHJ

= Area of square AEGJ – area of square DIHJ

$= a^2 - b^2$

Thus, $\mathbf{(a + b)(a - b) = a^2 - b^2}$

The process of multiplying the sum of two numbers by their difference is very useful in arithmetic. For example,

$64 \times 56 = (60 + 4) \times (60 - 4)$

$= 60^2 - 4^2$

$= 3600 - 16$

$= 3584$

(4) We, now find the product $(x + a)(x + b)$

$(x + a)(x + b) = x(x + b) + a(x + b)$ [Distributive law]

$= x^2 + bx + ax + ab$

$= x^2 + (a + b)x + ab$

Thus, $\mathbf{(x + a)(x + b) = x^2 + (a + b)x + ab}$

Deductions: (i) $\mathbf{(x - a)(x - b) = x^2 - (a + b)x + ab}$

(ii) $\mathbf{(x - a)(x + b) = x^2 + (b - a)x - ab}$

Students are advised to verify these results.

(5) Let us, now, find the product (ax + b) (cx + d)

$$(ax + b)(cx + d) = ax(cx + d) + b(cx + d)$$
$$= acx^2 + adx + bcx + bd$$
$$= acx^2 + (ad + bc)x + bd$$

Thus, $\mathbf{(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd}$

Deductions: (i) $\mathbf{(ax - b)(cx - d) = acx^2 - (ad + bc)x + bd}$

(ii) $\mathbf{(ax - b)(cx + d) = acx^2 - (bc - ad)x - bd}$

Students should verify these results.

Let us, now, consider some examples based on the special products mentioned above.

Example 4.1: Find the following products:

(i) $(2a + 3b)^2$ (ii) $\left(\frac{3}{2}a - 6b\right)^2$

(iii) $(3x + y)(3x - y)$ (iv) $(x + 9)(x + 3)$

(v) $(a + 15)(a - 7)$ (vi) $(5x - 8)(5x - 6)$

(vii) $(7x - 2a)(7x + 3a)$ (viii) $(2x + 5)(3x + 4)$

Solution:

(i) Here, in place of a, we have 2a and in place of b, we have 3b.

$$(2a + 3b)^2 = (2a)^2 + 2(2a)(3b) + (3b)^2$$
$$= 4a^2 + 12ab + 9b^2$$

(ii) Using special product (2), we get

$$\left(\frac{3}{2}a - 6b\right)^2 = \left(\frac{3}{2}a\right)^2 - 2\left(\frac{3}{2}a\right)(6b) + (6b)^2$$

$$= \frac{9}{4}a^2 - 18ab + 36b^2$$

(iii) $(3x + y)(3x - y) = (3x)^2 - y^2$ [using speical product (3)]

$$= 9x^2 - y^2$$

(iv) $(x + 9)(x + 3) = x^2 + (9 + 3)x + 9 \times 3$ [using speical product (4)]

$= x^2 + 12x + 27$

(v) $(a + 15)(a - 7) = a^2 + (15 - 7)a - 15 \times 7$

$= a^2 + 8a - 105$

(vi) $(5x - 8)(5x - 6) = (5x)^2 - (8 + 6)(5x) + 8 \times 6$

$= 25x^2 - 70x + 48$

(vii) $(7x - 2a)(7x + 3a) = (7x)^2 + (3a - 2a)(7x) - (3a)(2a)$

$= 49x^2 + 7ax - 6a^2$

(viii) $(2x + 5)(3x + 4) = (2 \times 3)x^2 + (2 \times 4 + 5 \times 3)x + 5 \times 4$

$= 6x^2 + 23x + 20$

Numerical calculations can be performed more conveniently with the help of special products, often called **algebraic formulae.** Let us consider the following example.

Example 4.2: Using special products, calculate each of the following:

(i) 101 × 101 (ii) 98 × 98 (iii) 68 × 72

(iv) 107 × 103 (v) 56 × 48 (vi) 94 × 99

Solution: (i) $101 \times 101 = 101^2 = (100 + 1)^2$

$= 100^2 + 2 \times 100 \times 1 + 1^2$

$= 10000 + 200 + 1$

$= 10201$

(ii) $98 \times 98 = 98^2 = (100 - 2)^2$

$= 100^2 - 2 \times 100 \times 2 + 2^2$

$= 10000 - 400 + 4$

$= 9604$

(iii) $68 \times 72 = (70 - 2) \times (70 + 2)$

$= 70^2 - 2^2$

$= 4900 - 4$

$= 4896$

(iv) $107 \times 103 = (100 + 7)(100 + 3)$

$= 100^2 + (7 + 3) \times 100 + 7 \times 3$

$= 10000 + 1000 + 21$

$= 11021$

(v) $56 \times 48 = (50 + 6)(50 - 2)$

$= 50^2 + (6 - 2) \times 50 - 6 \times 2$

$= 2500 + 200 - 12$

$= 2688$

(vi) $94 \times 99 = (100 - 6)(100 - 1)$

$= 100^2 - (6 + 1) \times 100 + 6 \times 1$

$= 10000 - 700 + 6$

$= 9306$

CHECK YOUR PROGRESS 4.1

1. Find each of the following products:

(i) $(5x + y)^2$
(ii) $(x - 3)^2$
(iii) $(ab + cd)^2$
(iv) $(2x - 5y)^2$
(v) $\left(\frac{x}{3} + 1\right)^2$
(vi) $\left(\frac{z}{2} - \frac{1}{3}\right)^2$
(vii) $(a^2 + 5)(a^2 - 5)$
(viii) $(xy - 1)(xy + 1)$
(ix) $\left(x + \frac{4}{3}\right)\left(x + \frac{3}{4}\right)$
(x) $\left(\frac{2}{3}x^2 - 3\right)\left(\frac{2}{3}x^2 + \frac{1}{3}\right)$
(xi) $(2x + 3y)(3x + 2y)$
(xii) $(7x + 5y)(3x - y)$

2. Simplify:

(i) $(2x^2 + 5)^2 - (2x^2 - 5)^2$
(ii) $(a^2 + 3)^2 + (a^2 - 3)^2$
(iii) $(ax + by)^2 + (ax - by)^2$
(iv) $(p^2 + 8q^2)^2 - (p^2 - 8q^2)^2$

3. Using special products, calculate each of the following:

(i) 102 × 102
(ii) 108 × 108
(iii) 69 × 69
(iv) 998 × 998
(v) 84 × 76
(vi) 157 × 143
(vii) 306 × 294
(viii) 508 × 492
(ix) 105 × 109
(x) 77 × 73
(xi) 94 × 95
(xii) 993 × 996

4.2 SOME OTHER SPECIAL PRODUCTS

(6) Consider the binomial (a + b). Let us find its cube.

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)(a^2+2ab+b^2) \text{ [using laws of exponents)} \\ &= a(a^2+2ab+b^2)+b(a^2+2ab+b^2) \quad \text{[Distributive laws)} \\ &= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 \\ &= a^3+3a^2b+3ab^2+b^3 \\ &= a^3+3ab(a+b)+b^3\end{aligned}$$

Thus, $\mathbf{(a+b)^3=a^3+3ab(a+b)+b^3}$

(7) We now find the cube of (a – b).

$$\begin{aligned}(a-b)^3 &= (a-b)(a-b)^2 \\ &= (a-b)(a^2-2ab+b^2) \text{ [using laws of exponents)} \\ &= a(a^2-2ab+b^2)-b(a^2-2ab+b^2) \quad \text{[Distributive laws)} \\ &= a^3-2a^2b+ab^2-a^2b+2ab^2-b^3 \\ &= a^3-3a^2b+3ab^2-b^3 \\ &= a^3-3ab(a-b)-b^3\end{aligned}$$

Thus, $\mathbf{(a-b)^3=a^3-3ab(a-b)-b^3}$

Note: You may also get the same result on replacing b by –b in

$(a+b)^3=a^3+3ab(a+b)+b^3$

(8) $(a+b)(a^2-ab+b^2)=a(a^2-ab+b^2)+b(a^2-ab+b^2)$ [Distributive law]

$$= a^3-a^2b+ab^2+a^2b-ab^2+b^3$$

$$= a^3+b^3$$

Thus, $\mathbf{(a+b)(a^2-ab+b^2)=a^3+b^3}$

(9) $(a-b)(a^2+ab+b^2)=a(a^2+ab+b^2)-b(a^2+ab+b^2)$ [Distributive law]

$$= a^3+a^2b+ab^2-a^2b-ab^2-b^3$$

$$= a^3-b^3$$

Thus, $\mathbf{(a-b)(a^2+ab+b^2)=a^3-b^3}$

Let us, now, consider some examples based on the above mentioned special products:

Example 4.3: Find each of the following products:

(i) $(7x + 9y)^3$ (ii) $(px - yz)^3$ (iii) $(x - 4y^2)^3$

(iv) $(2a^2 + 3b^2)^3$ (v) $\left(\frac{2}{3}a - \frac{5}{3}b\right)^3$ (vi) $\left(1 + \frac{4}{3}c\right)^3$

Solution: (i) $(7x + 9y)^3 = (7x)^3 + 3(7x)(9y)(7x + 9y) + (9y)^3$

$= 343\,x^3 + 189\,xy\,(7x + 9y) + 729y^3$

$= 343x^3 + 1323x^2y + 1701xy^2 + 729y^3$

(ii) $(px - yz)^3 = (px)^3 - 3(px)(yz)(px - yz) - (yz)^3$

$= p^3x^3 - 3pxyz\,(px - yz) - y^3z^3$

$= p^3x^3 - 3p^2x^2yz + 3pxy^2z^2 - y^3z^3$

(iii) $(x - 4y^2)^3 = x^3 - 3x\,(4y^2)\,(x - 4y^2) - (4y^2)^3$

$= x^3 - 12xy^2\,(x - 4y^2) - 64y^6$

$= x^3 - 12x^2y^2 + 48xy^4 - 64y^6$

(iv) $(2a^2 + 3b^2)^3 = (2a^2)^3 + 3(2a^2)(3b^2)\,(2a^2 + 3b^2) + (3b^2)^3$

$= 8a^6 + 18a^2b^2\,(2a^2 + 3b^2) + 27b^6$

$= 8a^6 + 36a^4b^2 + 54a^2b^4 + 27b^6$

(v) $$\left(\frac{2}{3}a - \frac{5}{3}b\right)^3 = \left(\frac{2}{3}a\right)^3 - 3\left(\frac{2}{3}a\right)\left(\frac{5}{3}b\right)\left(\frac{2}{3}a - \frac{5}{3}b\right) - \left(\frac{5}{3}b\right)^3$$

$$= \frac{8}{27}a^3 - \frac{10}{3}ab\left(\frac{2}{3}a - \frac{5}{3}b\right) - \frac{125}{27}b^3$$

$$= \frac{8}{27}a^3 - \frac{20}{9}a^2b + \frac{50}{9}ab^2 - \frac{125}{27}b^3$$

(vi) $$\left(1 + \frac{4}{3}c\right)^3 = (1)^3 + 3(1)\left(\frac{4}{3}c\right)\left(1 + \frac{4}{3}c\right) + \left(\frac{4}{3}c\right)^3$$

$$= 1 + 4c\left(1 + \frac{4}{3}c\right) + \frac{64}{27}c^3$$

$$= 1 + 4c + \frac{16}{3}c^2 + \frac{64}{27}c^3$$

Example 4.4: Using special products, find the cube of each of the following:

(i) 19 (ii) 101 (iii) 54 (iv) 47

Solution: (i) $19^3 = (20 - 1)^3$

$= 20^3 - 3 \times 20 \times 1\ (20 - 1) - 1^3$

$= 8000 - 60\ (20 - 1) - 1$

$= 8000 - 1200 + 60 - 1$

$= 6859$

(ii) $101^3 = (100 + 1)^3$

$= 100^3 + 3 \times 100 \times 1\ (100 + 1) + 1^3$

$= 1000000 + 300 \times 100 + 300 + 1$

$= 1030301$

(iii) $54^3 = (50 + 4)^3$

$= 50^3 + 3 \times 50 \times 4\ (50 + 4) + 4^3$

$= 125000 + 600\ (50 + 4) + 64$

$= 125000 + 30000 + 2400 + 64$

$= 157464$

(iv) $47^3 = (50 - 3)^3$

$= 50^3 - 3 \times 50 \times 3\ (50 - 3) - 3^3$

$= 125000 - 450\ (50 - 3) - 27$

$= 125000 - 22500 + 1350 - 27$

$= 103823$

Example 4.5: Without actual multiplication, find each of the following products:

(i) $(2a + 3b)\ (4a^2 - 6ab + 9b^2)$

(ii) $(3a - 2b)\ (9a^2 + 6ab + 4b^2)$

Solution: (i) $(2a + 3b)\ (4a^2 - 6ab + 9b^2) = (2a + 3b)\ [(2a)^2 - (2a)\ (3b) + (3b)^2]$

$= (2a)^3 + (3b)^3$

$= 8a^3 + 27b^3$

(ii) $(3a - 2b)\ (9a^2 + 6ab + 4b^2) = (3a - 2b)\ [(3a)^2 + (3a)\ (2b) + (2b)^2]$

$= (3a)^3 - (2b)^3$

$= 27a^3 - 8b^3$

Example 4.6: Simplify:

(i) $(3x - 2y)^3 + 3\,(3x - 2y)^2\,(3x + 2y) + 3(3x - 2y)\,(3x + 2y)^2 + (3x + 2y)^3$

(ii) $(2a - b)^3 + 3\,(2a - b)\,(2b - a)\,(a + b) + (2b - a)^3$

Solution: (i) Put $3x - 2y = a$ and $3x + 2y = b$

The given expression becomes

$a^3 + 3a^2b + 3ab^2 + b^3$

$= (a + b)^3$

$= (3x - 2y + 3x + 2y)^3$

$= (6x)^3$

$= 216x^3$

(ii) Put $2a - b = x$ and $2b - a = y$ so that $a + b = x + y$

The given expression becomes

$x^3 + 3xy\,(x + y) + y^3$

$= (x + y)^3$

$= (a + b)^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

Example 4.7: Simplify:

(i) $\dfrac{857 \times 857 \times 857 - 537 \times 537 \times 537}{857 \times 857 + 857 \times 537 + 537 \times 537}$

(ii) $\dfrac{674 \times 674 \times 674 + 326 \times 326 \times 326}{674 \times 674 - 674 \times 326 + 326 \times 326}$

Solution: The given expression can be written as

$$\frac{857^3 - 537^3}{857^2 + 857 \times 537 + 537^2}$$

Let $857 = a$ and $537 = b$, then the expression becomes

$$\frac{a^3 - b^3}{a^2 + ab + b^2} = \frac{(a - b)(a^2 + ab + b^2)}{a^2 + ab + b^2} = a - b$$

$= 857 - 537$

$= 320$

(ii) The given expression can be written as

$$\frac{674^3 + 326^3}{674^2 - 674 \times 326 + 326^2}$$

$$= \frac{(674 + 326)(674^2 - 674 \times 326 + 326^2)}{674^2 - 674 \times 326 + 326^2}$$

$= 674 + 326$

$= 1000$

CHECK YOUR PROGRESS 4.2

1. Write the expansion of each of the following:

   (i) $(3x + 4y)^3$ (ii) $(p - qr)^3$ (iii) $\left(a + \frac{b}{3}\right)^3$

   (iv) $\left(\frac{a}{3} - b\right)^3$ (v) $\left(\frac{1}{2}a^2 + \frac{2}{3}b^2\right)^3$ (vi) $\left(\frac{1}{3}a^2x^3 - 2b^3y^2\right)^3$

2. Using special products, find the cube of each of the following:

   (i) 8 (ii) 12 (iii) 18 (iv) 23

   (v) 53 (vi) 48 (vii) 71 (viii) 69

   (ix) 97 (x) 99

3. Without actual multiplication, find each of the following products:

   (i) $(2x + y)(4x^2 - 2xy + y^2)$ (ii) $(x - 2)(x^2 + 2x + 4)$

   (iii) $(1 + x)((1 - x + x^2)$ (iv) $(2y - 3z^2)(4y^2 + 6yz^2 + 9z^4)$

   (v) $(4x + 3y)(16x^2 - 12xy + 9y^2)$ (vi) $\left(3x - \frac{1}{7}y\right)\left(9x^2 + \frac{3}{7}xy + \frac{1}{49}y^2\right)$

4. Find the value of:

   (i) $a^3 + 8b^3$ if $a + 2b = 10$ and $ab = 15$

   [Hint: $(a + 2b)^3 = a^3 + 8b^3 + 6ab(a + 2b) \Rightarrow a^3 + 8b^3 = (a + 2b)^3 - 6ab(a + 2b)$]

   (ii) $x^3 - y^3$ when $x - y = 5$ and $xy = 66$

5. Find the value of $64x^3 - 125z^3$ if

   (i) $4x - 5z = 16$ and $xz = 12$

   (ii) $4x - 5z = \frac{3}{5}$ and $xz = 6$

6. Simplify:

   (i) $(2x + 5)^3 - (2x - 5)^3$

   (ii) $(7x + 5y)^3 - (7x - 5y)^3 - 30y\,(7x + 5y)\,(7x - 5y)^3$

   [Hint put $7x + 5y = a$ and $7x - 5y = b$ so that $a - b = 10y$]

   (iii) $(3x + 2y)\,(9x^2 - 6xy + 4y^2) - (2x + 3y)\,(4x^2 - 6xy + 9y^2)$

   (iv) $(2x - 5)\,(4x^2 + 10x + 25) - (5x + 1)\,(25x^2 - 5x + 1)$

7. Simplify:

   (i) $\frac{875 \times 875 \times 875 + 125 \times 125 \times 125}{875 \times 875 - 875 \times 125 + 125 \times 125}$

   (ii) $\frac{678 \times 678 \times 678 - 234 \times 234 \times 234}{678 \times 678 + 678 \times 234 + 234 \times 234}$

4.3 FACTORIZATION OF POLYNOMIALS

Recall that from 3 × 4 = 12, we say that 3 and 4 are factors of the product 12. Similarly, in algebra, since $(x + y)\,(x - y) = x^2 - y^2$, we say that $(x + y)$ and $(x - y)$ are factors of the product $(x^2 - y^2)$.

Factorization of a polynomial is a process of writing the polynomial as a product of two (or more) polynomials. Each polynomial in the product is called a factor of the given polynomial.

In factorization, we shall restirct ourselves, unless otherwise stated, to finding factors of the polynomials over integers, i.e. polynomials with integral coefficients. In such cases, it is required that the factors, too, be polynomials over integers. Polynomials of the type $2x^2 - y^2$ will not be considered as being factorable into $\left(\sqrt{2}x + y\right)\left(\sqrt{2}x - y\right)$ because these factors are not polynomials over integers.

A polynomial will be said to be completely factored if none of its factors can be further expressed as a product of two polynomials of lower degree and if the integer coefficients have no common factor other than 1 or –1. Thus, complete factorization of $(x^2 - 4x)$ is $x(x–4)$. On the other hand the factorization $(4x^2 - 1)\,(4x^2 + 1)$ of $(16x^4 - 1)$ is not complete since the factor $(4x^2 - 1)$ can be further factorised as $(2x - 1)\,(2x + 1)$. Thus, complete factorization of $(16x^4 - 1)$ is $(2x - 1)\,(2x + 1)\,(4x^2 + 1)$.

In factorization, we shall be making full use of special products learnt earlier in this lesson. Now, in factorization of polynomials we take various cases separately through examples.

(1) Factorization by Distributive Property

Example 4.8: Factorise:

(i) $10a - 25$ (ii) $x^2y^3 + x^3y^2$

(iii) $5ab\,(ax^2 + y^2) - 6mn(ax^2 + y^2)$ (iv) $a(b - c)^2 + b(b - c)$

Solution: (i) $10a - 25 = 5 \times 2a - 5 \times 5$

$= 5\,(2a - 5)$ [Since 5 is common to the two terms]

Thus, 5 and $2a - 5$ are factors of $10a - 25$

(ii) In $x^2y^3 + x^3y^2$, note that x^2y^2 is common (with greatest degree) in both the terms.

$$\therefore\ x^2y^3 + x^3y^2 = x^2y^2 \times y + x^2y^2 \times x$$

$$= x^2y^2\,(y + x)$$

Therefore, x, x^2, y, y^2, xy, x^2y, xy^2, x^2y^2 and $y + x$ are factors of $x^2y^3 + x^3y^2$

(iii) Note that $ax^2 + y^2$ is common in both the terms

$$\therefore\ 5ab\,(ax^2 + y^2) - 6mn(ax^2 + y^2) = (ax^2 + y^2)\,(5ab - 6mn)$$

(iv) $a(b - c)^2 + b(b - c) = (b - c) \times [a(b - c)] + (b - c) \times b$

$= (b - c) \times [a(b - c) + b]$

$= (b - c) \times [ab - ac + b]$

(2) Factorization Involving the Difference of Two Squares

You know that $(x + y)\,(x - y) = x^2 - y^2$. Therefore $x + y$ and $x - y$ are factors of $x^2 - y^2$.

Example 4.9: Factorise:

(i) $9x^2 - 16y^2$ (ii) $x^4 - 81y^4$

(iii) $a^4 - (2b - 3c)^2$ (iv) $x^2 - y^2 + 6y - 9$

Solution: (i) $9x^2 - 16y^2 = (3x)^2 - (4y)^2$ which is a difference of two squares.

$= (3x + 4y)\,(3x - 4y)$

(ii) $x^4 - 81y^4 = (x^2)^2 - (9y^2)$

$= (x^2 + 9y^2)\,(x^2 - 9y^2)$

Note that $x^2 - 9y^2 = (x)^2 - (3y)^2$ is again a difference of the two squares.

$x^4 - 81y^4 = (x^2 + 9y^2)\,[(x)^2 - (3y)^2]$

$= (x^2 + 9y^2)\,(x + 3y)\,(x - 3y)$

(iii) $a^4 - (2b - 3c)^2 = (a^2)^2 - (2b - 3c)^2$

$= [a^2 + (2b - 3c)]\,[a^2 - (2b - 3c)]$

$= (a^2 + 2b - 3c)\,(a^2 - 2b + 3c)$

(iv) $x^2 - y^2 + 6y - 9 \quad = x^2 - (y^2 - 6y + 9)$ [Note this step)

$= (x)^2 - [(y)^2 - 2 \times y \times 3 + (3)^2]$

$= (x)^2 - (y - 3)^2$

$= [x + (y - 3)]\,[x - (y - 3)]$

$= (x + y - 3)\,(x - y + 3)$

(3) Factorization of a Perfect Square Trinomial

Example 4.10 : Factorise

(i) $9x^2 + 24xy + 16y^2$ (ii) $x^6 - 8x^3 + 16$

Solution: (i) $9x^2 + 24xy + 16y^2 \quad = (3x)^2 + 2\,(3x)\,(4y) + (4y)^2$

$= (3x + 4y)^2$

$= (3x + 4y)\,(3x + 4y)$

Thus, the two factors of the given polynomial are identical, each being $(3x + 4y)$.

(ii) $x^6 - 8x^3 + 16 \quad = (x^3)^2 - 2(x^3)\,(4) + (4)^2$

$= (x^3 - 4)^2$

$= (x^3 - 4)\,(x^3 - 4)$

Again, the two factors of the given polynomial are identical, each being $(x^3 - 4)$.

(4) Factorization of a Polynomial Reducible to the Difference of Two Squares

Example 4.11: Factorise

(i) $x^4 + 4y^4$ (ii) $x^4 + x^2 + 1$

Solution: (i) $x^4 + 4y^4 \quad = (x^2)^2 + (2y^2)^2$

$= (x^2)^2 + (2y^2)^2 + 2\,(x^2)\,(2y^2) - 2\,(x^2)\,(2y^2)$

[Adding and subtracting $2\,(x^2)\,(2y^2)$]

$= (x^2 + 2y^2)^2 - (2xy)^2$

$= (x^2 + 2y^2 + 2xy)\,(x^2 + 2y^2 - 2xy)$

(ii) $x^4 + x^2 + 1 \;=\; (x^2)^2 + (1)^2 + 2x^2 - x^2$

[Adding and subtracting x^2]

$= (x^2 + 1)^2 - (x)^2$

$= (x^2 + x + 1)\,(x^2 - x + 1)$

CHECK YOUR PROGRESS 4.3

Factorise:

1. $10\,xy - 15xz$
2. $abc^2 - ab^2c$
3. $6p^2 - 15pq + 27\,p$
4. $a^2\,(b - c) + b\,(c - b)$
5. $2a(4x - y)^3 - b\,(4x - y)^2$
6. $x(x + y)^3 - 3xy\,(x + y)$
7. $100 - 25p^2$
8. $1 - 256y^8$
9. $(2x + 1)^2 - 9x^2$
10. $(a^2 + bc)^2 - a^2\,(b + c)^2$
11. $25x^2 - 10x + 1 - 36y^2$
12. $49x^2 - 1 - 14xy + y^2$
13. $m^2 + 14m + 49$
14. $4x^2 - 4x + 1$
15. $36a^2 + 25 + 60a$
16. $x^6 - 8x^3 + 16$
17. $a^8 - 47a^4 + 1$
18. $4a^4 + 81b^4$
19. $x^4 + 4$
20. $9a^4 - a^2 + 16$
21. Find the value of n if

    (i) $6n = 23 \times 23 - 17 \times 17$ (ii) $536 \times 536 - 36 \times 36 = 5n$

(5) Factorization of Perfect Cube Polynomials

Example 4.12: Factorise:

(i) $x^3 + 6x^2y + 12xy^2 + 8y^3$ (ii) $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$

Solution: (i) $x^3 + 6x^2y + 12xy^2 + 8y^3$

$= (x)^3 + 3x^2\,(2y) + 3x\,(2y)^2 + (2y)^3$

$= (x + 2y)^3$

Thus, the three factors of the given polynomial are identical, each being x + 2y.

(ii) Given polynomial is equal to

$(x^2)^3 - 3x^2y^2\,(x^2 - y^2) - (y^2)^3$

$= (x^2 - y^2)^3$

$= [(x + y)\,(x - y)]^3$ [Since $x^2 - y^2 = (x + y)\,(x - y)$]

$= (x + y)^3\,(x - y)^3$

(6) Factorization of Polynomials Involving Sum or Difference of Two Cubes

In special products you have learnt that

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3$$

and $$(x - y)(x^2 + xy + y^2) = x^3 - y^3$$

Therefore, the factors of $x^3 + y^3$ are $x + y$ and $x^2 - xy + y^2$ and

those of $x^3 - y^3$ are $x - y$ and $x^2 + xy + y^2$

Now, consider the following example:

Example 4.13: Factorise

(i) $64a^3 + 27b^3$ (ii) $8x^3 - 125y^3$

(iii) $8(x + 2y)^3 - 343$ (iv) $a^4 - a^{13}$

Solution: (i) $64a^3 + 27b^3 = (4a)^3 + (3b)^3$

$$= (4a + 3b)[(4a)^2 - (4a)(3b) + (3b)^2]$$

$$= (4a + 3b)(16a^2 - 12ab + 9b^2)$$

(ii) $8x^3 - 125y^3 = (2x)^3 - (5y)^3$

$$= (2x - 5y)[(2x)^2 + (2x)(5y) + (5y)^2]$$

$$= (2x - 5y)(4x^2 + 10xy + 25y^2)$$

(iii) $8(x + 2y)^3 - 343 = [2(x + 2y)]^3 - (7)^3$

$$= [2(x + 2y) - 7][2^2(x + 2y)^2 + 2(x + 2y)(7) + 7^2]$$

$$= (2x + 4y - 7)(4x^2 + 16xy + 16y^2 + 14x + 28y + 49)$$

(iv) $a^4 - a^{13} = a^4(1 - a^9)$ [Since a^4 is common to the two terms]

$$= a^4[(1)^3 - (a^3)^3]$$

$$= a^4(1 - a^3)(1 + a^3 + a^6)$$

$$= a^4(1 - a)(1 + a + a^2)(1 + a^3 + a^6)$$

[Since $1 - a^3 = (1 - a)(1 + a + a^2)$]

CHECK YOUR PROGRESS 4.4

Factorise:

1. $a^3 + 216b^3$
2. $a^3 - 343$
3. $x^3 + 12x^2y + 48xy^2 + 64y^3$
4. $8x^3 - 36x^2y + 54xy^2 - 27y^3$

5. $8x^3 - 125y^3 - 60x^2y + 150xy^2$	6. $64k^3 - 144k^2 + 108k - 27$
7. $729\ x^6 - 8$	8. $x^2 + x^2\ y^6$
9. $16a^7 - 54ab^6$	10. $27b^3 - a^3 - 3a^2 - 3a - 1$
11. $(2a - 3b)^3 + 64c^3$	12. $64x^3 - (2y - 1)^3$

(7) Factorising Trinomials by Splitting the Middle Term

You have learnt that

$$(x + a)(x + b) = x^2 + (a + b)x + ab = 1.x^2 + (a + b)x + ab$$

and $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$

In general, the expressions given here on the right are of the form $Ax^2 + Bx + C$ which can be factorised by multiplying the coefficient of x^2 in the first term with the last term and finding two such factors of this product that their sum is equal to the coefficient of x in the second (middle) term. In other words, we are to determine two such factors of AC so that their sum is equal to B. The example, given below, will clarify the process further.

Example 4.14: Factorise:

(i) $x^2 + 3x + 2$ (ii) $x^2 - 10xy + 24y^2$

(iii) $5x^2 + 13x - 6$ (iv) $3x^2 - x - 2$

Solution: (i) Here, A = 1, B = 3 and C= 2; so AC = 1 × 2 = 2

Therefore we are to determine two factors of 2 whose sum is 3

Obviously, $1 + 2 = 3$

(i.e. two factors of AC i.e. 2 are 1 and 2)

∴ We write the polynomial as

$$x^2 + (1 + 2)\,x + 2$$
$$= x^2 + x + 2x + 2$$
$$= x(x + 1) + 2(x + 1)$$
$$= (x + 1)(x + 2)$$

(ii) Here, $AC = 24y^2$ and $B = -10y$

Two factors of $24y^2$ whose sum is $-10y$ are $-4y$ and $-6y$

∴ We write the given polynomial as

$$x^2 - 4xy - 6xy + 24\,y^2$$
$$= x(x - 4y) - 6y(x - 4y)$$
$$= (x - 4y)(x - 6y)$$

(iii) Here, $AC = 5 \times (-6) = -30$ and $B = 13$

Two factors of -30 whose sum is 13 are 15 and -2

∴ We write the given polynomial as

$5x^2 + 15x - 2x - 6$

$= 5x(x + 3) - 2(x + 3)$

$= (x + 3)(5x - 2)$

(iv) Here, $AC = 3 \times (-2) = -6$ and $B = -1$

Two factors of -6 whose sum is (-1) are (-3) and 2.

∴ We write the given polynomial as

$3x^2 - 3x + 2x - 2$

$= 3x(x - 1) + 2(x - 1)$

$= (x - 1)(3x + 2)$

CHECK YOUR PROGRESS 4.5

Factorise:

1. $x^2 + 11x + 24$
2. $x^2 - 15xy + 54y^2$
3. $2x^2 + 5x - 3$
4. $6x^2 - 10xy - 4y^2$
5. $2x^4 - x^2 - 1$
6. $x^2 + 13xy - 30y^2$
7. $2x^2 + 11x + 14$
8. $10y^2 + 11y - 6$
9. $2x^2 - x - 1$
10. $(m - 1)(1 - m) + m + 109$
11. $(2a - b)^2 - (2a - b) - 30$

Hint put $2a - b = x$

12. $(2x + 3y)^2 - 2(2x + 3y)(3x - 2y) - 3(3x - 2y)^2$

Hint: Put $2x + 3y = a$ and $3x - 2y = b$

4.4 HCF AND LCM OF POLYNOMIALS

(1) HCF of Polynomials

You are already familiar with the term HCF (Highest Common Factor) of natural numbers in arithmetic. It is the largest number which is a factor of each of the given numbers. For instance, the HCF of 8 and 12 is 4 since the common factors of 8 and 12 are 1, 2 and 4 and 4 is the largest i.e. highest among them.

On similar lines in algebra, *the Highest Common Factor (HCF) of two or more given* *polynomials is the product of the polynomial(s) of highest degree and greatest numerical coefficient each of which is a factor of each of the given polynomials.*

For example, the HCF of $4(x + 1)^2$ and $6(x + 1)^3$ is $2(x + 1)^2$.

The HCF of monomials is found by multiplying the HCF of numerical coefficients of each of the monomials and the variable(s) with highest power(s) common to all the monomials. For example, the HCF of monomials $12x^2y^3$, $18xy^4$ and $24x^3y^5$ is $6xy^3$ since HCF of 12, 18 and 24 is 6; and the highest powers of variable factors common to the polynomials are x and y^3.

Let us now consider some examples.

Example 4.15: Find the HCF of

(i) $4x^2y$ and x^3y^2 (ii) $(x - 2)^3 (2x - 3)$ and $(x - 2)^2 (2x - 3)^3$

Solution: (i) HCF of numerical coefficients 4 and 1 is 1.

Since x occurs as a factor at least twice and y at least once in the given polynomials, therefore, their HCF is

$1 \times x^2 \times y$ i.e. x^2y

(ii) HCF of numerical coefficients 1 and 1 is 1.

In the given polynomials, $(x - 2)$ occurs as a factor at least twice and $(2x - 3)$ at least once. So the HCF of the given polynomials is

$1 \times (x - 2)^2 \times (2x - 3)$ i.e. $(x - 2)^2 (2x - 3)$

In view of Example 4.15 (ii), we can say that to determine the HCF of polynomials, which can be easily factorised, we express each of the polynomials as the product of the factors. Then the HCF of the given polynomials is the product of the HCF of numerical coefficients of each of the polynomials and factor (s) with highest power(s) common to all the polynomials. For further clarification, concentrate on the Example 4.16 given below.

Example 4.16: Find the HCF of

(i) $x^2 - 4$ and $x^2 + 4x + 4$

(ii) $4x^4 - 16x^3 + 12x^2$ and $6x^3 + 6x^2 - 72x$

Solution: (i) $x^2 - 4 = (x + 2)(x - 2)$

$x^2 + 4x + 4 = (x + 2)^2$

HCF of numerical coefficients = 1

HCF of other factors = $(x + 2)^1 = x + 2$

Hence, the required HCF = $x + 2$

(ii) $4x^4 - 16x^3 + 12x^2 = 4x^2 (x^2 - 4x + 3)$

$= 4x^2 (x - 1)(x - 3)$

$$6x^3 + 6x^2 - 72x = 6x(x^2 + x - 12)$$

$$= 6x(x + 4)(x - 3)$$

Required HCF $= 2x(x - 3)$ [Since HCF of numerical coefficient is 2)

$$= 2x^2 - 6x$$

(2) LCM of Polynomials

Like HCF, you are also familiar with the LCM (Lowest Common Multiple or Least Common Multiple) of natural numbers in arithmetic. It is the smallest number which is a multiple of each of the given numbers. For instance, the LCM of 8 and 12 is 24 since 24 is the smallest among common multiples of 8 and 12 as given below:

Multiples of 8:	8, 16, $\underline{24}$, 32, 40, $\underline{48}$, 56, 64, $\underline{72}$, 80, ...
Multiples of 12:	12, $\underline{24}$, 36, $\underline{48}$, 60, $\underline{72}$, 84, 96, ...
Common multiple of 8 and 12:	24, 48, 72, ...

On similar lines in Algebra, *the Lowest Common Multiple (LCM) of two or more polynomials is the product of the polynomial(s) of the lowest degree and the smallest numerical coefficient which are multiples of the corresponding elements of each of the given polynomials.*

For example, the LCM of $4(x + 1)^2$ and $6(x + 1)^3$ is $12(x + 1)^3$.

The LCM of monomials is found by multiplying the LCM of numerical coefficients of each of the monomials and all variable factors with highest powers. For example, the LCM of $12x^2y^2z$ and $18x^2yz$ is $36x^2y^2z$ since the LCM of 12 and 18 is 36 and highest powers variable factors x, y and z are x^2, y^2 and z respectively.

Let us, now, consider some examples to illustrate.

Example 4.17: Find the LCM of

(i) $4x^2y$ and x^3y^2 (ii) $(x - 2)^3 (2x - 3)$ and $(x - 2)^2 (2x - 3)^3$

Solution: (i) LCM of numerical coefficient 4 and 1 is 4.

Since highest power of x is x^3 and that of y is y^2,

the required LCM is $4x^3y^2$

(ii) Obviously LCM of numerical coefficients 1 and 1 is 1.

In the given polynomials, highest power of the factor $(x - 2)$ is $(x - 2)^3$

and that of $(2x - 3)$ is $(2x - 3)^3$.

LCM of the given polynomials $= 1 \times (x - 2)^3 \times (2x - 3)^3$

$$= (x - 2)^3 (2x - 3)^3$$

In view of Example 4.17 (ii), we can say that to determine the LCM of polynomials, which can be easily factorised, we express each of the polynomials as the product of factors. Then, the LCM of the given polynomials is the product of the LCM of the numerical coefficients and all other factors with their highest powers which occur in factorization of any of the polynoials. For further clarification, we take Example 4.18 given below.

Example 4.18: Find the LCM of

(i) $(x – 2)(x^2 – 3x + 2)$ and $x^2 – 5x + 6$

(ii) $8(x^3 – 27)$ and $12(x^5 + 27x^2)$

Solution: (i) $(x – 2)(x^2 – 3x + 2) = (x – 2)(x – 2)(x – 1)$

$= (x – 2)^2 (x – 1)$

Also $x^2 – 5x + 6 = (x – 2)(x – 3)$

LCM of numerical coefficients = 1

LCM of other factors = $(x – 2)^2 (x – 1)(x – 3)$

Hence, the LCM of given polynomials = $(x – 1)(x – 2)^2 (x – 3)$

(ii) $8(x^3 – 27) = 8(x – 3)(x^2 + 3x + 9)$

$12(x^5 + 27x^2) = 12x^2 (x^3 + 27)$

$= 12x^2 (x + 3)(x^2 – 3x + 9)$

LCM of numerical coefficient 8 and 12 = 24

LCM of other factors = $x^2 (x – 3)(x + 3)(x^2 + 3x + 9)(x^2 – 3x + 9)$

Hence, required LCM = $24x^2 (x – 3)(x + 3)(x^2 + 3x + 9)(x^2 – 3x + 9)$

CHECK YOUR PROGRESS 4.6

1. Find the HCF of the following polynomials:

(i) $27x^4y^2$ and $3xy^3$

(ii) $48y^7x^9$ and $12y^3x^5$

(iii) $(x + 1)^3$ and $(x + 1)^2 (x – 1)$

(iv) $x^2 + 4x + 4$ and $x + 2$

(v) $18(x + 2)^3$ and $24(x^3 + 8)$

(vi) $(x + 1)^2 (x + 5)^3$ and $x^2 + 10x + 25$

(vii) $(2x – 5)^2 (x + 4)^3$ and $(2x – 5)^3 (x – 4)$

(viii) $x^2 – 1$ and $x^4 – 1$

(ix) $x^3 – y^3$ and $x^2 – y^2$

(x) $6(x^2 – 3x + 2)$ and $18(x^2 – 4x + 3)$

2. Find the LCM of the following polynomials:

(i) $25x^3y^2$ and $15xy$

(ii) $30xy^2$ and $48x^3y^4$

(iii) $(x + 1)^3$ and $(x + 1)^2 (x – 1)$

(iv) $x^2 + 4x + 4$ and $x + 2$

(v) $18(x + 2)^3$ and $24(x^3 + 8)$

(vi) $(x + 1)^2 (x + 5)^3$ and $x^2 + 10x + 25$

(vii) $(2x – 5)^2 (x + 4)^2$ and $(2x – 5)^3 (x – 4)$

(viii) $x^2 – 1$ and $x^4 – 1$

(ix) $x^3 – y^3$ and $x^2 – y^2$

(x) $6(x^2 – 3x + 2)$ and $18(x^2 – 4x + 3)$

4.5 RATIONAL EXPRESSIONS

You are already familiar with integers and rational numbers. Just as a number, which can be expressed in the form $\frac{p}{q}$ where p and q (≠0) are integers, is called a rational number, an algebraic expression, which can be expressed in the from $\frac{P}{Q}$, where P and Q (non-zero polynomials) are polynomials, is called a **rational expression.** Thus, each of the expressions

$$\frac{x+1}{x-1}, \frac{x^2-3x+5}{x^2-5}, \frac{\frac{1}{2}a^2+b^2-\frac{5}{6}}{a+b}, \frac{x^2+\sqrt{2}y^2}{\sqrt{3}x-y}$$

is a rational expression in one or two variables.

Notes:

(1) The polynomial '$x^2 + 1$' is a rational expresion since it can be written as $\frac{x^2+1}{1}$ and you have learnt that the constant 1 in the denominator is a polynomial of degree zero.

(2) The polynomial 7 is a rational expresion since it can be written as $\frac{7}{1}$ where both 7 and 1 are polynomials of degree zero.

(3) Obvioulsy a rational expression need not be a polynomial. For example rational expression $\frac{1}{x}\left(=x^{-1}\right)$ is not a polynomial. On the contrary every polynomial is also a rational expression.

None of the expressions $\frac{\sqrt{x}+2}{1-x}$, $x^2+2\sqrt{x}+3$, $\frac{a^{\frac{2}{3}}-\frac{1}{b}}{a^2+ab+b^2}$ is a rational expression.

CHECK YOUR PROGRESS 4.7

1. Which of the following algebraic expressions are rational expressions?

(i) $\frac{2x-3}{4x-1}$ (ii) $\frac{8}{x^2+y^2}$

(iii) $\frac{2\sqrt{3}x^2+\sqrt{5}}{\sqrt{7}}$ (iv) $\frac{2x^2-\sqrt{x}+3}{6x}$

(v) $200+\sqrt{11}$ (vi) $\left(a+\frac{1}{b}\right)\div b^{\frac{1}{3}}$

(vii) $y^3+3yz(y+z)+z^3$ (vii) $5\div(a+3b)$

2. For each of the following, cite two examples:
   - (i) A rational expression is one variable
   - (ii) A rational expression is two variables
   - (iii) A rational expression whose numerator is a binomial and whose denominator is trinomial
   - (iv) A rational expression whose numerator is a constant and whose denominator is a quadratic polynomial
   - (v) A rational expression in two variables whose numerator is a polynomial of degree 3 and whose denominator is a polynomial of degree 5
   - (vi) An algebraic expression which is not a rational expression

4.6 OPERATIONS ON RATIONAL EXPRESSIONS

Four fundamental operations on rational expressions are performed in exactly the same way as in case of rational numbers.

(1) Addition and Subtraction of Rational Expressions

For observing the analogy between addition of rational numbers and that of rational expressions, we take the following example. Note that the analogy will be true for subtraction, multiplication and division of rational expressions also.

Example 4.19: Find the sum:

(i) $\frac{5}{6}+\frac{3}{8}$ (ii) $\frac{2x+1}{x-1}+\frac{x+2}{x+1}$

Solution: (i) $\frac{5}{6}+\frac{3}{8} = \frac{5\times4+3\times3}{24}$ ←LCM of 6 and 8.

$$= \frac{20+9}{24}$$

$$= \frac{29}{24}$$

(ii) $\frac{2x+1}{x-1}+\frac{x+2}{x+1}$ $= \frac{(2x+1)(x+1)+(x+2)(x-1)}{(x-1)(x+1) \leftarrow \text{LCM of } (x-1) \text{ and } (x+1)}$

$$= \frac{2x^2+3x+1+x^2+x-2}{x^2-1}$$

$$= \frac{3x^2+4x-1}{x^2-1}$$

Example 4.20: Subtract $\frac{x-1}{x+1}$ from $\frac{3x-2}{3x+1}$

Solution: $\frac{3x-2}{3x+1}-\frac{x-1}{x+1}$ $= \frac{(x+1)(3x-2)-(x-1)(3x+1)}{(3x+1)(x+1)}$

$$= \frac{3x^2+x-2-(3x^2-2x-1)}{3x^2+4x+1}$$

$$= \frac{3x-1}{3x^2+4x+1}$$

Note: Observe that the sum and difference of two rational expressions are also rational expressions.

Since the sum and difference of two rational expressions are rational expressions, $x+\frac{1}{x}(x \neq 0)$ and $x-\frac{1}{x}(x \neq 0)$ are both rational expressions as x and $\frac{1}{x}$ are both rational expressions. Similarly, each of $x^2+\frac{1}{x^2}, x^3+\frac{1}{x^3}, x^2-\frac{1}{x^2}, x^3-\frac{1}{x^3}$, etc. is a rational expression. These expresions create interest as for given value of $x+\frac{1}{x}$ or $x-\frac{1}{x}$, we can determine values of $x^2+\frac{1}{x^2}, x^2-\frac{1}{x^2}, x^3+\frac{1}{x^3}, x^3-\frac{1}{x^3}$ etc. and in some case vice versa also. Let us concentrate on the following example.

Example 4.21: Find the value of

(i) $x^2+\frac{1}{x^2}$ if $x-\frac{1}{x}=1$ (ii) $x^4+\frac{1}{x^4}$ if $x+\frac{1}{x}=4$

(iii) $x-\frac{1}{x}$ if $x^4+\frac{1}{x^4}=119$ (iv) $x^3+\frac{1}{x^3}$ if $x+\frac{1}{x}=3$

(v) $x^3 - \frac{1}{x^3}$ if $x - \frac{1}{x} = 5$

Solution: (i) We have $x - \frac{1}{x} = 1$

$$\therefore \left(x - \frac{1}{x}\right)^2 = (1)^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} - 2 \times x \times \frac{1}{x} = 1$$

$$\Rightarrow x^2 + \frac{1}{x^2} - 2 = 1$$

Hence, $x^2 + \frac{1}{x^2} = 3$

(ii) $x + \frac{1}{x} = 4$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = (4)^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} + 2 = 16$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 14$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = (14)^2$$

$$\Rightarrow x^4 + \frac{1}{x^4} + 2 = 196$$

So, $x^4 + \frac{1}{x^4} = 194$

(iii) We have $x^4 + \frac{1}{x^4} = 119$

$$\Rightarrow (x^2)^2 + \left(\frac{1}{x^2}\right)^2 + 2 = 119 + 2 = 121$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^2=(11)^2$$

$$\Rightarrow x^2+\frac{1}{x^2}=11 \qquad \text{[since both } x^2 \text{ and } \frac{1}{x^2} \text{ are positive]}$$

$$\Rightarrow x^2+\frac{1}{x^2}-2=9$$

$$\Rightarrow \left(x-\frac{1}{x}\right)^2=(3)^2$$

$$\therefore x-\frac{1}{x}=\pm 3$$

(iv) We have $x+\frac{1}{x}=3$

$$\therefore \left(x+\frac{1}{x}\right)^3=(3)^3$$

$$\Rightarrow x^3+\frac{1}{x^3}+3\times x\times\frac{1}{x}\left(x+\frac{1}{x}\right)=27$$

$$\Rightarrow x^3+\frac{1}{x^3}+3(3)=27$$

$$\therefore x^3+\frac{1}{x^3}=18$$

(v) We have $x-\frac{1}{x}=5$

$$\therefore \left(x-\frac{1}{x}\right)^3=(5)^3$$

$$\Rightarrow x^3-\frac{1}{x^3}-3\times x\times\frac{1}{x}\left(x-\frac{1}{x}\right)=125$$

$$\Rightarrow x^3-\frac{1}{x^3}-3(5)=125$$

$$\therefore x^3-\frac{1}{x^3}=140$$

CHECK YOUR PROGRESS 4.8

1. Find the sum of rational expressions:

(i) $\frac{x^2+1}{x-2}$ and $\frac{x^2-1}{x-2}$

(ii) $\frac{x+2}{x+3}$ and $\frac{x-1}{x-2}$

(iii) $\frac{x+1}{(x-1)^2}$ and $\frac{1}{x+1}$

(iv) $\frac{3x+2}{x^2-16}$ and $\frac{x-5}{(x+4)^2}$

(v) $\frac{x-2}{x+3}$ and $\frac{x+2}{x+3}$

(vi) $\frac{x+2}{x-2}$ and $\frac{x-2}{x+2}$

(vii) $\frac{x+1}{x+2}$ and $\frac{x^2-1}{x^2+1}$

(vii) $\frac{3\sqrt{2}x+1}{3x^2}$ and $\frac{-2\sqrt{2}x+1}{2x^2}$

2. Subtract

(i) $\frac{x-1}{x-2}$ from $\frac{x+4}{x+2}$

(ii) $\frac{2x-1}{2x+1}$ from $\frac{2x+1}{2x-1}$

(iii) $\frac{1}{x}$ from x

(iv) $\frac{2}{x}$ from $\frac{x+1}{x^2-1}$

(v) $\frac{x^2+1}{x-4}$ from $\frac{2x^2+3}{x-4}$

(vi) $\frac{1}{x^2+2}$ from $\frac{2x^3+x^2+3}{(x^2+2)^2}$

(vii) $\frac{x+2}{2(x^2-9)}$ from $\frac{x-2}{(x+3)^2}$

(vii) $\frac{x+1}{x-1}$ from $\frac{4x}{x^2-1}$

3. Find the value of

(i) $a^2+\frac{1}{a^2}$ when $a+\frac{1}{a}=2$

(ii) $a^2+\frac{1}{a^2}$ when $a-\frac{1}{a}=2$

(iii) $a^3+\frac{1}{a^3}$ when $a+\frac{1}{a}=2$

(iv) $a^3+\frac{1}{a^3}$ when $a+\frac{1}{a}=5$

(v) $a^3-\frac{1}{a^3}$ when $a-\frac{1}{a}=\sqrt{5}$

(vi) $8a^3+\frac{1}{27a^3}$ when $2a+\frac{1}{3a}=5$

(vii) $a^3+\frac{1}{a^3}$ when $a+\frac{1}{a}=\sqrt{3}$

(viii) $a^3+\frac{1}{a^3}$ when $a^2+\frac{1}{a^2}=7, a>0$

(ix) $a-\frac{1}{a}$ when $a^4+\frac{1}{a^4}=727$ (x) $a^3-\frac{1}{a^3}$ when $a^4+\frac{1}{a^4}=34, a>0$

---

(2) Multiplication and Division of Rational Expressions

You know that the product of two rational numbers, say, $\frac{2}{3}$ and $\frac{5}{7}$ is given as $\frac{2}{3}\times\frac{5}{7}=\frac{2\times5}{3\times7}=\frac{10}{21}$. Similarly, the product of two rational expressions, say, $\frac{P}{Q}$ and $\frac{R}{S}$ where P, Q, R, S $(Q, S \neq 0)$ are polynomials is given by $\frac{P}{Q}\times\frac{R}{S}=\frac{PR}{QS}$. You may observe that the product of two rational expressions is again a rational expression.

Example 4.22: Find the product:

(i) $\frac{5x+3}{5x-1}\times\frac{2x-1}{x+1}$ (ii) $\frac{2x+1}{x-1}\times\frac{x-1}{x+3}$

(iii) $\frac{x^2-7x+10}{(x-4)^2}\times\frac{x^2-7x+12}{x-5}$

Solution: (i) $\frac{5x+3}{5x-1}\times\frac{2x-1}{x+1}=\frac{(5x+3)(2x-1)}{(5x-1)(x+1)}$

$$=\frac{10x^2+x-3}{5x^2+4x-1}$$

(ii) $\frac{2x+1}{x-1}\times\frac{x-1}{x+3}=\frac{(2x+1)(x-1)}{(x-1)(x+3)}$

$=\frac{2x+1}{x+3}$ [Cancelling common factor $(x-1)$ from numerator and denominator]

(iii) $\frac{x^2-7x+10}{(x-4)^2}\times\frac{x^2-7x+12}{x-5}=\frac{(x^2-7x+10)(x^2-7x+12)}{(x-4)^2(x-5)}$

$$=\frac{(x-2)(x-5)(x-3)(x-4)}{(x-4)^2(x-5)}$$

$$= \frac{(x-2)(x-3)}{(x-4)}$$

[Cancelling common factor (x –4) (x – 5) from numerator and denominator]

$$= \frac{x^2-5x+6}{x-4}$$

Note: The result (product) obtained after cancelling the HCF from its numerator and denominator is called the result (product) **in lowest terms** or **in lowest form.**

You are also familiar with the division of a rational number, say, $\frac{2}{3}$ by a rational number, say, $\frac{5}{7}$ is given as $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5}$ where $\frac{7}{5}$ is the reciprocal of $\frac{5}{7}$. Similarly, division of a rational expression $\frac{P}{Q}$ by a non-zero rational expression $\frac{R}{S}$ is given by $\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \times \frac{S}{R}$ where P, Q, R, S are polynomials and $\frac{S}{R}$ is the **reciprocal expression** of $\frac{R}{S}$.

Example 4.23: Find the reciprocal of each of the following rational expressions:

(i) $\frac{x^2+20}{x^3+5x+6}$ (ii) $-\frac{2y}{y^2-5}$ (iii) x^3+8

Solution: (i) Reciprocal of $\frac{x^2+20}{x^3+5x+6}$ is $\frac{x^3+5x+6}{x^2+20}$

(ii) Reciprocal of $-\frac{2y}{y^2-5}$ is $-\frac{y^2-5}{2y} = \frac{5-y^2}{2y}$

(iii) Since $x^3+8 = \frac{x^3+8}{1}$, the reciprocal of x^3+8 is $\frac{1}{x^3+8}$

Example 4.24: Divide:

(i) $\frac{x^2+1}{x-1}$ by $\frac{x-1}{x+2}$

(ii) $\frac{x^2-1}{x^2-25}$ by $\frac{x^2-4x-5}{x^2+4x-5}$ and express the result in lowest form.

Solution: (i) $\dfrac{x^2+1}{x-1} \div \dfrac{x-1}{x+2} = \dfrac{x^2+1}{x-1} \times \dfrac{x+2}{x-1}$

$$= \frac{(x^2+1)(x+2)}{(x-1)^2} = \frac{x^3+2x^2+x+2}{x^2-2x+1}$$

(ii) $\dfrac{x^2-1}{x^2-25} \div \dfrac{x^2-4x-5}{x^2+4x-5} = \dfrac{(x^2-1)(x^2+4x-5)}{(x^2-25)(x^2-4x-5)}$

$$= \frac{(x-1)(x+1)(x+5)(x-1)}{(x-5)(x+5)(x+1)(x-5)}$$

$$= \frac{(x-1)(x-1)}{(x-5)(x-5)}$$

[Cancelling HCF (x+1)(x+5)]

$$= \frac{x^2-2x+1}{x^2-10x+25}$$

The result $\dfrac{x^2-2x+1}{x^2-10x+25}$ is in lowest form.

CHECK YOUR PROGRESS 4.9

1. Find the product and express the result in lowest terms:

(i) $\dfrac{7x+2}{2x^2+3x+1} \times \dfrac{x+1}{7x^2-5x-2}$

(ii) $\dfrac{x^3+1}{x^4+1} \times \dfrac{x^3-1}{x^4-1}$

(iii) $\dfrac{3x^2-15x+18}{2x-4} \times \dfrac{17x+3}{x^2-6x+9}$

(iv) $\dfrac{5x-3}{5x+2} \times \dfrac{x+2}{x+6}$

(v) $\dfrac{x^2+1}{x-1} \times \dfrac{x+1}{x^2-x+1}$

(vi) $\dfrac{x^3+1}{x-1} \times \dfrac{x-1}{2x}$

(vii) $\dfrac{x-3}{x-4} \times \dfrac{x^2-5x+4}{x^2-2x-3}$

(viii) $\dfrac{x^2-7x+12}{x^2-2x-3} \times \dfrac{x^2-2x-24}{x^2-16}$

2. Find the reciprocal of each of the following rational expressions:

(i) $\dfrac{x^2+2}{x-1}$

(ii) $-\dfrac{3a}{1-a}$

(iii) $-\dfrac{7}{1-2x-x^2}$ (iv) x^4+1

3. Divide and express the result as a rational expression in lowest terms:

(i) $\dfrac{x^2+11x+18}{x^2-4x-117} \div \dfrac{x^2+7x+10}{x^2-12x-13}$ (ii) $\dfrac{6x^2+x-1}{2x^2-7x-15} \div \dfrac{4x^2+4x+1}{4x^2-9}$

(iii) $\dfrac{x^2+x+1}{x^2-9} \div \dfrac{x^3-1}{x^2-4x+3}$ (iv) $\dfrac{x^2+2x-24}{x^2-x-12} \div \dfrac{x^2-x-6}{x^2-9}$

(v) $\dfrac{3x^2+14x-5}{x^2-3x+2} \div \dfrac{3x^2+2x-1}{3x^2-3x-2}$ (vi) $\dfrac{2x^2+x-3}{(x-1)^2} \div \dfrac{2x^2+5x+3}{x^2-1}$

LET US SUM UP

- Special products, given below, occur very frequently in algebra:

  (i) $(x+y)^2 = x^2 + 2xy + y^2$ (ii) $(x-y)^2 = x^2 - 2xy + y^2$

  (iii) $(x+y)(x-y) = x^2 - y^2$ (iv) $(x+a)(x+b) = x^2 + (a+b)x + ab$

  (v) $(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$

  (vi) $(x+y)^3 = x^3 + 3xy(x+y) + y^3$ (vii) $(x-y)^3 = x^3 - 3xy(x-y) - y^3$

  (viii) $(x+y)(x^2-xy+y^2) = x^3 + y^3$ (ix) $(x-y)(x^2+xy+y^2) = x^3 - y^3$

- Factorization of a polynomial is a process of writing the polynomial as a product of two (or more) polynomials. Each polynomial in the product is called a factor of the given polynomial.

- A polynomial is said to be completely factorised if it is expressed as a product of factors, which have no factor other than itself, its negative, 1 or –1.

- Apart from the factorization based on the above mentioned special products, we can factorise a polynomial by taking monomial factor out which is common to some or all of the terms of the polynomial using distributive laws.

- HCF of two or more given polynomials is the product of the polynomial of the highest degree and greatest numerical coefficient each of which is a factor of each of the given polynomials.

- LCM of two or more given polynomials is the product of the polynomial of the lowest degree and the smallest numerical coefficient which are multiples of corresponding elements of each of the given polynomials.

- An algebraic expression, which can be expressed in the form $\frac{P}{Q}$ where P and Q are polynomials, Q being a non-zero polynomial, is called a rational expression.
- Operations on rational expressions are performed in the way, they are performed in case of rational numbers. Sum, Difference, Product and Quotient of two rational expressions are also rational expressions.
- Expressing a rational expression into lowest terms means cancellation of common factor, if any, from the numerator and denominator of the rational exprssion.

TERMINAL EXERCISE

1. Mark a tick ☑ against the correct alternative:

(i) If $120^2 - 20^2 = 25p$, then p is equal to

(A) 16 (B) 140 (C) 560 (D) 14000

(ii) $(2a^2 + 3)^2 - (2a^2 - 3)^2$ is equal to

(A) $24a^2$ (B) $24a^4$ (C) $72a^2$ (D) $72a^4$

(iii) $(a^2 + b^2)^2 + (a^2 - b^2)^2$ is equal to

(A) $2(a^2 + b^2)$ (B) $4(a^2 + b^2)$

(C) $4(a^4 + b^4)$ (D) $2(a^4 + b^4)$

(iv) If $m - \frac{1}{m} = -\sqrt{3}$, then $m^3 - \frac{1}{m^3}$ is equal to

(A) 0 (B) $6\sqrt{3}$ (C) $-6\sqrt{3}$ (D) $-3\sqrt{3}$

(v) $\frac{327 \times 327 - 323 \times 323}{327 + 323}$ is equal to

(A) 650 (B) 327 (C) 323 (D) 4

(vi) $8m^3 - n^3$ is equal to:

(A) $(2m - n)(4m^2 - 2mn + n^2)$ (B) $(2m - n)(4m^2 + 2mn + n^2)$

(C) $(2m - n)(4m^2 - 4mn + n^2)$ (D) $(2m - n)(4m^2 + 4mn + n^2)$

(vii) $\frac{467 \times 467 \times 467 + 533 \times 533 \times 533}{467 \times 467 - 467 \times 533 + 533 \times 533}$ is equal to

(A) 66 (B) 198 (C) 1000 (D) 3000

(viii) The HCF of $36a^5b^2$ and $90a^3b^4$ is

(A) $36a^3b^2$ (B) $18a^3b^2$

(C) $90a^3b^4$ (D) $180a^5b^4$

(ix) The LCM of $x^2 - 1$ and $x^2 - x - 2$ is

(A) $(x^2 - 1)(x - 2)$ (B) $(x^2 - 1)(x + 2)$

(C) $(x - 1)^2 (x + 2)$ (D) $(x + 1)^2 (x - 2)$

(x) Which of the following is not a rational expression?

(A) $\sqrt{33}$ (B) $x + \frac{1}{\sqrt{5x}}$

(C) $8\sqrt{x} + 6\sqrt{y}$ (D) $\frac{x - \sqrt{3}}{x + \sqrt{3}}$

2. Find each of the following products:

(i) $(a^m + a^n)(a^m - a^n)$ (ii) $(x + y + 2)(x - y + 2)$

(iii) $(2x + 3y)(2x + 3y)$ (iv) $(3a - 5b)(3a - 5b)$

(v) $(5x + 2y)(25x^2 - 10xy + 4y^2)$ (vi) $(2x - 5y)(4x^2 + 10xy + 25y^2)$

(vii) $\left(a + \frac{5}{4}\right)\left(a + \frac{4}{5}\right)$ (viii) $(2z^2 + 3)(2z^2 - 5)$

(ix) $99 \times 99 \times 99$ (x) $103 \times 103 \times 103$

(xi) $(a + b - 5)(a + b - 6)$ (xii) $(2x + 7z)(2x + 5z)$

3. If $x = a - b$ and $y = b - c$, show that

$(a - c)(a + c - 2b) = x^2 - y^2$

4. Find the value of $64x^3 - 125z^3$ if $4x - 5z = 16$ and $xz = 12$.

5. Factorise:

(i) $x^7 y^6 + x^{22}y^{20}$ (ii) $3a^5b - 243ab^5$

(iii) $3a^6 + 12a^4b^2 + 12a^2b^4$ (iv) $a^4 - 8a^2b^3 + 16b^6$

(v) $3x^4 + 12y^4$ (vi) $x^8 + 14x^4 + 81$

(vii) $x^2 + 16x + 63$ (viii) $x^2 - 12x + 27$

(ix) $7x^2 + xy - 6y^2$ (x) $5x^2 - 8x - 4$

(xi) $x^6 - 729y^6$ (xii) $125a^6 + 64b^6$

6. Find the HCF of

(i) $x^3 - x^5$ and $x^4 - x^7$

(ii) $30(x^2 - 3x + 2)$ and $50(x^2 - 2x + 1)$

7. Find the LCM of

(i) $x^3 + y^3$ and $x^2 - y^2$

(ii) $x^4 + x^2y^2 + y^4$ and $x^2 + xy + y^2$

8. Perform the indicated operation:

(i) $\dfrac{x+1}{(x-1)^2} + \dfrac{1}{x+1}$

(ii) $\dfrac{2x^2+2x-7}{x^2+x-6} - \dfrac{x-1}{x-2}$

(iii) $\dfrac{x-1}{x-2} \times \dfrac{3x+1}{x^2-4}$

(iv) $\dfrac{x^2-1}{x^2-25} \div \dfrac{x^2-4x-5}{x^2+4x-5}$

9. Simplify: $\dfrac{2}{a-1} - \dfrac{2}{a+1} - \dfrac{4}{a^2+1} - \dfrac{8}{a^4+1}$

[Hint : $\dfrac{2}{a-1} - \dfrac{2}{a+1} = \dfrac{4}{a^2-1}$; now combine next term and so on]

10. If $m = \dfrac{x+1}{x-1}$ and $n = \dfrac{x-1}{x+1}$, find $m^2 + n^2 - mn$.

ANSWERS TO CHECK YOUR PROGRESS

4.1

1. (i) $25x^2 + 20xy + y^2$ (ii) $x^2 - 6x + 9$ (iii) $a^2b^2 + 2abcd + c^2d^2$

(iv) $4x^2 - 20xy + 5y^2$ (v) $\dfrac{x^2}{9} + \dfrac{2}{3}x + 1$ (vi) $\dfrac{z^2}{4} - \dfrac{1}{3}z + \dfrac{1}{9}$

(vii) $a^4 - 25$ (viii) $x^2y^2 - 1$ (ix) $x^2 + \dfrac{25}{12}x + 1$

(x) $\frac{4}{9}x^4 - \frac{25}{9}x^2 - 1$ (xi) $6x^2 + 13xy + 6y^2$ (xii) $21x^2 + 8xy - 5y^2$

2. (i) $40x^2$ (ii) $2a^6 + 18$ (iii) $2(a^2x^2 + b^2y^2)$ (iv) $32p^2q^2$

3. (i) 10404 (ii) 11664 (iii) 4761 (iv) 996004
(v) 6384 (vi) 22451 (vii) 89964 (viii) 249936
(ix) 11445 (x) 5621 (xi) 8930 (xii) 989028

4.2

1. (i) $27x^3 + 36x^2y + 36xy^2 + 64y^3$ (ii) $p^3 - 3p^2qr + 3pq^2r^2 - q^3r^3$
(iii) $a^3 + a^2b + \frac{ab^2}{3} + \frac{b^3}{27}$ (iv) $\frac{a^3}{27} - \frac{a^2b}{3} + ab^2 - b^3$
(v) $\frac{a^6}{8} + \frac{1}{2}a^4b^2 + \frac{2}{3}a^2b^4 + \frac{8}{27}b^6$ (vi) $\frac{a^6x^9}{27} - \frac{2}{3}a^4b^3x^6y^2 + 4a^2b^6x^3y^4 - 8b^9y^6$

2. (i) 512 (ii) 1728 (iii) 5832 (iv) 12167 (v) 148877
(vi) 110592 (vii) 357911 (viii) 328509 (ix) 912663 (x) 970299

3. (i) $8x^3 + y^3$ (ii) $x^3 - 8$ (iii) $x^3 + 1$
(iv) $8y^3 - 27z^6$ (v) $64x^3 + 27y^3$ (vi) $27x^3 - \frac{1}{343}y^3$

4. (i) 100 (ii) 1115

5. (i) 15616 (ii) $\frac{27027}{125}$

6. (i) $120x^2 + 250$ (ii) $1000y^3$ (iii) $19x^3 - 19y^3$ (iv) $-117x^3 - 126$

7. (i) 1000 (ii) 444

4.3

1. $5x(2y - 3z)$ 2. $abc(c - b)$
3. $3p(2p - 5q + 9)$ 4. $(b - c)(a^2 - b)$
5. $(4x - y)^2(8ax - 2ay - b)$ 6. $x(x + y)(x^2 - xy + y^2)$
7. $25(2 + 5p)(2 - 5p)$ 8. $(1 + 16y^4)(1 + 4y^2)(1 + 2y)(1 - 2y)$
9. $(5x + 1)(1 - x)$ 10. $(a^2 + bc + ab + ac)(a^2 + bc - ab - ac)$

11. $(5x + 6y - 1)(5x - 6y - 1)$ 12. $(7x - y + 1)(7x - y - 1)$

13. $(m + 7)^2$ 14. $(2x - 1)^2$

15. $(6a + 5)^2$ 16. $(x^3 - 4)^2$

17. $(a^4 + 7a^2 + 1)(a^2 + 3a + 1)(a^2 - 3a + 1)$

18. $(2a^2 + 6ab + 9b^2)(2a^2 - 6ab + 9b^2)$

19. $(x^2 + 2x + 2)(x^2 - 2x + 2)$

20. $(3a^2 + 5a + 4)(3a^2 - 5a + 4)$ 21. (i) 40 (ii) 57200

4.4

1. $(a + 6b)(a^2 - 6ab + 36b^2)$ 2. $(a - 7)(a^2 + 7a + 49)$

3. $(x + 4y)^3$ 4. $(2x - 3y)^3$

5. $(2x - 5y)^3$ 6. $(4k - 3)^3$

7. $(9x^2 - 2)(81x^4 + 18x^2 + 4)$ 8. $x^2(1 + y^2)(1 - y^2 + y^4)$

9. $2a(2a^2 - 3b^2)(4a^2 + 6a^2b^2 + 9b^4)$ 10. $(3b - a - 1)(9b^2 + 3ab + 3b + a^2 + a + 1)$

11. $(2a - 3b + 4c)(4a^2 + 9b^2 - 6ab - 8ac + 12bc + 16c^2$

12. $(4x - 2y + 1)(16x^2 + 8xy - 4x + 4y^2 - 4y + 1)$

4.5

1. $(x + 3)(x + 8)$ 2. $(x - 6y)(x - 9y)$ 3. $(x + 3)(2x - 1)$

4. $2(x - 2y)(3x + y)$ 5. $(2x^2 + 1)(x + 1)(x - 1)$ 6. $(x + 15y)(x - 2y)$

7. $(x + 2)(2x + 7)$ 8. $(2y - 3)(5y - 2)$ 9. $(x - 1)(2x + 1)$

10. $(12 - m)(m + 9)$ 11. $(2a - b - 6)(2a - b + 5)$ 12. $(9y - 7)(5x + y)$

4.6

1. (i) $3xy^2$ (ii) $12y^3x^5$ (iii) $(x + 1)^2$ (iv) $x + 2$ (v) $6(x + 2)$

(vi) $(x + 5)^2$ (vii) $(2x - 5)^2$ (viii) $x^2 - 1$ (ix) $x - y$ (x) $6(x - 1)$

2. (i) $75x^3y^2$ (ii) $240x^3y^4$ (iii) $(x - 1)(x + 1)^3$

(iv) $x^2 + 4x + 4$ (v) $72(x + 2)^3(x^2 - 2x + 4)$ (vi) $(x + 1)^2(x + 5)^3$

(vii) $(x - 4)(x + 4)^2(2x - 5)^3$ (viii) $x^4 - 1$ (ix) $(x - 1)(x + 1)(x^2 + x + 1)$

(x) $18(x - 1)(x - 2)(x - 3)$

4.7

1. (i), (ii), (iii), (v), (vii) and (viii)

4.8

1. (i) $\frac{2x^2}{x-2}$ (ii) $\frac{2x^2+2x-7}{x^2+x-6}$ (iii) $\frac{2x^2+2}{x^3-x^2-x+1}$

(iv) $\frac{4x^2+5x+28}{x^3+4x^2-16x+64}$ (v) $\frac{2x}{x+3}$ (vi) $\frac{2x^2+8}{x^2-4}$

(vii) $\frac{2x^3+3x^2-1}{x^3+2x^2+x+2}$ (viii) $\frac{5}{6x^2}$

2. (i) $\frac{x-6}{x^2-4}$ (ii) $\frac{8x}{4x^2-1}$ (iii) $\frac{x^2-1}{x}$

(iv) $\frac{2-x}{x^2-x}$ (v) $\frac{x^2+2}{x-4}$ (vi) $\frac{2x^3+1}{(x^2+2)^2}$

(vii) $\frac{x^2-15x+16}{2(x^3+3x^2-9x-27)}$ (viii) $\frac{1-x}{1+x}$

3. (i) 2 (ii) 6 (iii) 2 (iv) 110 (v) $8\sqrt{15}$

(vi) 115 (vii) 0 (viii) 18 (ix) ±5 (x) 14

4.9

1. (i) $\frac{1}{2x^2-x-1}$ (ii) $\frac{x^4+x^2+1}{x^6+x^4+x^2+1}$ (iii) $\frac{51x+9}{2x-6}$

(iv) $\frac{5x^2+7x-6}{5x^2+32x+12}$ (v) $\frac{x^3+x^2+x+1}{x^3-2x^2+2x-1}$ (vi) $\frac{x^3+1}{2x}$

(vii) $\frac{x-1}{x+1}$ (viii) $\frac{x-6}{x+1}$

2. (i) $\frac{x-1}{x^2+2}$ (ii) $\frac{a-1}{3a}$ (iii) $\frac{x^2+2x-1}{7}$ (iv) $\frac{1}{x^4+1}$

3. (i) $\frac{x+1}{x+5}$ (ii) $\frac{6x^2-11x+3}{2x^2-9x-5}$ (iii) $\frac{1}{x+3}$

(iv) $\frac{x+6}{x+2}$ (v) $\frac{2x^2+11x+5}{x^2-1}$ (vi) 1

ANSWERS TO TERMINAL EXERCISE

1. (i) C (ii) A (iii) D (iv) A (v) D (vi) B (vii) C (viii) B (ix) A (x) C

2. (i) $a^{2m} - a^{2n}$ (ii) $x^2 - y^2 + 4x + 4$ (iii) $4x^2 + 12xy + 9y^2$

(iv) $9a^2 - 30ab + 25b^2$ (v) $125x^3 + 8y^3$ (vi) $8x^3 - 125y^3$

(vii) $a^2 + \frac{41}{20}a + 1$ (viii) $4z^4 - 4z^2 - 15$ (ix) 970299

(x) 1092727 (xi) $a^2 + 2ab - 11a + 30$ (xii) $4x^2 + 24xz + 35z^2$

4. 15616

5. (i) $x^7y^6(1 + x^{15}y^{14})$ (ii) $3ab(a - 3b)(a + 3b)(a^2 + 9b^2)$

(iii) $3a^2(a^2 + 2b^2)^2$ (iv) $(a^2 - 4b^3)^2$

(v) $3(x^2 + 2xy + 2y^2)$ (vi) $(x^4 - 2x^2 + 9)(x^4 + 2x^2 + 9)$

(vii) $(x + 9)(x + 7)$ (viii) $(x - 3)(x - 9)$

(ix) $(x + y)(7x - 6y)$ (x) $(x - 2)(5x + 2)$

(xi) $(x - 3y)(x + 3y)(x^2 - 3xy + 9y^2)(x^2 + 3xy + 9y^2)$

(xii) $(5a^2 + 4b^2)(25a^4 - 20a^2b^2 + 16b^4)$

6. (i) $x^3(1 - x)$ (ii) $10(x - 1)$

7. (i) $(x^2 - y^2)(x^2 - xy + y^2$ (ii) $x^4 + x^2y^2 + y^4$

8. (i) $\frac{2x^2+2}{x^3-x^2-x+1}$ (ii) $\frac{x+2}{x+3}$

(iii) $\frac{3x^2-2x-1}{x^3+2x^2-4x-8}$ (iv) $\frac{x^2-2x+1}{x^2-10x+25}$

9. $\frac{16}{a^8-1}$

10. $\frac{x^4+14x^2+1}{x^4-2x^2+1}$

5

LINEAR EQUATIONS

You have learnt about basic concept of a variable and a constant. You have also learnt about algebraic exprssions, polynomials and their zeroes. We come across many situations such as six added to twice a number is 20. To find the number, we have to assume the number as x and formulate a relationship through which we can find the number. We shall see that the formulation of such expression leads to an equation involving variables and constants. In this lesson, you will study about linear equations in one and two variables. You will learn how to formulate linear equations in one variable and solve them algebraically. You will also learn to solve linear equations in two variables using graphical as well as algebraic methods.

OBJECTIVES

After studying this lesson, you will be able to

- *identify linear equations from a given collection of equations;*
- *cite examples of linear equations;*
- *write a linear equation in one variable and also give its solution;*
- *cite examples and write linear equations in two variables;*
- *draw graph of a linear equation in two variables;*
- *find the solution of a linear equation in two variables;*
- *find the solution of a system of two linear equations graphically as well as algebraically;*
- *Translate real life problems in terms of linear equations in one or two variables and then solve the same.*

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a variable and constant

- Algebraic expressions and operations on them
- Concept of a polynomial, zero of a polynomial and operations on polynomials

5.1 LINEAR EQUATIONS

You are already familiar with the algebraic expressions and polynomials. The value of an algebraic expression depends on the values of the variables involved it. You have also learnt about polynomial in one variable and their degrees. A polynomials in one variable whose degree is one is called a **linear polynomial** in one variable. When two expressions are separated by an equality sign, it is called an **equation**. Thus, in an equation, there is always an equality sign. The equality sign shows that the expression to the left of the sign (the left had side or LHS) is equal to the expression to the right of the sign (the right hand side or RHS). For example,

$3x + 2 = 14$ ...(1)

$2y - 3 = 3y + 4$ ...(2)

$z^2 - 3z + 2 = 0$ ...(3)

$3x^2 + 2 = 1$ ...(4)

are all equations as they contain equality sign and also contain variables. In (1), the LHS = $3x + 2$ and RHS = 14 and the variable involved is x. In (2), LHS = $2y - 3$, RHS = $3y + 4$ and both are linear polynomials in one variable. In (3) and (4), LHS is a polynomial of degree two and RHS is a number.

You can also observe that in equation (1), LHS is a polynomial of degree one and RHS is a number. In (2), both LHS and RHS are linear polynomials and in (3) and (4), LHS is a quadratic polynomial. The equations (1) and (2) are linear equations and (3) and (4) are not linear equations.

In short, an equation is a condition on a variable. The condition is that two expressions, i.e., LHS and RHS should be equal. It is to be noted that atleast one of the two expressions must contain the variable.

It should be noted that the equation $3x - 4 = 4x + 6$ is the same as $4x + 6 = 3x - 4$. Thus, an equation remains the same when the expressions on LHS and RHS are interchanged. This property is often use in solving equations.

An equation which contains two variables and the exponents of each variable is one and has no term involving product of variables is called a linear equation in two variables. For example, $2x + 3y = 4$ and $x - 2y + 2 = 3x + y + 6$ are linear equations in two variables. The equation $3x^2 + y = 5$ is not a linear equation in two variables and is of degree 2, as the exponent of the variable x is 2. Also, the equation $xy + x = 5$ is not a linear equation in two variables as it contains the term xy which is the product of two variables x and y.

The general form of a linear equation in one variable is $ax + b = 0$, $a \neq 0$, a and b are constants. The general form of a linear equation in two variables is $ax + by + c = 0$ where a, b and c are real numbers such that at atleast one of a and b is non-zero.

Example 5.1: Which of the following are linear equations in one variable? Also write their LHS and RHS.

(i) $2x + 5 = 8$

(ii) $3y - z = y + 5$

(iii) $x^2 - 2x = x + 3$

(iv) $3x - 7 = 2x + 3$

(v) $2 + 4 = 5 + 1$

Solution:

(i) It is a linear equation in x as the exponent of x is 1. LHS = $2x + 5$ and RHS = 8

(ii) It is not a linear equation in one variable as it contains two variables y and z. Here, LHS = $3y - z$ and RHS = $y + 5$

(iii) It is not a linear equation as highest exponent of x is 2. Here, LHS = $x^2 - 2x$ and RHS = $x + 3$.

(iv) It is a linear equation in x as the exponent of x in both LHS and RHS is one.
LHS = $3x - 7$, RHS = $2x + 3$

(v) It is not a linear equation as it does not contain any variable. Here LHS = $2 + 4$ and RHS = $5 + 1$.

Example 5.2: Which of the following are linear equations in two variables.

(i) $2x + z = 5$

(ii) $3y - 2 = x + 3$

(iii) $3t + 6 = t - 1$

Solution:

(i) It is a linear equation in two variables x and z.

(ii) It is a linear equation in two variables y and x.

(iii) It is not a linear equation in two variables as it contains only one variable t.

CHECK YOUR PROGRESS 5.1

1. Which of the following are linear equations in one variable?

(i) $3x - 6 = 7$

(ii) $2x - 1 = 3z + 2$

(iii) $5 - 4 = 1$

(iv) $y^2 = 2y - 1$

2. Which of the following are linear equations in two variables:

(i) $3y - 5 = x + 2$

(ii) $x^2 + y = 2y - 3$

(iii) $x + 5 = 2x - 3$

5.2 FORMATION OF LINEAR EQUATIONS IN ONE VARIABLE

Consider the following situations:

(i) 4 more than x is 11

(ii) A number y divided by 7 gives 2.

(iii) Reena has some apples with her. She gave 5 apples to her sister. If she is left with 3 apples, how many apples she had.

(iv) The digit at tens place of a two digit number is two times the digit at units place. If digits are reversed, the number becomes 18 less than the original number. What is the original number?

In (i), the equation can be written as $x + 4 = 11$. You can verify that $x = 7$ satisfies the equation. Thus, $x = 7$ is a solution.

In (ii), the equation is $\frac{y}{7} = 2$.

In (iii), You can assume the quantity to be found out as a variable say x, i.e., let Reena has x apples. She gave 5 apples to her sister, hence she is left with $x - 5$ apples. Hence, the required equation can be written as $x - 5 = 3$, or $x = 8$.

In (iv), Let the digit in the unit place be x. Therefore, the digit in the tens place should be 2x. Hence, the number is

$$10\,(2x) + x = 20x + x = 21x$$

When the digit are reversed, the tens place becomes x and unit place becomes 2x. Therefore, the number is $10x + 2x = 12x$. Since original number is 18 more than the new number, the equation becomes

$$21x - 12x = 18$$

or $$9x = 18$$

CHECK YOUR PROGRESS 5.2

Form a linear equation using suitable variables for the following situations:

1. Twice a number subtracted from 15 is 7.
2. A motor boat uses 0.1 litres of fuel for every kilometer. One day, it made a trip of x km. Form an equation in x, if the total consumption of fuel was 10 litres.
3. The length of rectangle is twice its width. The perimeter of rectangle is 96m. [Assume width of rectangle as y m]
4. After 15 years, Salma will be four times as old as she is now. [Assume present age of Salma as t years]

5.3 SOLUTION OF LINEAR EQUATIONS IN ONE VARIABLE

Let us consider the following linear equation in one variable,

$$x - 3 = -2$$

Here LHS = $x - 3$ and RHS = -2

Now, we evaluate RHS and LHS for some values of x

x	LHS	RHS
0	– 3	– 2
1	– 2	– 2
3	0	– 2
4	1	– 2

We observe that LHS and RHS are equal only when x = 1. For all other values of x, LHS ≠ RHS. We say that the value of x equal to 1 **satisfies** the equation or **x = 1 is a solution of the equation**.

A number, which when substituted for the variable in the equation makes LHS equal to RHS, is called its solution. We can find the solution of an equation by trial and error method by taking different values of the variable. However, we shall learn a systematic way to find the solution of a linear equation.

An equation can be compared with a balance for weighing, its sides are two pans and the equality symbol '=' tells us that the two pans are in balance.

We have seen the working of balance, If we put equal (and hence add) or remove equal weights, (and hence subtract) from both pans, the two pans remain in balance. Thus we can translate for an equation in the following way:

1. Add same number to both sides of the equation.
2. Subtract same number from both sides of the equation.
3. Multiply both sides of the equation by the same non-zero number.
4. Divide both sides of the equation by the same non-zero number.

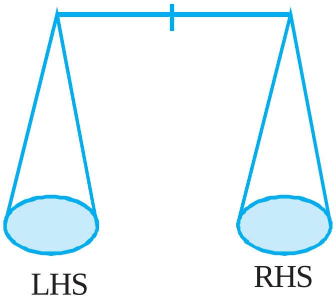


Fig 5.1

We now consider some examples:

Example 5.3: Solve $5 + x = 8$.

Solution: Subtracting 5 from both sides of the equation.

We get $5 + x - 5 = 8 - 5$

or $x + 0 = 3$

or $x = 3$

So, $x = 3$ is the solution of the given equation.

Check: When $x = 3$, LHS $= 5 + x = 5 + 3 = 8$ and R.H.S. $= 8$.
Therefore, LHS = RHS.

Example 5.4: Solve: $y - 2 = 7$.

Solution: Adding 2 to both sides of the equation, we get

$$y - 2 + 2 = 7 + 2$$

or $y = 9$

Hence, $y = 9$ is the solution.

Check: When $y = 9$, LHS $= y - 2 = 9 - 2 = 7$ and RHS $= 7$. Therefore, LHS = RHS.

Example 5.5: Solve: $7x + 2 = 8$.

Solution: Subtracting 2 from both sides of the equation, we get

$$7x + 2 - 2 = 8 - 2$$

or $7x = 6$

or $\frac{7x}{7} = \frac{6}{7}$ (dividing both sides by 7)

or $x = \frac{6}{7}$

Therefore, $x = \frac{6}{7}$ is the solution of the equation.

Example 5.6: Solve: $\frac{3y}{2} - 3 = 9$

Solution: Adding 3 to both sides of the equation, we get

$$\frac{3y}{2} - 3 + 3 = 9 + 3$$

or $\frac{3y}{2} = 12$

or $\frac{3y}{2} \times 2 = 12 \times 2$ (Multiplying both sides by 2)

or $3y = 24$

or $\frac{3y}{3} = \frac{24}{3}$ (Dividing both sides by 3)

or $y = 8$

Hence, $y = 8$ is the solution.

Example 5.7: Solve the equation $2(x + 3) = 3(2x - 7)$

Solution: The equation can be written as

$2x + 6 = 6x - 21$

or $6x - 21 = 2x + 6$ [Interchanging LHS and RHS]

or $6x - 21 + 21 = 2x + 6 + 21$ [Adding 21 on both sides]

or $6x = 2x + 27$

or $6x - 2x = 2x + 27 - 2x$ [Subtracting 2x from both sides]

or $4x = 27$

or $x = \frac{27}{4}$

Thus, $x = \frac{27}{4}$ is the solution of the equation.

Note:

1. It is not necessary to write the details of what we are adding, subtracting, multiplying or dividing each time.
2. The process of taking a term from LHS to RHS or RHS to LHS, is called transposing.
3. When we transpose a term from one side to other side, sign '+' changes to '–', '–' to '+'.

4. A linear equation in one variable can be written as ax + b = 0, where a and b are constants and x is the variable. Its solution is $x = -\frac{b}{a},\ a \neq 0$.

Example 5.8: Solve 3x – 5 = x +3

Solution: We have 3x – 5 = x + 3

or 3x = x + 3 + 5

or 3x – x = 8

or 2x = 8

or x = 4

Therefore, x = 4 is the solution of the given equation.

CHECK YOUR PROGRESS 5.3

Solve the following equations:

1. x – 5 = 8
2. 19 = 7 + y
3. 3z + 4 = 5z + 4
4. $\frac{1}{3}y + 9 = 12$
5. 5(x – 3) = x + 5

5.4 WORD PROBLEMS

You have learnt how to form linear equations in one variable. We will now study some applications of linear equations.

Example 5.9: The present age of Jacob's father is three times that of Jacob. After 5 years, the difference of their ages will be 30 years. Find their present ages.

Solution: Let the present age of Jacob be x years.

Therefore, the present age of his father is 3x years.

After 5 years, the age of Jacob = (x + 5) years.

After 5 years, the age of his father = (3x + 5) years.

The difference of their ages = (3x + 5) – (x + 5) years, which is given to be 30 years, therefore

$3x + 5 - (x + 5) = 30$

or $3x + 5 - x - 5 = 30$

or $3x - x = 30$

or $2x = 30$

or $x = 15$

Therefore, the present age of Jacob is 15 years and the present age of his father = 3x = 3 × 15 = 45 years.

Check: After 5 years, age of Jacob = 15 + 5 = 20 years

After 5 years, age of his father = 45 + 5 =50 years

Difference of their ages = 50 – 20 = 30 years

Example 5.10 : The sum of three consecutive even integers is 36. Find the integers.

Solution: Let the smallest integer be x.

Therefore, other two integers are x + 2 and x + 4.

Since, their sum is 36, we have

$x + (x + 2) + (x + 4) = 36$

or $3x + 6 = 36$

or $3x = 36 - 6 = 30$

or $x = 10$

Therefore, the required integers are 10, 12 and 14.

Example 5.11: The length of a rectangle is 3 cm more than its breadth. If its perimeter is 34 cm find its length and breadth.

Solution: Let the breadth of rectangle be x cm

Therefore, its length = x + 3

Now, since perimeter = 34 cm

We have $2(x + 3 + x) = 34$

or $2x + 6 + 2x = 34$

or $4x = 34 - 6$

or $4x = 28$

or $x = 7$

Therefore, breadth = 7 cm, and length = 7 + 3 = 10 cm.

CHECK YOUR PROGRESS 5.4

1. The sum of two numbers is 85. If one number exceeds the other by 7, find the numbers.
2. The age of father is 20 years more than twice the age of the son. If sum of their ages is 65 years, find the age of the son and the father.
3. The length of a rectangle is twice its breadth. If perimeter of rectangle is 66 cm, find its length and breadth.
4. In a class, the number of boys is $\frac{2}{5}$ of the number of girls. Find the number of girls in the class, if the number of boys is 10.

5.5 LINEAR EQUATIONS IN TWO VARIABLES

Neha went to market to the purchase pencils and pens. The cost of one pencil is Rs 2 and cost of one pen is Rs 4. If she spent Rs 50, how many pencils and pens she purchased?

Since, we want to find the number of pencils and pens, let us assume that she purchased x pencils and y pens. Then,

Cost of x pencils = Rs 2 x

Cost of y pens = Rs 4 y

Since, total cost in Rs 50, we have

$$2x + 4y = 50 \qquad \ldots(1)$$

This is a linear equation in two variables x and y as it is of the form $ax + by + c = 0$

We shall now take different values of x and y to find the solution of the equation (1)

1. If x =1, y = 12, then LHS = $2 \times 1 + 4 \times 12 = 2 + 48 = 50$ and RHS = 50. Therefore, x = 1 and y = 12 is a solution.
2. If x = 3, y = 11, then LHS = $2 \times 3 + 4 \times 11 = 50$ and RHS = 50. Therefore, x = 3, y = 11 is also a solution.
3. If x = 4, y = 10, then LHS = $9 \times 4 + 4 \times 10 = 48$ and RHS = 50. Therefore, x = 4, y = 10 is **not** a solution of the equation.

Thus, a linear equation in two variables has more than one solution.

We have seen that a linear equation in one variable 'x' is of the form $ax + b = 0, a \neq 0$. It has only one solution i.e., $x = -\frac{b}{a}$. However, a linear equation in two variables x and y is of the form

$$ax + by + c = 0 \qquad ...(1)$$

where a, b and c are constants and atleast one of a or b is non-zero. Let $a \neq 0$, then (1) can be written as

$$ax = -by - c$$

or $$x = -\frac{b}{a}y - \frac{c}{a}$$

Now, for each value of y, we get a unique value of x. Thus, a linear equation in two variables will have **infinitely many solutions**.

Note: A linear equation $ax + c = 0$, $a \neq 0$, can be considered as a linear equation in two variables by expressing it as

$$ax + 0y + c = 0$$

i.e., by taking the coefficient of y as zero. It still has many solutions such as

$$x = -\frac{c}{a},\ y = 0;\ \ x = -\frac{c}{a},\ y = 1 \text{ etc.}$$

i.e., for each value of y, the value of x will be equal to $-\frac{c}{a}$.

Example 5.12: The sum of two integers is 15. Form a linear equation in two variables.

Solution: Let the two integers be x and y. Therefore, their sum = x + y. It is given that the sum is 15.

Hence, required equation is $x + y = 15$.

Example 5.13: For the equation $4x - 5y = 2$, verify whether (i) $x = 3$, $y = 2$ and (ii) $x = 4$, $y = 1$ are solutions or not.

Solution: (i) We have $4x - 5y = 2$

When $x = 3$, $y = 2$, $\text{LHS} = 4x - 5y = 4 \times 3 - 5 \times 2$

$$= 12 - 10 = 2$$

$$= \text{RHS}$$

Therefore, $x = 3$, $y = 2$ is a solution of the given equation.

(ii) When $x = 4$, $y = 1$, $\text{LHS} = 4 \times 4 - 5 \times 1 = 16 - 5 = 11$

But RHS = 2. Therefore, $\text{LHS} \neq \text{RHS}$

Hence, $x = 4$, $y = 1$ is not a solution.

CHECK YOUR PROGRESS 5.5

1. Form linear equations in two variables using suitable variables for the unknowns.
   - (i) The perimeter of a rectangle is 98 cm. [Take length as x and breadth as y.]
   - (ii) The age of father is 10 years more than twice the age of son.
   - (iii) A number is 10 more than the other number.
   - (iv) The cost of 2kg apples and 3 kg oranges is Rs. 120. [Take x and y as the cost per kg of apples and oranges respectively.]

Write True or False for the following:

2. $x = 0, y = 3$ is a solution of the equation
   $3x + 2y - 6 = 0$
3. $x = 2, y = 5$ is a solution of the equation
   $5x + 2y = 10$

5.6 GRAPH OF A LINEAR EQUATION IN TWO VARIABLES

You will now learn to draw the graph of a linear equation in two variables. Consider the equation $2x + 3y = 12$. It can be written as

$$2x = 12 - 3y \text{ or } 3y = 12 - 2x$$

$$x = \frac{12-3y}{2} \quad \text{or} \quad y = \frac{12-2x}{3}$$

Now, for each value of y or for each value of x, we get a unique corresponding value of x or y. We make the following table for the values of x and y which satisfy the equation:

$$2x + 3y = 12$$

x	0	6	3	9	– 3
y	4	0	2	– 2	6

Thus, $x = 0, y = 4$; $x = 6, y = 0$; $x = 3, y = 2$; $x = 9, y = -2$; $x = -3, y = 6$ are all solutions of the given equation.

We write these solutions as order pairs (0, 4), (6, 0), (3, 2), (9, – 2) and (– 3, 6).

Here, first entry gives the value of x and the corresponding second entry gives the value of y. We will now learn to draw the graph of this equation by plotting these ordered pairs in a plane and then join them. In the graph of $2x + 3y = 12$, the points representing the solutions will be on a line and a point which is not a solution, will not lie on this line. Each point also called orderd pair, which lies on the line will give a solution and a point which does not lie on the line will not be a solution of the equation.

To draw the graph of a linear equation in two variables, we will first plot these points in a plane. We proceed as follows:

Step 1: We take two perpendicular lines X'OX and YOY' intersecting at O. Mark the real numbers on X'OX and YOY' by considering them as number lines with the point O as the real number 0 as shown in Fig 5.2. These two lines divide the plane into four parts, called first quadrant, second quadrant, third quadrant and fourth quadrant. The number line X'OX is called **x-axis** and the line Y'OY is called **y-axis**. Since, we have taken x-axis and y-axis, perpendicular to each other in a plane, we call the plane as coordinate plane or **cartesian plane** in the honour of French mathematician Descartes who invented this system to plot a point in the plane.

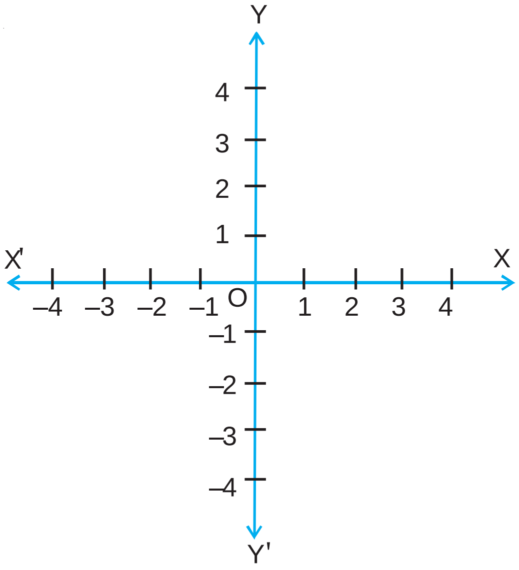


Fig 5.2

Step 2: To plot a point say (3, 2), take the point 3 on x-axis and through this point, draw a line '*l*' perpendicular to x-axis (i.e. parallel to y-axis). Now take the point 2 on y-axis and through 2, draw a line 'm' perpendicular to y-axis (i.e. parallel to x-axis) to meet *l* at P. The point P represents the point (3, 2) on the plane.

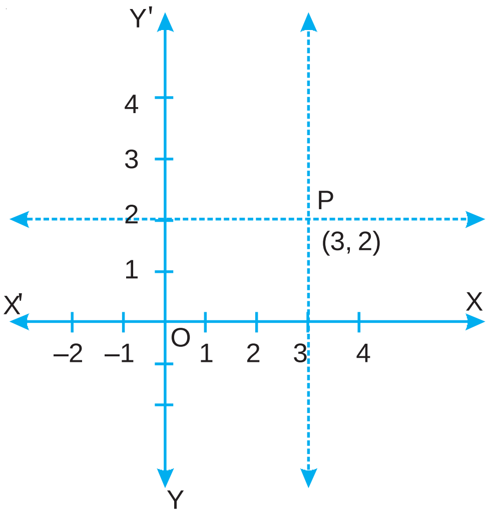


Fig. 5.3

Note 1: It may be noted that, for the ordered pair (a, b), a is called **x-coordinate** and b is called **y-coordinate**.

Note 2: Every point on x-axis can be written as (a, 0) i.e. its y-coordinate is zero and every point on y-axis is of the form (0, b) i.e., its x-coordinate is zero. The coordinates of the point O are (0, 0).

Note 3: In the first quadrant, both x and y coordinates are positive, in the second quadrant, x coordinate is negative and y coordinate is positive, in the third quadrant both x and y coordinates are negative and in the fourth quadrant, x-coordinate is positive and y-coordinate is negative.

Example 5.14: Represent the point (–2, 3) in the coordinate plane.

Solution: Draw x-axis and y-axis on the plane and mark the points on them. Take the point –2 on x-axis and draw the line *l* parallel to y-axis. Now take the point 3 on y-axis and draw the line 'm' parallel to x-axis to meet *l* at P. The point P represent (–2, 3), we say (–2 , 3) are coordinates of the point P.

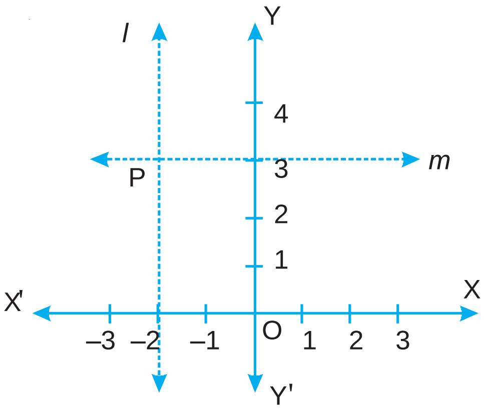


Fig 5.4

You will now learn to draw the graph of a linear equation in two variables. It should be noted that the graph of linear equation in two variables is a line and the coordinates of every point on the line satisfies the equation. If a point does not lie on the graph then its coordinates will not safisfy the equation. You also know that from two given points, one and only one line can be drawn. Therefore, it is sufficient to take any two points, i.e., values of the variables x and y which satisfy the equation. However, it is suggested that you should take three points to avoid any chance of a mistake occurring.

Example 5.15: Draw the graph of the equation 2x – 3y = 6.

Solution: Now choose values of x and y which satisfy the equation 2x – 3y = 6. It will be easy to write the equation by transforming it in any of the following form

$$2x = 3y + 6 \text{ or } 3y = 2x - 6$$

$$\Rightarrow \; x = \frac{3y+6}{2} \quad \text{or} \quad y = \frac{2x-6}{3}$$

Now by taking different values of x or y, you find the corresponding values of y or x. If we take different values of x in $y = \frac{2x-6}{3}$, we get corresponding values of y. If x = 0, we get y = – 2, x = 3 gives y = 0 and x = – 3 gives y = – 4.

You can represent these values in the following tabular form:

x	0	3	–3
y	– 2	0	– 4

The corresponding points in the plane are (0, –2), (3, 0) and (–3, –4). You can now plot these points and join them to get the line which represents the graph of the linear equation as shown here.

Note that all the three points must lie on the line.

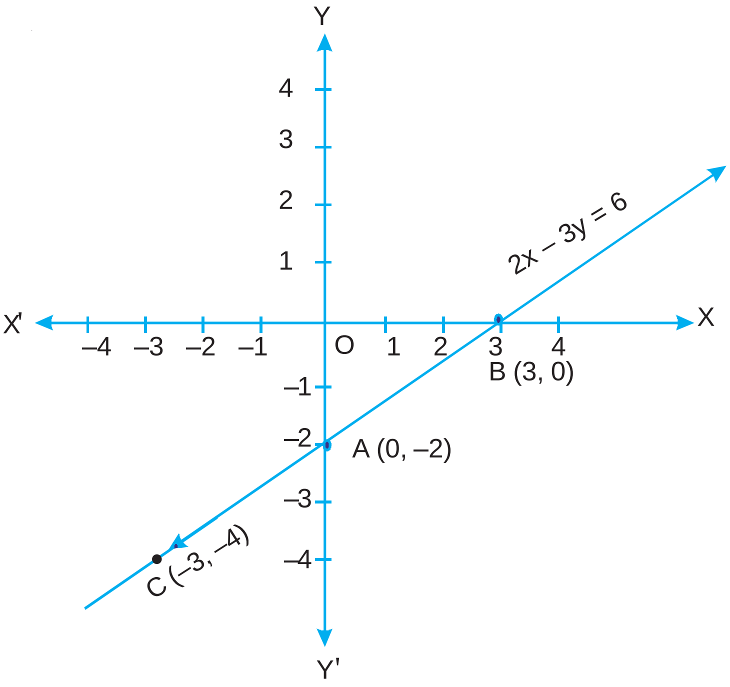


Fig 5.5

Example 5.16: Draw the graph of the equation x = 3.

Solution: It appears that it is a linear equation in one variable x. You can easily convert it into linear equation in two variables by writing it as

$$x + 0y = 3$$

Now you can have the following table for values of x and y.

x	3	3	3
y	3	0	1

Observe that for each value of y, the value of x is always 3. Thus, required points can be taken as (3, 3), (3, 0), (3, 1). The graph is shown in Fig. 5.6.

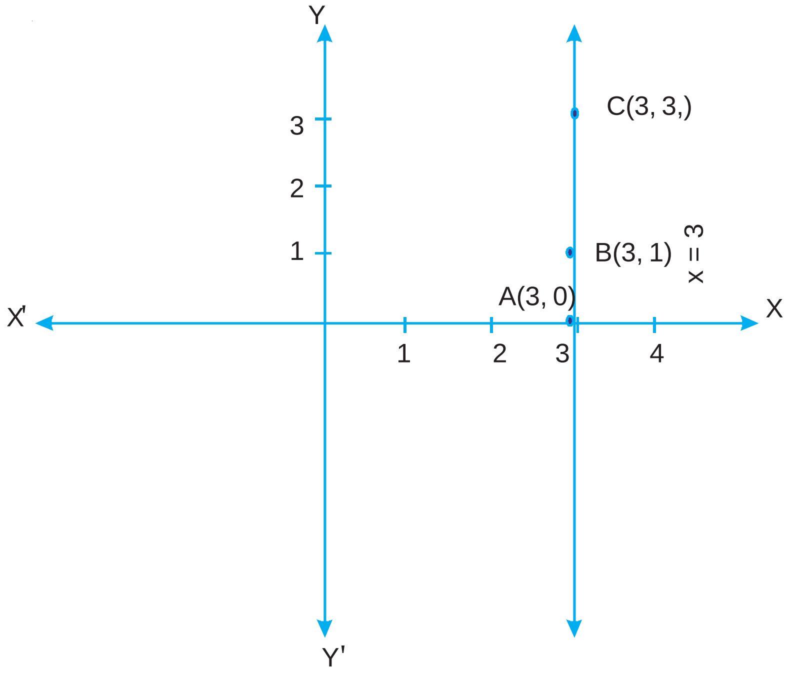


Fig. 5.6

CHECK YOUR PROGRESS 5.6

1. Plot the following points in the cartesian plane:

(i) (3, 4)	(ii) (–3, –2)	(iii) (–2, 1)
(iv) (2, –3)	(v) (4, 0)	(vi) (0, –3)

2. Draw the graph of each of the following linear equations in two variables:

(i) $x + y = 5$	(ii) $3x + 2y = 6$
(iii) $2x + y = 6$	(iv) $5x + 3y = 4$

5.7 SYSTEM OF LINEAR EQUATIONS IN TWO VARIABLES

Neha went to market and purchased 2 pencils and 3 pens for ₹ 19. Mary purchased 3 pencils and 2 pens for ₹ 16. What is the cost of 1 pencil and 1 pen? If the cost of one pencil is ₹ x and cost of one pen is ₹ y, then the linear equation in case of Neha is $2x + 3y = 19$ and for Mary it is $3x + 2y = 16$. To find the cost of 1 pencil and 1 pen, you have to find those values of x and y which satisfy both the equations, i.e.,

$$2x + 3y = 19$$

$$3x + 2y = 16$$

These two equations taken together are called system of linear equations in two variables and the values of x and y which satisfy both equations simultaneously is called the solution.

There are different methods for solving such equation. These are graphical method and algebraic method. You will first learn about graphical method and then algebraic method for solving such equations.

5.7.1 Graphical method

In this method, you have to draw the graphs of both linear equations on the same graph sheet. The graphs of the equations may be

(i) **Intersecting lines:** In this case, the point of intersection will be common solution of both simultaneous equations. The x-coordinate will give the value of x and y-coordinate will given value of y. In this case system will have a unique solution.

(ii) **Concident lines:** In this case each point on the common line will give the solution. Hence, system of equations will have infinitely many solutions.

(iii) **Parallel lines:** In this case, no point will be common to both equations. Hence, system of equations will have no solution.

Example 5.17: Solve the following system of equations:

$$x - 2y = 0 \qquad ...(1)$$

$$3x + 4y = 20 \qquad ...(2)$$

Solution: Let us draw the graphs of these equations. For this, you need atleast two solutions of each equation. We give these values in the following tables.

$x - 2y = 0$

x	0	2	–2
y	0	1	–1

$3x + 4y = 20$

x	0	4	6
y	5	2	1/2

Now plot these points on the same graph sheet as given below:

The two graphs intersect at the point P whose coordinates are (4, 2). Thus x = 4, y = 2 is the solution.

You can verify that x = 4, y = 2 satisfies both the equations.

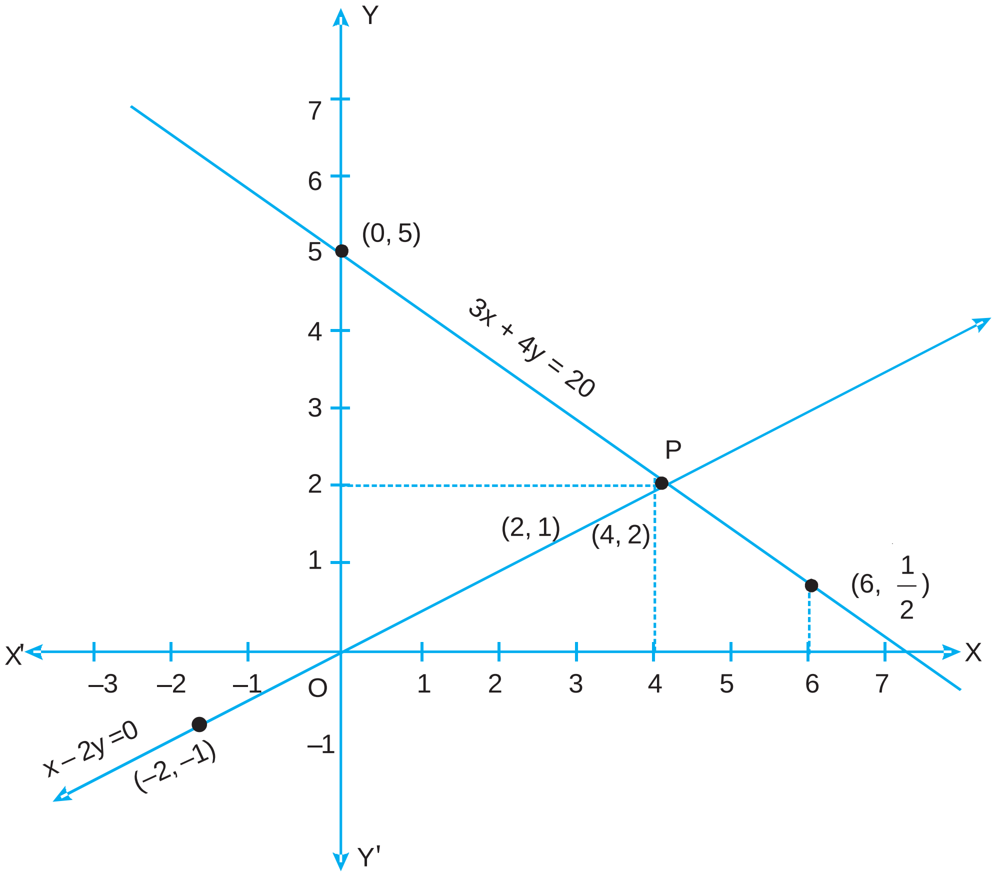


Fig. 5.7

Example 5.18: Solve the following system of equations:

$x + y = 8$ ...(1)

$2x - y = 1$ ...(2)

Solution: To draw the graph of these equation, make the following by selecting some solutions of each of the equation.

$x + y = 8$

x	3	4	5
y	5	4	3

$2x - y = 1$

x	0	1	2
y	–1	1	3

Now, plot the points (3, 5), (4, 4) and (5, 3) to get the graph of $x + y = 8$ and (0, –1), (1, 1) and (2, 3) to get the graph of $2x - y = 1$ on the same graph sheet. The two lines intersect at the point P whose coordinates are (3, 5). Thus $x = 3$, $y = 5$, is the solution of the system of equations. You can verify that $x = 3$, $y = 5$ satisfies both equations simultaneously.

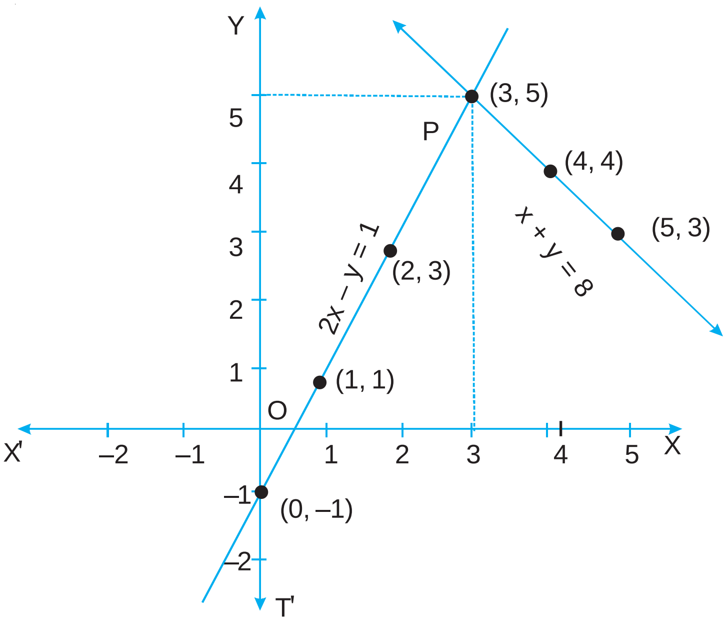


Fig. 5.8

Example 5.19: Solve the following system of equations:

$$x + y = 2 \qquad ...(1)$$

$$2x + 2y = 4 \qquad ...(2)$$

Solution: First make tables for some solutions of each of the equation.

$x + y = 2$

x	0	2	1
y	2	0	1

$2x + 2y = 4$

x	0	2	1
y	2	0	1

Now draw the graph of these equations by plotting the corresponding points.

You can see that graph of both the equations is the same. Hence, system of equations has infinitely many solutions. For example, x = 0, y = 2; x = 1, y = 1; x = 2, y = 0 etc. You can also observe that these two equations are essentially the same equation.

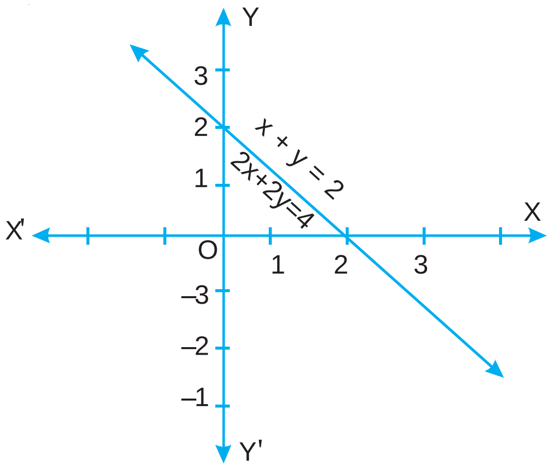


Fig. 5.9

Example 5.20: Solve the following system of equations:

$$2x - y = 4 \qquad ...(1)$$

$$4x - 2y = 6 \qquad ...(2)$$

Solution: Let us draw the graph of both equations by taking some solutions of each of the equation.

$2x - y = 4$

x	0	2	–1
y	–4	0	–6

$4x - 2y = 6$

x	0	1.5	2
y	–3	0	1

You can observe that these graphs are parallel lines. Since, they do not have any common point, the system of equations, therefore, has no solution.

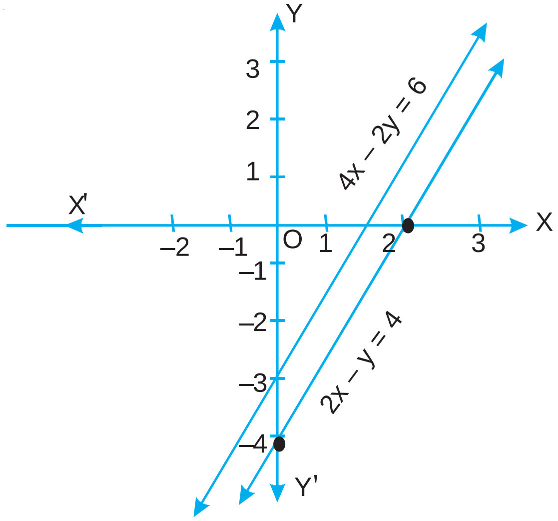


Fig. 5.10

CHECK YOUR PROGRESS 5.7

Solve the following system of equations graphically. Also, tell whether these have unique solution, infinitely many solutions or no solution.

1. $x - y = 3$
   $x + y = 5$
2. $2x + 3y = 1$
   $3x - y = 7$
3. $x + 2y = 6$
   $2x + 4y = 12$

4. $3x + 2y = 6$

   $6x + 4y = 18$

5. $2x + y = 5$

   $3x + 2y = 8$

5.7.2 Algebraic Method

There are several methods of solving system of two linear equations in two variables. You have learnt one method which is known as graphical method. We shall now discuss here two more methods, called algebraic methods. They are

(i) Substitution Method.

(ii) Elimination method.

Note: These methods are useful in case the system of equations has a unique solution.

Substitution Method: In this method, we find the value of one of the variable from one equation and substitute it in the second equation. This way, the second equation will be reduced to linear equaion in one variable which we have already solved. We explain this method through some examples.

Example 5.21: Solve the following system of equations by substitution method.

$$5x + 2y = 8 \qquad ...(1)$$

$$3x - 5y = 11 \qquad ...(2)$$

Solution: From (1), we get

$$2y = 8 - 5x$$

or $$y = \frac{1}{2}(8 - 5x) \qquad ...(3)$$

Substituting the value of y in (2), we get

$$3x - \frac{5}{2}(8 - 5x) = 11$$

or $6x - 5(8 - 5x) = 22$ [multiplying both sides by 2]

or $6x - 40 + 25x = 22$

or $31x = 40 + 22$

or $$x = \frac{62}{31} = 2$$

Substituting the value of x = 2 in (3), we get

$$y = \frac{1}{2}(8 - 5\times 2) = \frac{1}{2}(8 - 10)$$

or $$y = -\frac{2}{2} = -1$$

So, the solution to the system of equations is x = 2, y = – 1.

Example 5.22: Solve the following system of equations by substitution method:

$$2x + 3y = 7 \qquad ...(1)$$

$$3x + y = 14 \qquad ...(2)$$

Solution: From equation (2), we get

$$y = 14 - 3x \qquad ...(3)$$

Substituting the value of y in (1), we get

$$2x + 3\,(14 - 3x) = 7$$

or $$2x + 42 - 9x = 7$$

or $$2x - 9x = 7 - 42$$

or $$-7x = -35$$

Therefore $$x = \frac{-35}{-7} = 5$$

Substituting the value of x in (3), we get

$$y = 14 - 3x = 14 - 3 \times 5$$

or $$y = 14 - 15 = -1$$

Hence, x = 5, y = – 1 is the solution.

Check: You can verify that x = 5, y = – 1 satisfies both the equations.

CHECK YOUR PROGRESS 5.8

Solve the following system of equations by substitution method:

1. $x + y = 14$
   $x - y = 2$
2. $2x + 3y = 11$
   $2x - 4y = -24$

3. $3x + 2y = 11$ 4. $7x - 2y = 1$

$2x + 3y = 4$ $3x + 4y = 15$

Elimination Method: In this method, we eliminate one of the variable by multiplying both equations by suitable non-zero contants to make the coefficients of one of the variable numerically equal. Then we add or subtract one equation to or from the other so that one variable gets eliminated and we get an equation in one variable. We now consider some examples to illustrate this method.

Example 5.23: Solve the following system of equations using elimination method.

$$3x - 5y = 4 \qquad ...(1)$$

$$9x - 2y = 7 \qquad ...(2)$$

Solution: To elminate x, multiply equation (1) by 3 to make coefficient of x equal. You get the equations.

$$9x - 15y = 12 \qquad ...(3)$$

$$9x - 2y = 7 \qquad ...(4)$$

Subtracting (4) from (3), we get

$$9x - 15y - (9x - 2y) = 12 - 7$$

or $9x - 15y - 9x + 2y = 5$

or $-13\,y = 5$

or $y = -\frac{5}{13}$

Substituting $y = -\frac{5}{13}$ in equation (1), we get

$$3x - 5 \times \left(-\frac{5}{13}\right) = 4$$

or $3x + \frac{25}{13} = 4$

or $3x = 4 - \frac{25}{13} = \frac{27}{13}$

or $x = \frac{9}{13}$

Therefore, $x = \frac{9}{13}$ and $y = -\frac{5}{13}$ is the required solution of the given system of equations.

Example 5.24: Solve the following system of equations using elimination method.

$$2x + 3y = 13 \qquad ...(1)$$

$$5x - 7y = -11 \qquad ...(2)$$

Solution: To eliminate y, multiply equation (1) by 7 and equation (2) by 3, we get

$$14x + 21y = 91 \qquad ...(3)$$

$$15x - 21y = -33 \qquad ...(4)$$

Adding (3) and (4), we get

$$29\,x = 58$$

or $\quad x = \frac{58}{29} = 2$

Substituting x = 2 in (1), we get

$$2 \times 2 + 3y = 13$$

or $\quad 3y = 13 - 4 = 9$

or $\quad y = \frac{9}{3} = 3$

Therefore, x = 2 and y = 3 is the solution of the given system of equations.

CHECK YOUR PROGRESS 5.9

Solve the following systems of equations by elimination method:

1. $3x + 4y = -6$
   $3x - y = 9$
2. $x + 2y = 5$
   $2x + 3y = 8$
3. $x - 2y = 7$
   $3x + y = 35$
4. $3x + 4y = 15$
   $7x - 2y = 1$
5. $2x + 3y = 4$
   $3x + 2y = 11$
6. $3x - 5y = 23$
   $2x - 4y = 16$

5.8 WORD PROBLEMS

Example 5.25: The perimeter of a rectangular garden is 20 m. If the length is 4 m more than the breadth, find the length and breadth of the garden.

Solution: Let the length of the garden be x m. Therefore, breadth of garden = $(x - 4)$ m. Since, perimeter is 20 m, so

$2[x + (x - 4)] = 20$

or $2(2x - 4) = 20$

or $2x - 4 = 10$

or $2x = 10 + 4 = 14$

or $x = 7$

Hence, length = 7 m and breadth = $7 - 4 = 3$ m.

Alternatively, you can solve the problem using two variables. Proceed as follows:

Let the length of garden = x m

and width of garden = y m

Therefore $x = y + 4$ ...(1)

Also, perimeter is 20 m, therefore

$2(x + y) = 20$

or $x + y = 10$ ...(2)

Solving (1) and (2), we get $x = 7, y = 3$

Hence, length = 7 m and breadth = 3m

Example 5.26: Asha is five years older than Robert. Five years ago, Asha was twice as old as Robert was then. Find their present ages.

Solution: Let present age of Asha be x years

and present age of Robert be y years

Therefore, $x = y + 5$

or $x - y = 5$ ...(1)

5 years ago, Asha was $x - 5$ years and Robert was $(y - 5)$ years old.

Therefore, $x - 5 = 2(y - 5)$

or $x - 2y = -5$ ...(2)

Solving (1) and (2), we get $y = 10$ and $x = 15$

Hence, present age of Asha = 15 years and present age of Robert = 10 years.

Example 5.27: Two places A and B are 100 km apart. One car starts from A and another from B at the same time. If they travel in the same direction, they meet after 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of the two cars. Assume that the speed of car at A is more than the speed of car at B.

Solution: Let speed of the car starting from A be x km/h

and speed of the car starting from B be y km/h

Therefore, the distance travelled by car at A in 5 hours = $5x$ km

and the distance travelled by car at B in 5 hours = $5y$ km

Since they meet after 5 hours when they travel in the same direction, the car at A has travelled 100 km more than the car at B. Therefore,

$$5x - 5y = 100$$

or $$x - y = 20 \qquad ...(1)$$

When they travel towards each other, they meet after 1 hour. It means, total distance travelled by car at A and car at B in 1 hour is 100 km

Therefore $$x + y = 100 \qquad ...(2)$$

Solving (1) and (2), we get $x = 60$ and $y = 40$

Therefore, the speed of car at A = 60 km/h and

the speed of car at B = 40 km/h.

CHECK YOUR PROGRESS 5.10

1. Rahim's father is three times as old as Rahim. If sum of their ages is 56 years, find their ages.
2. Rita has 18m of cloth. She cut it into two pieces in such a way that one piece is 4 m longer than the other. What is the length of shorter piece.
3. A total of Rs 50000 is to be distributed among 200 persons as prizes. A prize is either Rs 500 or Rs 100. Find the number of each type of prizes.
4. A purse contain Rs 2500 in notes of denominations of 100 and 50. If the number of 100 rupee notes is one more than that of 50 rupee notes, find the number of notes of each denomination.

LET US SUM UP

- An equation in one variable of degree one is called a linear equation in variable.
- The general form of a linear equation in one variable is $ax + b = 0$, $a \neq 0$, a and b are real numbers.
- The value of the variable which satisfies the linear equation is called its solution or root.
- To solve a word problem, it is first translated into algebriac statements and then solved.
- The general form of a linear equation in two variables is $ax + by + c = 0$, where a, b, c are real numbers and atleast one of a or b is non zero.
- The equation $ax + c = 0$ can be expressed as linear equation in two variables as $ax + 0y + c = 0$.
- To draw the graph of a linear equation in two variables, we find atleast two points in plane whose coordinates are solutions of the equation and plot them.
- The graph of a linear equaion in two variables is a line.
- To solve two simultaneous equations in two variables, we draw their graphs on the same graph paper.
  (i) if graph is intersecting lines, point of intersection gives unique solution.
  (ii) If graph is the same line, system has infinitely many solutions
  (iii) If graph is parallel lines, system of equation has no solution
- Algebraic methods of solving system of linear equations are
  (i) Substitution method
  (ii) Elimination method
- To solve word problems, we tranlate the given information (data) into linear equations and solve them.

TERMINAL EXERCISE

1. Choose the correct option:

   (i) Which one of the following is a linear equation in one variable?

   (A) $2x + 1 = y - 3$ (B) $3t - 1 = 2t + 5$

   (C) $2x - 1 = x^2$ (D) $x^2 - x + 1 = 0$

   (ii) Which one of the following is not a linear equation?

   (A) $5 + 4x = y + 3$ (B) $x + 2y = y - x$

(C) $3 - x = y^2 + 4$ (D) $x + y = 0$

(iii) Which of the following numbers is the solution of the equation $2(x + 3) = 18$?

(A) 6 (B) 12

(C) 13 (D) 21

(iv) The value of x, for which the equation $2x - (4 - x) = 5 - x$ is satisfied, is:

(A) 4.5 (B) 3

(C) 2.25 (D) 0.5

(v) The equation $x - 4y = 5$ has

(A) no solution (B) unique solution

(C) two solutions (D) infinitely many solutions

2. Solve each of the following equations

(i) $2z + 5 = 15$ (ii) $\frac{x+2}{3} = -2$

(iii) $\frac{4-2y}{3} + \frac{y+1}{2} = 1$ (iv) $2.5x - 3 = 0.5x + 1$

3. A certain number increased by 8 equals 26. Find the number.

4. Present ages of Reena and Meena are in the ration 4 : 5. After 8 years, the ratio of their ages will be 5 : 6. Find their present ages.

5. The denominator of a rational number is greater than its numerator by 8. If the denominaor is decreased by 1 and numerator is increased by 17, the number obtained is $\frac{3}{2}$. Find the rational number

6. Solve the following system of equations graphically:

(i) $x - 2y = 7$
$x + y = -2$

(ii) $4x + 3y = 24$
$3y - 2x = 6$

(iii) $x + 3y = 6$
$2x - y = 5$

(iv) $2x - y = 1$
$x + y = 8$

7. Solve the following system of equations :

(i) $x + 2y - 3 = 0$
$x - 2y + 1 = 0$

(ii) $2x + 3y = 3$
$3x + 2y = 2$

(iii) $3x - y = 7$
$4x - 5y = 2$

(iv) $5x - 2y = -7$
$2x + 3y = -18$

8. The sum of the digits of a two-digit number is 11. If the digits are reversed, the new number is 27 less than the original number. Find the original number.

9. Three years ago Atul's age was four times Parul's age. After 5 years from now, Atul's age will be two times Parul's age. Find their present ages.

10. The perimeter of a rectangular plot of land is 32 m. If the length is increased by 2m and breadth is decreased by 1 m, the area of the plot remains the same. Find the length and breadth of the plot.

ANSWERS TO CHECK YOUR PROGRESS

5.1

1. (i) 2. (i)

5.2

1. $15 - 2x = 7$
2. $0.1x = 10$
3. $6y = 96$
4. $t + 15 = 4t$

5.3

1. $x = 13$ 2. $y = 12$ 3. $z = 0$

4. $y = 9$ 5. $x = 5$

5.4

1. 39, 46
2. 15 years, 50 years
3. 22 cm, 11 cm
4. 25

5.5

1. (i) $2(x + y) = 98$

   (ii) $y = 2x + 10$, where age of son = x years, age of father = y years

   (iii) $x + 10 = y$

   (iv) $2x + 3y = 120$

2. True 3. False

5.7

1. x = 4, y = 1, unique solution
2. x = 2, y = – 1, unique solution
3. Infinitely many solutions
4. No solution
5. x= 2, y = 1, unique solution

5.8

1. x = 8, y = 6 2. x = –2, y = 5

3. x = 5, y = –2 4. x = 1, y = 3

5.9

1. x = 2, y = –3 2. x = 1, y = 2

3. x = 11, y = 2 4. x = 1, y = 3

5. x = 5, y = –2 6. x = 6, y = –1

5.10

1. 14 years, 42 years
2. 7 m
3. 75 prizes Rs 500 and 125 prizes of Rs 100 each.
4. 17 of Rs 100 each and 16 of Rs 50 each.

ANSWERS TO TERMINAL EXERCISE

1. (i) (B) (ii) (C) (iii) (A) (iv) (C) (v) (D)
2. (i) z = 5 (ii) x = – 8 (iii) y = 5 (iv) x = 2
3. 18
4. Age of Reena = 32 years, age of Meena = 40 years
5. $\frac{13}{21}$

6. (i) $x = 1, y = -3$ (ii) $x = 3, y = 4$
   (iii) $x = 3, y = 1$ (iv) $x = 3, y = 5$
7. (i) $x = 1, y = 1$ (ii) $x = 0, y = 1$
   (iii) $x = 3, y = 2$ (iv) $x = -3, y = -4$
8. 74
9. Atul: 19 years, Parul: 7 years
10. 10 m, 6m

6

QUADRATIC EQUATIONS

In this lesson, you will study about quadratic equations. You will learn to identify quadratic equations from a collection of given equations and write them in standard form. You will also learn to solve quadratic equations and translate and solve word problems using quadratic equations.

OBJECTIVES

After studying this lesson, you will be able to

- *identify a quadratic equation from a given collection of equations;*
- *write quadratic equations in standard form;*
- *solve quadratic equations by (i) factorization and (ii) using the quadratic formula;*
- *solve word problems using quadratic equations.*

EXPECTED BACKGROUND KNOWLEDGE

- Polynomials
- Zeroes of a polynomial
- Linear equations and their solutions
- Factorisation of a polynomial

6.1 QUADRATIC EQUATIONS

You are already familiar with a polynomial of degree two. A polynomial of degree two is called a quadratic polynomial. When a quadratic polynomial is equated to zero, it is called **a quadratic equation**. In this lesson, you will learn about quadratic equations in one variable only. Let us consider some examples to identify a quadratic equation from a collection of equations

Example 6.1: Which of the following equations are quadratic equations?

(i) $3x^2 = 5$ (ii) $x^2 + 2x + 3 = 0$

(iii) $x^3 + 1 = 3x^2$ (iv) $(x + 1)(x + 3) = 2x + 1$

(v) $x + \frac{1}{x} = \frac{5}{2}$ (v) $x^2 + \sqrt{x} + 1 = 0$

Solution:

(i) It is a quadratic equation since $3x^2 = 5$ can be written as $3x^2 – 5 = 0$ and $3x^2 – 5$ is a quadratic polynomial.

(ii) $x^2 + 2x + 3 = 0$ is a quadratic equation as $x^2 + 2x + 3$, is a polynomial of degree 2.

(iii) $x^3 + 1 = 3x^2$ can be written as $x^3 – 3x^2 + 1 = 0$. LHS is not a quadratic polynomial since highest exponent of x is 3. **So, the equation is not a quadratic equation.**

(iv) $(x + 1)(x + 3) = 2x + 1$ is a quadratic equation, since $(x + 1)(x + 3) = 2x + 1$ can be written as

$$x^2 + 4x + 3 = 2x + 1$$

or $x^2 + 2x + 2 = 0$

Now, LHS is a polynomial of degree 2, hence $(x + 1)(x + 3) = 2x + 1$ is a quadratic equation.

(v) $x + \frac{1}{x} = \frac{5}{2}$ is not a quadratic equation.

However, it can be reduced to quadratic equation as shown below:

$$x + \frac{1}{x} = \frac{5}{2}$$

or $\frac{x^2 + 1}{x} = \frac{5}{2}, x \neq 0$

or $2(x^2 + 1) = 5x, x \neq 0$

or $2x^2 – 5x + 2 = 0, x \neq 0$

(vi) $x^2 + \sqrt{x} + 1 = 0$ is not a quadratic equation as $x^2 + \sqrt{x} + 1$ is not a quadratic polynomial (Why?)

CHECK YOUR PROGRESS 6.1

1. Which of the following equations are quadratic equations?

   (i) $3x^2 + 5 = x^3 + x$

   (ii) $\sqrt{3}\,x^2 + 5x + 2 = 0$

   (iii) $(5y + 1)(3y - 1) = y + 1$

   (iv) $\dfrac{x^2+1}{x+1} = \dfrac{5}{2}$

   (v) $3x + 2x^2 = 5x - 4$

6.2 STANDARD FORM OF A QUADRATIC EQUATION

A quadratic equation of the form $ax^2 + bx + c = 0$, $a > 0$ where a, b, c, are constants and x is a variable is called a quadratic equation in the standard form. Every quadratic equation can always be written in the standard form.

Example 6.2: Which of the following quadratic equations are in standard form? Those which are not in standard form, express them in standard form.

(i) $2 + 3x + 5x^2 = 0$

(ii) $3x^2 - 5x + 2 = 0$

(iii) $7y^2 - 5y = 2y + 3$

(iv) $(z + 1)(z + 2) = 3z + 1$

Solution: (i) It is not in the standard form. Its standard form is $5x^2 + 3x + 2 = 0$

(ii) It is in standard form

(iii) It is not in the standard form. It can be written as

$7y^2 - 5y = 2y + 3$

or $7y^2 - 5y - 2y - 3 = 0$

or $7y^2 - 7y - 3 = 0$

which is now in the standard form.

(iv) It is not standard form. It can be rewritten as

$(z + 1)(z + 2) = 3z + 1$

or $z^2 + 3z + 2 = 3z + 1$

or $z^2 + 3z - 3z + 2 - 1 = 0$

or $z^2 + 1 = 0$

or $z^2 + 0z + 1 = 0$

which is now in the standard form.

CHECK YOUR PROGRESS 6.2

1. Which of the following quadratic equations are in standard form? Those, which are not in standard form, rewrite them in standard form:

   (i) $3y^2 - 2 = y + 1$ (ii) $5 - 3x - 2x^2 = 0$

   (iii) $(3t - 1)(3t + 1) = 0$ (iv) $5 - x = 3x^2$

6.3 SOLUTION OF A QUADRATIC EQUATION

You have learnt about the zeroes of a polynomial. A zero of a polynomial is that real number, which when substituted for the variable makes the value of the polynomial zero. In case of a quadratic equation, the value of the variable for which LHS and RHS of the equation become equal is called a root or solution of the quadratic equation. You have also learnt that if α is a zero of a polynomial p(x), then $(x - \alpha)$ is a factor fo p(x) and conversely, if $(x - \alpha)$ is a factor of a polynomial, then α is a zero of the polynomial. You will use these results in finding the solution of a quadratic equation. There are two algebraic methods for finding the solution of a quadratic equation. These are (i) Factor Method and (ii) Using the Quadratic Formula.

Factor Method

Let us now learn to find the solutions of a quadratic equation by factorizing it into linear factors. The method is illustrated through examples.

Example 6.3: Solve the equation $(x - 4)(x + 3) = 0$

Solution: Since, $(x - 4)(x + 3) = 0$, therefore,

either $x - 4 = 0$, or $x + 3 = 0$

or $x = 4$ or $x = -3$

Therefore, $x = 4$ and $x = -3$ are solutions of the equation.

Example 6.4: Solve the equation $6x^2 + 7x - 3 = 0$ by factorisation.

Solution: Given $6x^2 + 7x - 3 = 0$

By breaking middle term, we get

$6x^2 + 9x - 2x - 3 = 0$ [since, $6 \times (-3) = -18$ and $-18 = 9 \times (-2)$ and $9 - 2 = 7$]

or $3x(2x + 3) - 1(2x + 3) = 0$

or $(2x + 3)(3x - 1) = 0$

This gives $2x + 3 = 0$ or $3x - 1 = 0$

or $x = -\frac{3}{2}$ or $x = \frac{1}{3}$

Therefore, $x = -\frac{3}{2}$ and $x = \frac{1}{3}$ are solutions of the given equation.

Example 6.5: Solve $x^2 + 2x + 1 = 0$

Solution: We have $x^2 + 2x + 1 = 0$

or $(x + 1)^2 = 0$

or $x + 1 = 0$

which gives $x = -1$

Therefore, $x = -1$ is the only solution.

Note: In Examples 6.3 and 6.4, you saw that equations had two distinct solutions. However, in Example 6.5, you got only one solution. We say that it has two solutions and these are coincident.

CHECK YOUR PROGRESS 6.3

1. Solve the following equations using factor method.

(i) $(2x + 3)(x + 2) = 0$

(ii) $x^2 + 3x - 18 = 0$

(iii) $3x^2 - 4x - 7 = 0$

(iv) $x^2 - 5x - 6 = 0$

(v) $25x^2 - 10x + 1 = 0$

(vi) $4x^2 - 8x + 3 = 0$

Quadratic Formula

Now you will learn to find a formula to find the solution of a quadratic equation. For this, we will rewrite the general quadratic equation $ax^2 + bx + c = 0$ by completing the square.

We have $ax^2 + bx + c = 0$

Multiplying both sides by '4a' to make the coefficient of x^2 a perfect square, of an even number, we get

$$4a^2x^2 + 4abx + 4ac = 0$$

or $(2ax)^2 + 2(2ax)b + (b)^2 + 4ac = b^2$ [adding b^2 to both sides]

or $(2ax)^2 + 2(2ax)b + (b)^2 = b^2 - 4ac$

or $(2ax + b)^2 = \left\{\pm\sqrt{b^2 - 4ac}\right\}^2$

or $2ax + b = \pm\sqrt{b^2 - 4ac}$

or $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This gives two solutions of the quadratic equation $ax^2 + bx + c = 0$. The solutions (roots) are:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Here, the expression $(b^2 - 4ac)$, denoted by D, is called **Discriminant**, because it determines the number of solutions or nature of roots of a quadratic equation.

For a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, if

(i) $D = b^2 - 4ac > 0$, the equation has two real distinct roots, which are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

(ii) $D = b^2 - 4ac = 0$, then equation has two real equal roots, each equal to $\dfrac{-b}{2a}$

(iii) $D = b^2 - 4ac < 0$, the equation will not have any real root, since square root of a negative real number is not a real number.

Thus, a quadratic equation will have at the most two roots.

Example 6.6: Without determining the roots, comment on the nature (number of solutions) of roots of the following equations:

(i) $3x^2 - 5x - 2 = 0$

(ii) $2x^2 + x + 1 = 0$

(iii) $x^2 + 2x + 1 = 0$

Solution: (i) The given equation is $3x^2 - 5x - 2 = 0$. Comparing it with $ax^2 + bx + c = 0$, we get $a = 3$, $b = -5$ and $c = -2$.

Now $D = b^2 - 4ac \quad = (-5)^2 - 4 \times 3 \times (-2)$

$$= 25 + 24 = 49$$

Since, $D > 0$, the equation has two real distinct roots.

(ii) Comparing the equation $2x^2 + x + 1 = 0$ with $ax^2 + bx + c = 0$,

we get $a = 2$, $b = 1$, $c = 1$

Now $D = b^2 - 4ac = (1)^2 - 4 \times 2 \times 1 = 1 - 8 = -7$

Since, $D = b^2 - 4ac < 0$, the equation does not have any real root.

(iii) Comparing the equation $x^2 + 2x + 1 = 0$ with $ax^2 + bx + c = 0$,

we get $a = 1, b = 2, c = 1$

Now $D = b^2 - 4ac = (2)^2 - 4 \times 1 \times 1 = 0$

Since, $D = 0$, the equation has two equal roots.

Example 6.7: Using quadratic formula, find the roots of the equation $6x^2 - 19x + 15 = 0$

Solution: Comparing the given equation with $ax^2 + bx + c = 0$

We get, $a = 6, b = -19, c = 15$

Now $D = b^2 - 4ac = (-19)^2 - 4 \times 6 \times 15$

$= 361 - 360 = 1$

Therefore, roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{19 \pm \sqrt{1}}{12} = \frac{19 \pm 1}{12}$$

So, roots are $\frac{19+1}{12} = \frac{5}{3}$ and $\frac{19-1}{12} = \frac{3}{2}$

Thus, the two roots are $\frac{5}{3}$ and $\frac{3}{2}$.

Example 6.8: Find the value of m sothat the equation $3x^2 + mx - 5 = 0$ has equal roots.

Solution: Comparing the given equation with $ax^2 + bx + c = 0$

We have, $a = 3, b = m, c = -5$

For equal roots

$D = b^2 - 4ac = 0$

or $m^2 - 4 \times 3 \times (-5) = 0$

or $m^2 = 60$

This gives $m = \pm 2\sqrt{15}$

Hence, for $m = \pm 2\sqrt{15}$, the equation will have equal roots.

CHECK YOUR PROGRESS 6.4

1. Without determining the roots, comment on nature of roots of following equations:

   (i) $3x^2 - 7x + 2 = 0$ (ii) $4x^2 - 12x + 9 = 0$ (iii) $25x^2 + 20x + 4 = 0$

   (iv) $x^2 - x + 1$

2. Solve the following equations using quadratic formula:

   (i) $y^2 - 14y - 12 = 0$ (ii) $x^2 - 5x = 0$ (iii) $x^2 - 15x + 50 = 0$

3. Find the value of m so that the following equations have equal roots:

   (i) $2x^2 - mx + 1 = 0$ (ii) $mx^2 + 3x - 5 = 0$

   (iii) $3x^2 - 6x + m = 0$ (iv) $2x^2 + mx - 1 = 0$

6.4 WORD PROBLEMS

We will now solve some problems which involve the use of quadratic equations.

Example 6.9: The sum of squares of two consecutive odd natural numbers is 74. Find the numbers.

Solution: Let two consecutive odd natural numbers be x and x + 2. Since, sum of their squares is 74. we have

$$x^2 + (x + 2)^2 = 74$$

or $$x^2 + x^2 + 4x + 4 = 74$$

or $$2x^2 + 4x - 70 = 0$$

or $$x^2 + 2x - 35 = 0$$

or $$x^2 + 7x - 5x - 35 = 0$$

or $$x(x + 7) - 5(x + 7) = 0$$

or $$(x + 7)(x - 5) = 0$$

Therefore $x + 7 = 0$ or $x - 5 = 0$

or $$x = -7 \text{ or } x = 5$$

Now, x can not be negative as it is a natural number. Hence $x = 5$

So, the numbers are 5 and 7.

Example 6.10: The sum of the areas of two square fields is 468 m^2. If the difference of their perimeter is 24 m, find the sides of the two squares.

Solution: Let the sides of the bigger square be x and that of the smaller square be y.

Hence, perimeter of bigger square = 4x

and perimeter of smaller square = 4y

Therefore, $4x - 4y = 24$

or $x - y = 6$

or $x = y + 6$ ....(1)

Also, since sum of areas of two squares is 468 m^2

Therefore, $x^2 + y^2 = 468$ ....(2)

Substituting value of x from (1) into (2), we get

$$(y+6)^2 + y^2 = 468$$

or $y^2 + 12y + 36 + y^2 = 468$

or $2y^2 + 12y - 432 = 0$

or $y^2 + 6y - 216 = 0$

Therefore $y = \dfrac{-6 \pm \sqrt{36+864}}{2} = \dfrac{-6 \pm \sqrt{900}}{2}$

or $y = \dfrac{-6 \pm 30}{2}$

Therefore, $y = \dfrac{-6+30}{2}$ or $\dfrac{-6-30}{2}$

or $y = 12$ or -18

Since, side of square can not be negative, so y = 12

Therefore, $x = y + 6 = 12 + 6 = 18$

Hence, sides of squares are 18 m and 12 m.

Example 6.11: The product of digits of a two digit number is 12. When 9 is added to the number, the digits interchange their places. Determine the number.

Solution: Let the digit at ten's place be x

and digit at unit's place be y

Therefore, number = $10x + y$

When digits are interchanged, the number becomes $10y + x$

Therefore $10x + y + 9 = 10y + x$

or $10x - x + y - 10y = -9$

or $9x - 9y = -9$

or $x - y = -1$

or $x = y - 1$ ...(1)

Also, product of digits is 12

Hence, $xy = 12$ ...(2)

Substituting value of x from (1) into (2), we get

$(y - 1)y = 12$

or $y^2 - y - 12 = 0$

or $(y - 4)(y + 3) = 0$

Hence, $y = 4$ or $y = -3$

Since, digit can not be negative, $y = 4$

Hence $x = y - 1 = 4 - 1 = 3$

Therefore, the number is 34.

Example 6.12: The sum of two natural numbers is 12. If sum of their reciprocals is $\frac{4}{9}$, find the numbers.

Solution: Let one number be x

Therefore, other number = $12 - x$

Since, sum of their reciprocals is $\frac{4}{9}$, we get

$$\frac{1}{x} + \frac{1}{12 - x} = \frac{4}{9}, \quad x \neq 0, \; 12 - x \neq 0$$

or $$\frac{12 - x + x}{x(12 - x)} = \frac{4}{9}$$

or $$\frac{12}{12x - x^2} = \frac{4}{9}$$

or $$\frac{12 \times 9}{4} = 12x - x^2$$

or $27 = 12x - x^2$

or $x^2 - 12x + 27 = 0$

or $(x - 3)(x - 9) = 0$

It gives $x = 3$ or $x = 9$

When first number x is 3, other number is $12 - 3 = 9$ and when first number x is 9, other number is $12 - 9 = 3$.

Therefore, the required numbers are 3 and 9.

CHECK YOUR PROGRESS 6.5

1. The sum of the squares of two consecutive even natural numbers is 164. Find the numbers.
2. The length of a rectangular garden is 7 m more than its breadth. If area of the garden is 144 m^2, find the length and breadth of the garden.
3. The sum of digits of a two digit number is 13. If sum of their squares is 89, find the number.
4. The digit at ten's place of a two digit number is 2 more than twice the digit at unit's place. If product of digits is 24, find the two digit number.
5. The sum of two numbers is 15. If sum of their reciprocals is $\frac{3}{10}$, find the two numbers.

LET US SUM UP

- An equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ and a, b, c are real numbers is called a quadratic equation in standard form.
- The value(s) of the variable which satisfy a quadratic equation are called it roots or solutions.
- The zeros of a quadratic polynomial are the roots or solutions of the corresponding quadratic equation.
- If you can factorise $ax^2 + bx + c = 0$, $a \neq 0$, into product of linear factors, then the roots of the quadratic equation $ax^2 + bx + c = 0$, can be obtained by equating each factor to zero.
- Roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- $b^2 - 4ac$ is called discriminant of the quadratic equation. $ax^2 + bx + c = 0, a \neq 0$ It is usually denoted by D.

  (i) If $D > 0$, then the quadratic equation has two real unequal (distinct) roots.

  (ii) If $D = 0$, then the quadratic equation has two equal (coincident) roots.

  (iii) If $D < 0$, then the quadratic equation has no real root.

TERMINAL EXERCISE

1. Which of the following are quadratic equations?

   (i) $y\left(\sqrt{5}y - 3\right) = 0$ (ii) $5x^2 - 3\sqrt{x} + 8 = 0$

   (iii) $3x - \frac{1}{x} = 5$ (iv) $x(2x + 5) = x^2 + 5x + 7$

2. Solve the following equations by factorisation method:

   (i) $(x - 8)(x + 4) = 13$ (ii) $3y^2 - 7y = 0$

   (iii) $x^2 + 3x - 18 = 0$ (iv) $6x^2 + x - 15 = 0$

3. Find the value of m for which $5x^2 - 3x + m = 0$ has equal roots.
4. Find the value of m for which $x^2 - mx - 1 = 0$ has equal roots.
5. Solve the following quadratic equations using quadratic formula:

   (i) $6x^2 - 19x + 15 = 0$ (ii) $x^2 + x - 1 = 0$

   (iii) $21 + x = 2x^2$ (iv) $2x^2 - x - 6 = 0$

6. The sides of a right angled triangle are $x - 1$, x and $x + 1$. Find the value of x and hence the sides of the triangle.
7. the sum of squares of two consecutive odd integers is 290. Find the integers.
8. The hypotenuse of a right angled triangle is 13 cm. If the difference of remaining two sides is 7 cm, find the remaining two sides.
9. The sum of the areas of two squares is 41 cm^2. If the sum of their perimeters is 36 cm, find the sides of the two squares.
10. A right angled isosceles triangle is inscribed in a circle of radius 5 cm. Find the sides of the triangle.

ANSWERS TO CHECK YOUR PROGRESS

6.1

1. (ii), (iii), (v)

6.2

1. (i) No, $3y^2 - y - 3 = 0$ (ii) No, $2x^2 + 2x - 5 = 0$

   (iii) No, $6t^2 + t - 1 = 0$ (iv) No, $3x^2 + x - 5 = 0$

6.3

1. (i) $\frac{3}{2}, -2$ (ii) 3, – 6   (iii) $\frac{7}{3}, -1$

   (iv) 2, 3   (v) $\frac{1}{5}, \frac{1}{5}$ (vi) $\frac{3}{2}, \frac{1}{2}$

6.4

1. (i) Two real distinct roots

   (ii) Two real equal roots

   (iii) Two real equal roots

   (iv) No real roots

2. (i) $7 \pm \sqrt{37}$ (ii) 0, 5   (iii) 5, 10

3. (i) $\pm 2\sqrt{2}$ (ii) $\frac{9}{20}$ (iii) 3   (iv) For no value of m

6.5

1. 8, 10   2. 16m, 9m   3. 85, 58

4. 83   (v) 5, 10

ANSWERS TO TERMINAL EXERCISE

1. (i), (iv)

2. (i) 8, 4   (ii) 0, $\frac{7}{3}$ (iii) 3, – 6   (iv) $\frac{3}{2}, -\frac{5}{3}$

3. $\frac{9}{20}$

4. For no value of m

5. (i) $\frac{3}{2}, \frac{5}{3}$ (ii) $\frac{-1 \pm \sqrt{5}}{2}$ (iii) $\frac{7}{2}, -3$ (iv) $2, \frac{3}{2}$

6. 3, 4, 5

7. 11, 13 or –13 , –11

8. 5 cm, 12 cm

9. 5 cm, 4 cm

10. $5\sqrt{2}$ cm, $5\sqrt{2}$ cm, 10 cm

7

ARITHMETIC PROGRESSIONS

In your daily life you must have observed that in nature, many things follow patterns such as petals of flowers, the holes of a honey-comb, the spirals on a pine apple etc. In this lesson, you will study one special type of number pattern called Arithmetic Progression (AP). You will also learn to find general term and the sum of first n terms of an arithmetic progression.

OBJECTIVES

After studying this lesson, you will be able to

- *identify arithmetic progression from a given list of numbers;*
- *determine the general term of an arithmetic progression;*
- *find the sum of first n terms of an arithmetic progression.*

PREVIOUS BACKGROUND KNOWLEDGE

- Knowledge of number system
- Operations on number system

7.1 SOME NUMBER PATTERNS

Let us consider some examples:

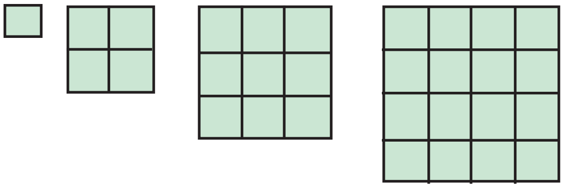
(i) Rita deposits ₹ 1000 in a bank at the simple interest of 10% per annum. The amount at the end of first, second, third and fourth years, in rupees will be respectively

1100, 1200, 1300, 1400

Do you observe any pattern? You can see that amount increases every year by a fixed amount of ₹ 100.

(ii) The number of unit squares in a square with sides 1, 2, 3, 4, ... units are respectively

1, 4, 9, 16, ...



Can you see any pattern in the list of these numbers? You can observe that

$1 = 1^2, 4 = 2^2, 9 = 3^2, 16 = 4^2, ...$

i.e., these are squares of natural numbers.

Now consider some more lists of numbers and try to recognise a pattern if possible:

1, 3, 5, 7, 9 ..... (1)
2, 4, 6, 8, 10 ... (2)
1, 4, 7, 10, 13 .... (3)
5, 3, 1, –1, –3... (4)
1, 3, 9, 27, 81, ... (5)
2, 3, 5, 7, 11, 13... (6)

You can observe that numbers in the list (1) are odd natural numbers. The first number is 1, second number is 3, third number is 5, etc. All these numbers follow a pattern. The pattern is that all these numbers, except the first is obtained by adding 2 to its previous number.

In lists (2), (3) and (4), each number except the first is obtained by adding 2, 3, and – 2 respectively to its previous number.

In (5), each number, except the first is obtained by multiplying 3 to its previous number. In the list (6), you can see that it is the list of prime numbers and it is not possible to give any rule till date, which gives the next prime number.

The numbers in a list are generally denoted by

$a_1, a_2, a_3, ..., a_n, ...$

or $t_1, t_2, t_3, ..., t_n, ...$

which are respectively called first, second, third and nth term in the list of numbers. We sometimes call each of these lists as **sequence** or **pattern of numbers.**

7.2 ARITHMETIC PROGRESSION

You have seen different type of patterns. Some patterns follow definite mathematical rules to generate next term in the pattern. You will now study one particular type of pattern of

numbers.Recall the following patterns.

1, 3, 5, 7, 9, ... (1)

2, 4, 6, 8, 10, ... (2)

1, 4, 7, 10, 13,... (3)

You have observed that in (1) and (2), each term except first is obtained by adding 2 to its previous number (term). In (3), each term except first is obtained by adding 3 to its previous term. The numbers appearing in a number pattern are called its terms. As already stated these terms are usually denoted by

$a_1, a_2, a_3, ..., a_n, ...$

or $t_1, t_2, t_3, ..., t_n, ...$ etc

The suffix denotes the position of the term in the pattern. Thus, a_n or t_n denotes 'n'th term of the pattern.

A particular type of pattern in which each term except the first is obtained by adding a fixed number (positive or negative) to the previous term is called an Arithmetic Progression (A.P.). The first term is usually denoted by '*a*' and the common difference is denoted by *d*. Thus, standard form of an Arithmetic Progression would be:

$a, a + d, a + 2d, a + 3d, ...$

Example 7.1: In the following list of numbers, find which are Arithmetic Progressions. In case of AP, find their respective first terms and common differences.

(i) 2, 7, 12, 17, 22, ...

(ii) 4, 0, –4, –8, –12 ...

(iii) 3, 7, 12, 18, 25 ...

(iv) 2, 6, 18, 54, 162 ...

Solution:

(i) It is an arithmetic progression (AP).

Since $7 – 2 = 5, 12 – 7 = 5, 17 – 12 = 5$ and $22 – 17 = 5$

Thus, each term except first is obtained by adding 5 to its previous term. Hence, first term $a = 2$ and common difference $d = 5$.

(ii) We observe that

$0 – 4 = – 4, – 4 – 0 = – 4, – 8 – (–4) = – 4, – 12 – (–8) = – 4$

Thus, it is an AP with first term $a = 4$

and common difference $d = – 4$.

(iii) You can see that in the list

3, 7, 12, 18, 25, ...

7 – 3 = 4, 12 – 7 = 5, 18 – 12 = 6, 25 – 18 = 7

Thus, difference of two consecutive terms is not the same. Hence, it is not an AP.

(iv) In the list of numbers

2, 6, 18, 54, 162, ...

6 – 2 = 4, 18 – 6 = 12

Therefore, difference of two consecutive terms is not the same. Hence, it is not an AP.

CHECK YOUR PROGRESS 7.1

Which of the following are AP? If they are in AP, find their first terms and common differences:

1. –5, –1, 3, 7, 11, ...
2. 6, 7, 8, 9, 10, ...
3. 1, 4, 6, 7, 6, 4, ...
4. –6, – 3, 0, 3, 6, 9, ...

7.3 GENERAL (nth) TERM OF AN AP

Let us consider an AP whose first term is '*a*' and common difference in '*d*'. Let us denote the terms of AP as $t_1, t_2, t_3, ..., t_n$, where t_n denotes the nth term of the AP. Since first term is a, second term is obtained by adding d to a i.e., a + d, the third term will be obtained by adding '*d*' to a + d. So, third term will be (a + d) + d = a + 2d and so on.

With this

First term, $t_1 = a \qquad = a + (1 - 1)\,d$

Second term, $t_2 = a + d \qquad = a + (2 - 1)\,d$

Third term, $t_3 = a + 2d \qquad = a + (3 - 1)\,d$

Fourth term, $t_4 = a + 3d \qquad = a + (4 - 1)\,d$

Can you see any pattern? We observe that each term is a + (term number – 1) d. What will be 10th term, say:

$$t_{10} = a + (10 - 1)d = a + 9d$$

Can you now say "what will be the nth term or general term?"

Clearly $t_n = a + (n - 1)\,d$

Example 7.2: Find the 15th and nth terms of the AP

16, 11, 6, 1, – 4, – 9, ...

Solution: Here a = 16 and d = 11 – 16 = – 5

Thus, $t_{15} = a + (15 - 1)d = a + 14d$

$= 16 + 14(-5) = 16 - 70$

$= -54$

Therefore, 15th term i.e., $t_{15} = -54$

Now $t_n = a + (n - 1)d$

$= 16 + (n - 1) \times (-5) = 16 - 5n + 5$

$= 21 - 5n$

Thus, nth term, i.e., $t_n = = 21 - 5n$

Example 7.3: The first term of an AP is – 3 and 12th term is 41. Determine the common difference.

Solution: Let first term of AP be a and commong difference be d.

Therefore, $t_{12} = a + (12 - 1)d = 41$

or $-3 + 11d = 41$ [Since a = –3]

or $11d = 44$

or $d = 4$

Therefore, common difference is 4.

Example 7.4: The common difference of an AP is 5 and 10th term is 43. Find its first term.

Solution: We have:

$t_{10} = a + (10 - 1)\,d$

So, $43 = a + 9 \times 5$ [Since $d = 5$]

or $43 = a + 45$

Hence, $a = -2$

Therefore, first term is – 2.

Example 7.5: The first term of an AP is – 2 and 11th term is 18. Find its 15th term.

Solution: To find 15th term, you need to find d.

Now $t_{11} = a + (11 - 1)d$

So, $18 = -2 + 10d$

or $10d = 20$

or $d = 2$

Now $t_{15} = a + 14d$

$= -2 + 14 \times 2 = 26$

Therefore, $t_{15} = 26$.

Example 7.6: If p times the pth term of an AP is equal to q times the qth term, prove that its (p + q)th term is zero, provided $p \neq q$.

Solution: We have:

$t_p = a + (p - 1)d$

$t_q = a + (q - 1)d$

Since $pt_p = qt_q$, therefore,

$p[a + (p - 1)d] = q[a + (q - 1)d]$

or $pa + p(p - 1)d - qa - q(q - 1)d = 0$

or $(p - q)a + (p^2 - q^2)d - pd + qd = 0$

or $(p - q)a + (p^2 - q^2)d - (p - q)d = 0$

or $(p - q)a + (p - q)(p + q)d - (p - q)d = 0$

or $(p - q)[a + (p + q)d - d] = 0$

or $a + (p + q)d - d = 0$ $[\text{as } p - q \neq 0]$

or $a + (p + q - 1)d = 0$

Since, LHS is nothing but (p + q)th term, therefore,

$t_{p+q} = 0$

CHECK YOUR PROGRESS 7.2

1. The first term of an AP is 4 and common difference is –3, find its 12th term.
2. The first term of an AP is 2 and 9th term is 26, find the common difference.
3. The 12th term of an AP is –28 and 18th term is –46. Find its first term and common difference.
4. Which term of the AP 5, 2, –1, ... is –22?
5. If the *p*th, *q*th and *r*th terms of an AP are x, y and z respectively, prove that:

   $x(q - r) + y(r - p) + z(p - q) = 0$

7.4 SUM OF FIRST n TERMS OF AN AP

Carl Friedrich Gauss, the great German mathematician, was in elementary school, when his teacher asked the class to find the sum of first 100 natural numbers. While the rest of the class was struggling with the problem, Gauss found the answer within no time. How Gauss got the answer? Probably, he did as follows:

$$S = 1 + 2 + 3 + ... + 99 + 100 \qquad (1)$$

Writing these numbers in reverse order, we get

$$S = 100 + 99 + 98 + ... + 2 + 1 \qquad (2)$$

Adding (1) and (2), term by term, we get

$$2S = 101 + 101 + 101 + ... + 101 + 101 \text{ (100 times)}$$

$$= 100 \times 101$$

$$\text{or } S = \frac{100\times101}{2} = 5050$$

We shall use the same method to find the sum of first ‘n’ terms of an AP.

The first ‘n’ terms of an AP are

$$a,\ a + d,\ a + 2d,\ ...,\ a + (n – 2)d,\ a + (n – 1)d$$

Let us denote the sum of n terms by S_n. Therefore,

$$S_n = a + (a + d) + (a + 2d) + ... + [a + (n – 2)d] + [a + (n – 1)d] \qquad (3)$$

Writing these terms in reverse order, we get

$$S_n = [a + (n – 1)d] + [a + (n – 2)d] + ... + (a + d) + a \qquad (4)$$

We now add (3) and (4), term by term. We can see that the sum of any term in (3) and the corresponding term in (4) is 2a + (n – 1)d. We get

$$2S_n = [2a + (n – 1)d] + [2a + (n – 1)d] + ... + [2a + (n – 1)d] + [2a + (n – 1)d] \text{, n times}$$

$$\text{or } 2S_n = n[2a + (n – 1)d]$$

$$\text{or } S_n = \frac{n}{2}[2a + (n – 1)d],$$

which gives general formula for finding the sum of first ‘*n*’ terms of an AP.

This can be rewritten as

$$S_n = \frac{n}{2}[a + \{a + (n – 1)d\}]$$

$$= \frac{n}{2}(a + t_n), \qquad [\text{as } n^{th} \text{ term } t_n = a + (n – 1)d]$$

Sometimes, *n*th term is named as last term and is denoted by '*l*'. Thus:

$$S_n = \frac{n}{2}(a + l) \qquad (4)$$

Example 7.7: Find the sum of the first 12 terms of the following AP

(i) 11, 16, 21, 26 ....

(ii) – 151, – 148, – 145, – 142

Solution: (i) The given AP is

11, 16, 21, 26 ....

Here, $a = 11$, $d = 16 - 11 = 5$ and $n = 12$.

You know that sum of first n terms of an AP is given by

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Therefore, $S_{12} = \frac{12}{2}[2 \times 11 + (12 - 1)5]$

$$= 6\,[22 + 55] = 6 \times 77 = 462$$

Hence, required sum is 462.

(ii) The given AP is

– 151, – 148, – 145, – 142

Here, $a = -151$, $d = -148 - (-151) = 3$ and $n = 12$.

We know that

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Hence, sum of first 12 terms is

$$S_{12} = \frac{12}{2}[2 \times (-151) + (12 - 1)3]$$

$$= 6[-302 + 33] = 6 \times (-269)$$

$$= -1614$$

Therefore, required sum is – 1614.

Example 7.8: How may terms of the AP 2, 4, 6, 8, 10 .... are needed to get sum 210?

Solution: For the given AP, $a = 2$, $d = 2$ and $S_n = 210$.

We have: $S_n = \frac{n}{2}[2a + (n-1)d]$

or $210 = \frac{n}{2}[2 \times 2 + (n-1)2]$

or $420 = n[2n + 2]$

or $420 = 2n^2 + 2n$

or $2n^2 + 2n - 420 = 0$

or $n^2 + n - 210 = 0$

or $n^2 + 15n - 14n - 210 = 0$

or $n(n + 15) - 14(n + 15) = 0$

or $(n + 15)(n - 14) = 0$

or $n = -15$ or $n = 14$

Since, n cannot be negative, so, $n = 14$

Therefore, first 14 terms are needed to get the sum 210.

Example 7.9: Find the following sum

$2 + 5 + 8 + 11 + ... + 59$

Solution: Here 2, 5, 8, 11, ... are in AP and $a = 2$, $d = 3$ and $t_n = 59$.

To find the sum, you need to find the value of n.

Now, $t_n = a + (n-1)d$

So, $59 = 2 + (n-1)3$

or $59 = 3n - 1$

or $60 = 3n$

Therefore, $n = 20$

Now, $S_n = \frac{n}{2}[2a + (n-1)d]$

or $S_{20} = \frac{20}{2}[2 \times 2 + (20-1)3]$

or $S_{20} = 10[4 + 57] = 610$

Therefore, required sum is 610.

Example 7.10: Find the sum of all natural numbers between 1 and 1000 which are divisible by 7.

Solution: Here, the first number which is divisible by 7 is 7 and last number, which is divisible by 7 is 994. Therefore, the terms to be added are

7, 14, 21, ..., 994

Here $a = 7, d = 7, t_n = 994$

Now $t_n = a + (n - 1)d$

or $994 = 7 + (n - 1)7$

or $994 = 7n$

This gives n = 142.

Now, $S_n = \frac{n}{2}[a + l]$

$$= \frac{142}{2}[7 + 994] = 71 \times 1001$$

$= 71071$

Therefore, required sum is 71071.

Example 7.11: The sum of first three terms of an AP is 36 and their product is 1620. Find the AP.

Solution: We can take three terms of the AP as a, a + d and a + 2d. However, the product will be rather difficult and solving the two equations simultaneoulsy will be time consuming. The elegant way is to assume the first three terms as a – d, a and a + d, so that the sum of three terms becomes 3a.

Let first three terms of the AP b a – d, a and a + d

Therefore, $a - d + a + a + d = 36$

or $3a = 36,$

which gives $a = 12$

Now, since product is 1620, we have:

$(a - d)\, a\, (a + d) = 1620$

or $(12 - d)\, 12\, (12 + d) = 1620$

or $12^2 - d^2 = 135$

or $144 - d^2 = 135$

or $d^2 = 9$

Therefore, $d = 3$ or -3

If $d = 3$, the numbers are $12 - 3$, 12 and $12 + 3$

i.e. 9, 12, 15 (Since $a = 12$)

If $d = -3$, the numbers are15, 12 and 9

Therefore, the first three terms of the AP 9, 12, 15 and 15, 12, 9 satisfy the given conditions.

CHECK YOUR PROGRESS 7.3

1. Find the sum of first 15 terms of the following APs:
   (i) 11, 6, 1, – 4, –9 ...
   (ii) 7, 12, 17, 22, 27 ...
2. How many terms of the AP 25, 28, 31, 34, ... are needed to give the sum 1070?
3. Find the following sum:
   $1 + 4 + 7 + 10 + \ldots + 118$
4. Find the sum of all natural numbers upto 100 which are divisible by 3.
5. The sum of any three consecutive terms of an AP is 21 and their product is 231. Find the three terms of the AP.
6. Of the l, a, n, d and S_n, determine the ones that are missing for each of the following arithmetic progression
   (i) $a = -2$, $d = 5$, $S_n = 568$.
   (ii) $l = 8$, $n = 8$, $S_8 = -20$
   (iii) $a = -3030$, $l = -1530$, $n = 5$
   (iv) $d = \frac{2}{3}$, $l = 10$, $n = 20$

LET US SUM UP

- A progression in which each term, except the first, is obtained by adding a constant to the previous term is called an AP.
- The first term of an AP is denoted by a and common difference by d.

- The 'n'th term of an AP is given by $t_n = a + (n - 1)d$.
- The sum of first n terms of an AP is given by $S_n = \frac{n}{2}[2a + (n - 1)d]$
- The sum of an AP whose first term is a and last term is l and number of terms is n is given by $S_n = \frac{n}{2}(a + l)$

TERMINAL EXERCISE

1. Which of the following patterns are arithmetic progressions?
    (i) 2, 5, 8, 12, 15, ...
    (ii) – 3, 0, 3, 6, 9 .....
    (iii) 1, 2, 4, 8, 16, ...
2. Write the nth term of each of the following arithmetic progressions:
    (i) 5, 9, 13, 17, ...
    (ii) – 7, – 11, – 15, – 19
3. The fourth term of an AP is equal to three times its first term and seventh term exceeds twice the third term by 1. Find the first term and common difference.
4. The 5th term of an AP is 23 and 12th term is 37. Find the first term and common difference.
5. The angles of a triangle are in AP. If the smallest angle is one-third the largest angle, find the angles of the triangle.
6. Which term of AP

    (i) 100, 95, 90, 85, ..., is – 25?

    (ii) $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$..... is $\frac{25}{4}$?
7. The nth term of an AP is given by $t_n = a + bn$. Show that it is an AP. Find its first term and common difference.
8. If 7 times the 7th term of an AP is equal to 11 times the 11th term, show that the 18th term is zero.
9. Each term of an AP whose first term is a and common difference is d, is doubled. Is the resulting pattern an AP? If so, find its first term and common difference.
10. If k + 2, 4k – 6 and 3k – 2 are three consecutive terms of an AP, find k.

11. How many terms of the AP:

    (i) 1, 4, 7, 10, ... are needed to get the sum 715?

    (ii) –10, –7, –4, –1, ... are needed to get the sum 104?

12. Find the sum of first 100 odd natural numbers.

13. In an AP, a = 2 and sum of the first five terms is one-fourth the sum of the next five terms. Show that its 20th term is –12.

    [**Hint:** If AP is a, a + d, a + 2d, ... , then $S_5 = \frac{5}{2}[a + (a + 4d)]$

    In the next five terms, the first term is a + 5d and last term is a + 9d.

14. If sum of first n terms of an AP is $2n + 3n^2$, find rth term of the A.P. [Hint $t_r = S_r - S_{r-1}$]

15. Find the sum of all 3-digit numbers which leave the remainder 1, when divided by 4.

    [**Hint:** First term = 101, last term = 997]

ANSWERS TO CHECK YOUR PROGRESS

7.1

1. a = – 5, d = 4
2. a = 6, d = 1
3. Not an AP
4. a = –6, d = 3

7.2

1. – 29   2. 3   3. 5, – 3   4. 10th term

7.3

1. (i) – 360   (ii) 630
2. 20
3. 2380
4. 1689
5. 3, 7, 11 or 11, 7, 3
6. (i) $n = 16, l = 73$ (ii) $a = -3, d = 3$

   (iii) $d = 375, S_n = -11400$ (iv) $a = -\frac{3}{8}, S_n = \frac{220}{3}$

ANSWERS TO TERMINAL EXERCISE

1. (ii)

2. (i) $t_n = 4n + 1$ (ii) $t_n = -4n - 3$

3. 3, 2

4. 15, 2

5. 30°, 60°, 90°

6. (i) 26th term (ii) 25th term

7. a + b, b

9. Yes, first term = 2a, common difference = 2d

10. 3 11. (i) 22 terms (ii) 13 terms

12. 10,000 14. 6r – 1 15. 123525

Secondary Course Mathematics

Practice Work-Algebra

Maximum Marks: 25 **Time : 45 Minutes**

Instructions:

1. Answer all the questions on a separate sheet of paper.
2. Give the following informations on your answer sheet

   Name

   Enrolment number

   Subject

   Topic of practice work

   Address
3. Get your practice work checked by the subject teacher at your study centre so that you get positive feedback about your performance.

1. The value of a if $(x - a)$ is a factor of $x^6 - ax^5 + x^4 - ax^3 + 3x - a + 2$, is 1

   (A) $a = 1$

   (B) $a = -1$

   (C) $a = 2$

   (D) $a = -2$

2. The reciprocal of $\frac{1}{(-3/5)^{-2}}$ is 1

   (A) $\left(-\frac{3}{5}\right)^2$

   (B) $\left(\frac{-5}{3}\right)^2$

(C) $(-5/3)^{-2}$

(D) $\left(\frac{3}{5}\right)^{-2}$

3. In an A.P., the sum of three numbers is 15 and their product is 45. Then the three numbers are 1

(A) 1, 3, 15

(B) 2, 4, 9

(C) 1, 5, 9

(D) 0, 5, 9

4. If $y=\frac{x-1}{x+1}$, then $2y-\frac{1}{2y}$ is equal to 1

(A) $\frac{3x^2-10x-3}{2(x^2-1)}$

(B) $\frac{3x^2-10x+1}{x^2-1}$

(C) $\frac{3x^2+10x+3}{2(x^2-1)}$

(D) $\frac{3x^2-10x+3}{2(x^2-1)}$

5. The lowest form of the expression $\frac{4x^2-25}{2x^2+11x=15}$ is 1

(A) $\frac{2x-5}{x+3}$

(B) $\frac{2x+5}{x+3}$

(C) $\frac{2x-5}{x-3}$

(D) $\dfrac{2x-5}{x-3}$

6. Find x, so that $\left(\dfrac{7}{8}\right)^{-3} \times \left(\dfrac{8}{7}\right)^{-11} = \left(\dfrac{7}{8}\right)^{x}$: 2

7. Find three irrational numbers between $\sqrt{3}$ and $\sqrt{8}$. 2

8. The HCF of two polynomials is (x–2) and their LCM is $x^4 + 2x^3 - 8x - 16$. If one of the polynomials is $x^3 - 8$, find the other polynomial. 2

9. The sum of a number and its reciprocal is $\dfrac{50}{7}$, find the number. 2

10. The length of a rectangle is 5 cm less than twice its breadth. If the perimeter is 110 cm, find the area of the rectangle. 2

11. Show that the sum of an AP whose first term is a, the second term is b and the last term is c, is equal to $\dfrac{(a+c)(b+c-2a)}{2(b-a)}$. 4

12. Had Ajay scored 10 more marks in his test out of 30 marks, 9 times these marks would have been the square of his actual marks. How many marks did he get in the test? 6

MODULE 2

Commercial Mathematics

It is a common saying by elders keep your expenditure, less than your income. The latent meaning of this is to save something for difficult times. You must have seen birds and animals saving eatables for rainy season, in their nests or caves. Taking the lead from this, the students have been told about the importance and need of savings in this module

Many Indian mathematicians have worked on the topic of commercial Mathematics. Yodoksu (370 B.C.) worked on fractions and ratio and proportion. In the reigns of Ashoka and Chandragupta, there is a description of levying taxes. There is a description of many mathematicians working on practice and proportion (like Aryabhatt, Mahavira, Brahmgupta, Sridharacharya). In 900 A.D., Bakhshali Manuscript was discovered which had a number of problems on Commercial mathematics.

To keep your savings safe is another tough task. Banks and other financial institutions keep the money of their customers and on the expiry of the period pay extra money, called interest, in addition to the money deposited. This encourages citizens to save and keep the money safe. This is why calculation of interest on deposits in banks is included for teaching.

The Government provides a number of facilities to the citizens. For that they levy certain taxes on citizens. One of these taxes is sales tax to which the learners are introduced in this "module. Financial transactions about buying and selling are generally done for profit. Due to greater supply of goods or sub-standard goods they are to be sold on loss. The learners are, therefore, introduced to percentage and profit and loss. Sometimes we have to buy articles on instalments because of non-availability of adequate funds. Due to this the students are taught to calculate interest when they buy articles on instalment plan. Sometimes when we are not able to return loaned money on time, the financer starts charging interest on interest also, which is called compound interest. Due to this the study of compound interest has been included in this module. The formulae of compound interest is also used in finding increase or decrease in prices of things. This is also taught under "Appreciation and Depreciation" of value.

8

PERCENTAGE AND ITS APPLICATIONS

You must have seen advertisements in newspapers, television and hoardings etc of the following type:

“Sale, up to 60% off”.

“Voters turnout in the poll was over 70%”.

“Ramesh got 93% aggregate in class XII examination”.

“Banks have lowered the rate of interest on fixed deposits from 8.5% to 7%”.

In all the above statements, the important word is ‘percent’. The word ‘percent’ has been derived from the Latin word ‘percentum’ meaning per hundred or out of hundred.

In this lesson, we shall study percent as a fraction or a decimal and shall also study its applications in solving problems of profit and loss, discount, simple interest, compound interest, rate of growth and depreciation etc.

OBJECTIVES

After studying this lesson, you will be able to

- *illustrate the concept of percentage;*
- *calculate specified percent of a given number or a quantity;*
- *solve problems based on percentage;*
- *solve problems based on profit and loss;*
- *calculate the discount and the selling price of an article, given marked price of the article and the rate of discount;*
- *solve inverse problems pertaining to discount;*
- *calculate simple interest and the amount, when a given sum of money is invested for a specified time period on a given rate of interest;*

- *illustrate the concept of compound interest vis-a-vis simple interest;*
- *calculate compound interest, the amount and the difference between compound and simple interest on a given sum of money at a given rate and for a given time period; and*
- *solve real life problems pertaining to rate of growth and decay, using the formula of compound interest, given a uniform or variable rate.*

EXPECTED BACKGROUND KNOWLEDGE

- Four fundamental operations on whole numbers, fractions and decimals.
- Comparison of two fractions.

8.1 PERCENT

Recall that a fraction $\frac{3}{4}$ means 3 out of 4 equal parts. $\frac{7}{13}$ means 7 out of 13 equal parts and $\frac{23}{100}$ means 23 out of 100 equal parts.

A fraction whose denominator is 100 is read as percent, for example $\frac{23}{100}$ is read as twenty three percent.

The symbol '%' is used for the term percent.

A ratio whose second term is 100 is also called a percent,

So, 33 : 100 is equivalent to 33%.

Recall that while comparing two fractions, $\frac{3}{5}$ and $\frac{1}{2}$, we first convert them to equivalent fractions with common denominator (L.C.M. of the denominators).

thus $\frac{3}{5}=\frac{3}{5}\times\frac{2}{2}=\frac{6}{10}$, and

$$\frac{1}{2}=\frac{1}{2}\times\frac{5}{5}=\frac{5}{10}$$

Now, because $\frac{6}{10}>\frac{5}{10}$ $\therefore \frac{3}{5}>\frac{1}{2}$

We could have changed these fractions with common denominator 100 as

$$\frac{3}{5}=\frac{3}{5}\times\frac{20}{20}=\frac{60}{100} \quad \text{or} \quad 60\%$$

$$\frac{1}{2}=\frac{1}{2}\times\frac{50}{50}=\frac{50}{100} \quad \text{or} \quad 50\%$$

and so, $\frac{3}{5}>\frac{1}{2}$ as 60% is greater than 50%.

8.2 CONVERSION OF A FRACTION INTO PERCENT AND VICE VERSA

In the above section, we have learnt that, to convert a fraction into percent, we change the fraction into an equivalent fraction with denominator 100 and then attach the symbol % with the changed numerator of the fraction. For example,

$$\frac{3}{4}=\frac{3}{4}\times\frac{25}{25}=\frac{75}{100}=75\times\frac{1}{100}=75\% \text{ and}$$

$$\frac{4}{25}=\frac{4}{25}\times\frac{4}{4}=\frac{16}{100}=16\times\frac{1}{100}=16\%$$

Note: To write a fraction as percent, we may multiply the fraction by 100, simplify it and attach % symbol. For example,

$$\frac{4}{25}=\frac{4}{25}\times 100\%=16\%$$

Conversely,

To write a percent as a fraction, we drop the % sign, multiply the number by $\frac{1}{100}$ (or divide the number by 100) and simplify it. For example,

$$47\%=47\times\frac{1}{100}=\frac{47}{100}, \qquad 17\%=17\times\frac{1}{100}=\frac{17}{100}, \qquad 3\%=\frac{3}{100}$$

$$45\%=45\times\frac{1}{100}=\frac{45}{100}=\frac{9}{20}, \quad 210\%=\frac{210}{100}=\frac{21}{10}, \quad x\%=\frac{x}{100}.$$

8.3 CONVERSION OF DECIMAL INTO A PERCENT AND VICE VERSA

Let us consider the following examples:

$$0.35 = \frac{35}{100} = 35 \times \frac{1}{100} = 35\%$$

$$4.7 = \frac{47}{10} = \frac{470}{100} = 470 \times \frac{1}{100} = 470\%$$

$$0.459 = \frac{459}{1000} = \frac{459}{10} \times \frac{1}{100} = 45.9\%$$

$$0.0063 = \frac{63}{10000} = \frac{63}{100} \times \frac{1}{100} = 0.63\%$$

Thus, **to write a decimal as a percent, we move the decimal point two places to the right and put the % sign**

Conversely,

To write a percent as a decimal, we drop the %sign and insert or move the decimal point two places to the left. For example,

$43\% = 0.43$ $75\% = 0.75$ $12\% = 0.12$

$9\% = 0.09$ $115\% = 1.15$ $327\% = 3.27$

$0.75\% = 0.0075$ $4.5\% = 0.045$ $0.2\% = 0.002$

Let us take a few more examples:

Example 8.1: Shweta obtained 18 marks in a test of 25 marks. What was her percentage of marks?

Solution: Total marks = 25

Marks obtained = 18

$$\therefore \text{ Fraction of marks obtained} = \frac{18}{25}$$

$$\therefore \text{ Marks obtained in percent} = \frac{18}{25} \times \frac{4}{4} = \frac{72}{100} = 72\%$$

Alternatively:

$$\text{Marks obtained in percent} = \frac{18}{25} \times 100\% = 72\%$$

Example 8.2: One-fourth of the total number of shoes in a shop were on discount sale. What percent of the shoes were there on normal price?

Solution: Fraction of the total number of shoes on sale = $\frac{1}{4}$

$\therefore$ Fraction of the total number of shoes on normal price = $1-\frac{1}{4}=\frac{3}{4}$

$$=\frac{3}{4}\times\frac{25}{25}=\frac{75}{100}=75\% \text{ or } \frac{3}{4}\times 100\% = 75\%$$

Example 8.3: Out of 40 students in a class, 32 opted to go for a picnic. What percent of students opted for picnic?

Solution: Total number of students in a class = 40

Number of students, who opted for picnic = 32

$\therefore$ Number of students, in percent, who opted for picnic

$$=\frac{32}{40}\times 100\% = 80\%$$

Example 8.4: In the word ARITHMETIC, what percent of the letters are I's?

Solution: Total number of letters = 10

Number of I's = 2

$\therefore$ Percent of I's = $\frac{2}{10}\times 100\% = 20\%$

Example 8.5: A mixture of 80 litres, of acid and water, contains 20 litres of acid. What percent of water is in the mixture?

Solution: Total volume of the mixture = 80 litres

Volume of acid = 20 litres

$\therefore$ Volume of water = 60 litres

$\therefore$ Percentage of water in the mixture = $\frac{60}{80}\times 100\% = 75\%$

CHECK YOUR PROGRESS 8.1

1. Convert each of the following fractions into a percent:

(a) $\frac{12}{25}$ (b) $\frac{9}{20}$ (c) $\frac{5}{12}$ (d) $\frac{6}{15}$ (e) $\frac{125}{625}$

(f) $\frac{3}{10}$ (g) $\frac{108}{300}$ (h) $\frac{189}{150}$ (i) $\frac{72}{25}$ (j) $\frac{1231}{1250}$

2. Write each of the following percents as a fraction:

(a) 53% (b) 85% (c) $16\frac{7}{8}\%$ (d) 3.425% (e) 6.25%

(f) 70% (g) $15\frac{3}{4}\%$ (h) 0.0025% (i) 47.35% (j) 0.525%

3. Write each of the following decimals as a percent:

(a) 0.97 (b) 0.735 (c) 0.03 (d) 2.07 (e) 0.8

(f) 1.75 (g) 0.0250 (h) 3.2575 (i) 0.152 (j) 3.0015

4. Write each of the following percents as a decimal:

(a) 72% (b) 41% (c) 4% (d) 125% (e) 9%

(f) 410% (g) 350% (h) 102.5% (i) 0.025% (j) 10.25%

5. Gurpreet got half the answers correct, in an examination. What percent of her answers were correct?

6. Prakhar obtained 18 marks in a test of total 20 marks. What was his percentage of marks?

7. Harish saves ₹ 900 out of a total monthly salary of ₹ 14400. Find his percentage of saving.

8. A candidate got 47500 votes in an election and was defeated by his opponent by a margin of 5000 votes. If there were only two candidates and no votes were declared invalid, find the percentage of votes obtained by the winning candidate.

9. In the word PERCENTAGE, what percent of the letters are E's?

10. In a class of 40 students, 10 secured first division, 15 secured second division and 13 just qualified. What percent of students failed.

8.4 CALCULATION OF PERCENT OF A QUANTITY OR A NUMBER

To determine a specified percent of a number or quantity, we first change the percent to a fraction or a decimal and then multiply it with the number or the quantity. For example:

$$25\% \text{ of } 90 = \frac{25}{100} \times 90 = 22.50$$

or $$25\% \text{ of } 90 = 0.25 \times 90 = 22.50$$

$$60\% \text{ of Rs. } 120 = 0.60 \times \text{Rs. } 120 = \text{Rs. } 72.00$$

$$120\% \text{ of } 80 \text{ kg} = 1.20 \times 80 \text{ kg} = 96 \text{ kg}$$

Let us take some examples from daily life:

Example 8.6: In an examination, Neetu scored 62% marks. If the total marks in the examination are 600, then what are the marks obtained by Neetu?

Solution: Here we have to find 62% of 600

$\therefore$ 62% of 600 marks = 0.62 × 600 marks = 372 marks

$\therefore$ Marks obtained by Neetu = 372

Example 8.7: Naresh earns ₹ 30800 per month. He keeps 50% for household expenses, 15% for his personal expenses, 20% for expenditure on his children and the rest he saves. What amount does he save per month?

Solution: Expenditure on Household = 50%

Expenditure on self = 15%

Expenditure on children = 20%

Total expenditure = (50 + 15 + 20)% = 85%

$\therefore$ Savings (100 – 85)% = 15%

$\therefore$ 15% of ₹ 30800 = ₹ (0.15 × 30800)

= ₹ 4620

Example 8.8: What percent of 360 is 144?

Solution: Let x% of 360 = 144

$$\therefore \frac{x}{100} \times 360 = 144$$

Or $$x = \frac{144}{360} \times 100 = 40\%$$

Alternatively, 144 out of 360 is equal to the fraction $\frac{144}{360}$

$$\therefore \text{Percent} = \frac{144}{360} \times 100\% = 40\%$$

Example 8.9: If 120 is reduced to 96, what is the reduction percent?

Solution: Here, reduction = 120 – 96 = 24

$$\therefore \text{Reduction percent} = \frac{24}{120} \times 100\% = 20\%$$

Example 8.10: The cost of an article has increased from ₹ 450 to ₹ 495. By what percent did the cost increased?

Solution: The increase in Cost Price = ₹ (495 – 450)

= ₹ 45

$$\text{Increase percent} = \frac{45}{450} \times 100 = 10\%$$

Example 8.11: 60% of the students in a school are girls. If the total number of girls in the school is 690, find the total number of students in the school. Also, find the number of boys in the school.

Solution: Let the total number of students in the school be x

Then, 60% of $x = 690$

$$\therefore \frac{60}{100} \times x = 690 \text{ or } x = \frac{690 \times 100}{60} = 1150$$

$\therefore$ Total number of students in the school = 1150

$\therefore$ Hence number of boys = 1150 – 690 = 460

Example 8.12: A's income is 25% more than that of B. B's income is 8% more than that of C. If A's income is ₹ 20250, then find the income of C.

Solution: Let income of C be ₹ x

Income of B $= x + 8\%$ of x

$$= x + \frac{8x}{100} = \frac{108}{100} \times x$$

$$\text{Income of A} = \frac{108x}{100} + 25\% \text{ of } \frac{108x}{100}$$

$$= \frac{108x}{100} \times \frac{125}{100}$$

$$\therefore \frac{108}{100} \times x \times \frac{125}{100} = 20250$$

$$\text{or } x = 20250 \times \frac{100}{108} \times \frac{100}{125} = 15000$$

$\therefore$ Income of C is ₹ 15000.

Example 8.13: A reduction of 10% in the price of tea enables a dealer to purchase 25 kg more tea for ₹ 22500. What is the reduced price per kg of tea? Also, find the original price per kg.

Solution: 10% of ₹ 22500 = $\frac{10}{100} \times 22500$ = ₹ 2250

∴ Reduced price of 25 kg tea = ₹ 2250

∴ Reduced price per kg = ₹ $\frac{2250}{25}$ = ₹ 90 per kg.

Since, the reduction was 10% so the original price = ₹ 100 per kg.

Example 8.14: A student got 45% marks in the first paper and 70% in the second paper. How much percent should he get in the third paper so as to get 60% as overall score?

Solution: Let each paper be of 100 marks.

∴ Marks obtained in first paper = 45% of 100 = 45

Marks obtained in second paper = 70% of 100 = 70

Total marks (in three papers) he wants to obtain = 60% of 300

$$= \frac{60}{100} \times 300 = 180$$

∴ Marks to be obtained in third paper = 180 – (45 + 70)

= 180 – 115 = 65

Example 8.15: Find the sum which when increased by 15% becomes ₹ 19320.

Solution: Let the sum be ₹ x

$\therefore\ x + 15\%$ of $x = 19320$

$$x + \frac{15x}{100} = 19320 \quad \text{or} \quad \frac{115x}{100} = 19320$$

$$\therefore x = \frac{19320 \times 100}{115} = 16800$$

Hence, the required sum = ₹ 16800.

CHECK YOUR PROGRESS 8.2

1. Find: (i) 16% of 1250 (ii) 47% of 1200
2. A family spends 35% of its monthly budget of ₹ 7500 on food. How much does the family spend on food?

3. In a garden, there are 500 plants of which 35% are trees, 20% are shrubs, 25% are herbs and the rest are creepers. Find out the number of each type of plants.
4. 60 is reduced to 45. What percent is the reduction?
5. If 80 is increased to 125, what is the increase percent?
6. Raman has to score a minimum 40% marks for passing the examination. He gets 178 marks and fails by 22 marks. Find the maximum marks.
7. It takes me 45 minutes to go to school and I spend 80% of the time travelling by bus. How long does the bus journey last?
8. In an election, between 2 candidates 25% voters did not cast their votes. A candidate scored 40% of the votes polled and was defeated by 900 votes. Find the total number of voters.
9. A rise of 25% in the price of sugar compels a person to buy 1.5 kg of sugar less for ₹ 240. Find the increased price as well as the original price per kg of sugar.
10. A number is first increased by 20% and then decreased by 20%. What is the net increase or decrease percent?
11. 'A' scored 12 marks, while B scored 10 marks, in the first terminal examination. If in the second terminal examination (with same total number of marks) 'A' scored 14 marks and 'B' scored 12 marks, which student showed more improvement?
12. 30,000 students appeared in a contest. Of them 40% were girls and the remaining boys. If 10% boys and 12% girls won the contest with prizes, find the percentage of students who won prizes.
13. Sunil earns 10% more than Shailesh and Shailesh earns 20% more than Swami. If Swami earns ₹ 3200 less than Sunil, find the earnings of each.

8.5 APPLICATION OF PERCENTAGE

In our day to day life, we come across a number of situations wherein we use the concept of percent. In the following section, we discuss the application of percentage in different fields, like problems in profit and loss, discount, simple interest, compound interest, rate of growth and depreciation.

8.5.1 Profit and Loss

Let us recall the terms and formulae related to profit and loss.

Cost Price (C.P.): The Price at which an article is purchased, is called its cost price.

Selling Price (S.P.): The Price at which an article is sold, is called its selling price.

Profit (Gain): When S.P. > C.P., then there is profit, and

$$\text{Profit} = \text{S.P.} - \text{C.P.}$$

Loss: When C.P. > S.P., then there is loss, and

Loss = C.P. – S.P.

Formulae $$\text{Profit }\% = \left(\frac{\text{Profit}}{\text{C.P.}}\times 100\right)\%, \qquad \text{Loss}\% = \left(\frac{\text{Loss}}{\text{C.P.}}\times 100\right)\%$$

$$\text{S.P.} = \frac{(\text{C.P.})\times(100+\text{Profit}\%)}{100} = \frac{(\text{C.P.})(100-\text{Loss}\%)}{100}$$

$$\text{C.P.} = \frac{\text{S.P.}\times 100}{(100+\text{Profit}\%)} = \frac{(\text{S.P.})\times 100}{(100-\text{Loss}\%)}$$

Note: Gain % or loss % is always calculated on C.P.

Let us take some examples to illustrate the applications of these formulae in solving problems related to profit and loss:

Example 8.16: A shopkeeper buys an article for Rs. 360 and sells it for Rs. 270. Find his gain or loss percent.

Solution: Here C.P. = Rs. 360, and S.P. = Rs. 270

Since C.P. > S.P., $\therefore$ there is a loss.

Loss = C.P. – S.P. = Rs (360 – 270) = Rs. 90

$$\text{Loss }\% = \left(\frac{\text{Loss}}{\text{C.P.}}\times 100\right)\%$$

$$= \frac{90}{360}\times 100 = 25\%$$

Example 8.17: Sudha purchased a house for ₹ 4,52,000 and spent ₹ 28,000 on its repairs. She had to sell it for ₹ 4,92,000. Find her gain or loss percent.

Solution: Here C.P. = Cost price + Overhead charges

= ₹ (452000 + 28000) = ₹ 4,80,000

S.P. = ₹ 4,92,000

Since, S.P. > C.P., $\therefore$ Gain = ₹ (492000 – 480000) = ₹ 12000

$$\text{Gain }\% = \frac{12000\times 100}{480000} = \frac{5}{2}\% = 2.5\%$$

Example 8.18: By selling a book for ₹ 258, a publisher gains 20%. For how much should he sell it to gain 30%?

Solution: S.P. = Rs. 258

Profit = 20%

$$\text{C.P.} = \frac{\text{S.P.}\times 100}{100+\text{Profit}\%} = ₹\ \frac{258\times 100}{120} = ₹\ 215$$

Now, if Profit = 30% and C.P. = Rs. 215, then,

$$\text{S.P.} = \frac{\text{C.P.}\times(100+\text{Profit}\%)}{100} = ₹\ \frac{215\times 130}{100} = ₹\ 279.50$$

Example 8.19: A man bought oranges at 25 for ₹ 100 and sold them at 20 for ₹ 100. Find his gain or loss percent.

Solution: C.P. of 25 oranges = ₹ 100

$$\therefore\ \text{C.P. of 1 orange} = ₹\ \frac{100}{25} = ₹\ 4$$

$$\text{and S.P. of 1 orange} = ₹\ \frac{100}{20} = ₹\ 5$$

$\therefore$ Profit on 1 orange = ₹ (5 – 4) = ₹ 1

$$\text{Profit }\% = \frac{1}{4}\times 100 = 25\%$$

Example 8.20: A man sold two horses for ₹ 29700 each. On one he lost 10% while he gained 10% on the other. Find his total gain or loss percent in the transaction.

Solution: S.P. of first horse = ₹ 29700

Loss = 10%

$$\therefore\ \text{C.P.} = ₹\ \frac{29700\times 100}{90} = ₹\ 33,000$$

S.P. of 2nd horse = ₹ 29700,

Profit = 10%

$$\text{C.P.} = ₹\ \frac{29700\times 100}{110} = ₹\ 27,000$$

Total CP = ₹ (33000 + 27000) = ₹ 60,000

Total SP = ₹ (2 × 29700) = ₹ 59400

Net Loss = ₹ (60000 – 59400) = ₹ 600

$$\text{Loss \%} = \frac{600}{60000} \times 100 = 1\%$$

Example 8.21: The cost price of 15 articles is equal to the selling price of 12 articles. Find the gain percent.

Solution: Let the C.P. of 15 articles be ₹ 15

then S.P. of 12 articles = ₹ 15

$$\text{S.P. of 15 articles} = ₹\ \frac{15}{12} \times 15 = ₹\ \frac{75}{4}$$

$$\text{Gain} = ₹\left(\frac{75}{4} - 15\right) = ₹\ \frac{15}{4}$$

$$\text{Gain \%} = \frac{15/4}{15} \times 100 = 25\%$$

Example 8.22: A watch was sold at a profit of 12%. Had it been sold for ₹ 33 more, the profit would have been 14%. Find the cost price of the watch.

Solution: Let the cost price of the watch be ₹ x

$$\therefore \text{S.P.} = \frac{x \times 112}{100} = \frac{112x}{100}$$

If the watch is sold for Rs. 33 more then S.P. $= \left(\frac{112x}{100} + 33\right)$

New profit = 14%

$$\therefore \text{C.P.} = x = \frac{\left(\frac{112x}{100} + 33\right) \times 100}{114}$$

or $114x = 112\,x + 3300$ or $2x = 3300$

$x = 1650 \quad \therefore$ C.P. = ₹ 1650

CHECK YOUR PROGRESS 8.3

1. A shopkeeper bought an almirah from a wholesale dealer for ₹ 4500 and sold it for ₹ 6000. Find his profit or loss percent.

2. A retailer buys a cooler for ₹ 3800 but had to spend ₹ 200 on its transport and repair. If he sells the cooler for ₹ 4400, determine, his profit percent.
3. A vendor buys lemons at the rate of 5 for ₹ 7 and sells them at ₹ 1.75 per lemon. Find his gain percent.
4. A man purchased a certain number of oranges at the rate of 2 for ₹ 5 and sold them at the rate of 3 for ₹ 8. In the process, he gained ₹ 20. Find the number of oranges he bought.
5. By selling a bi-cycle for ₹ 2024, the shopkeeper loses 12%. If he wishes to make a gain of 12% what should be the selling price of the bi-cycle?
6. By selling 45 oranges for ₹ 160, a woman loses 20%. How many oranges should she sell for ₹ 112 to gain 20% on the whole?
7. A dealer sold two machines at ₹ 2400 each. On selling one machine, he gained 20% and on selling the other, he lost 20%. Find the dealer's net gain or loss percent.
8. Harish bought a table for ₹ 960 and sold it to Raman at a profit of 5%. Raman sold it to Mukul at a profit of 10%. Find the money paid by Mukul for the table.
9. A man buys bananas at 6 for ₹ 5 and an equal number at ₹ 15 per dozen. He mixes the two lots and sells them at ₹ 14 per dozen. Find his gain or loss percent, in the transaction.
10. If the selling price of 20 articles is equal to the cost price of 23 articles, find the loss or gain percent.

8.5.2 Discount

You must have seen advertisements of the following types, especially during the festival season.

SALE	}	DIWALI BONANZA
discount upto 50%		20% discount on all items.

A discount is a reduction in the marked (or list) price of an article. "20% discount" means a reduction of 20% in the marked price of an article. For example, if the marked price of an article is ₹ 100, it is sold for ₹ 80, i.e. ₹ 20 less than the marked price. Let us define the terms, we shall use:

Marked Price (or List price): The marked price (M.P.) of an article is the price at which the article is listed for sale. Since this price is written (marked) on the article, so it is called the marked price.

Discount: The discount is the reduction from the marked price of the article.

Net selling price (S.P.): In case of discount selling, the price of the article obtained by subtracting discount from the marked price is called the Net Selling price or Selling price (S.P.). Let us take the following examples, to illustrate:

Example 8.23: A coat is marked at ₹ 2400. Find its selling price if a discount of 12% is offered.

Solution: Here, Marked Price (M.P.) of the coat = ₹ 2400

Discount = 12%

Net selling price (S.P.) = M.P. – Discount

= ₹ 2400 – 12% of ₹ 2400

$$= ₹\ 2400 - ₹\left(\frac{12}{100}\times 2400\right)$$

= ₹ (2400 – 288)

= ₹ 2112

Thus, the net selling price of coat is ₹ 2112.

Example 8.24: A machine listed at ₹ 8400 is available for ₹ 6300. Find the rate of discount offered.

Solution: Here, Marked Price (M.P.) = ₹ 8400

Net selling price (S.P.) = ₹ 6300

Discount offered = ₹ (8400 – 6300)

= ₹ 2100

$$\text{Discount }\% = \frac{2100}{8400}\times 100\% = 25\%$$

Note: Discount is always calculated on Marked Price.

Example 8.25: A wholesaler's list price of a fan is ₹ 1250 and is available to a retailer at a discount of 20%. For how much should the retailer sell it, to earn a profit of 15%.

Solution: M.P. = ₹ 1250

Discount = 20% of ₹ 1250

$$= ₹\ \frac{20}{100}\times 1250 = ₹\ 250$$

∴ Cost Price of the retailer = ₹ (1250 – 250)

= ₹ 1000

Profit = 15%

$$\therefore\ \text{S.P.} = \frac{\text{C.P.}(100+\text{Profit}\%)}{100} = ₹\ \frac{1000\times 115}{100}$$

= ₹ 1150

Example 8.26: A shopkeeper marks his goods 25% more than their cost price and allows a discount of 10%. Find his gain or loss percent.

Solution: Let the C.P. of an article = ₹ 100

∴ Marked Price (M.P.) = ₹ 100 + 25% of ₹ 100

= ₹ 125

Discount offered = 10%

∴ Net selling Price = ₹ 125 – 10% of ₹ 125

$$= ₹\ 125 - ₹ \left(\frac{10}{100} \times 125\right)$$

= ₹ (125 – 12.50) = ₹ 112.50

∴ Gain = ₹ (112.50 – 100) = ₹ 12.50

$$\text{Gain \%} = \frac{12.50}{100} \times 100 = 12.5\%$$

Example 8.27: An article listed at ₹ 5400 is offered at a discount of 15%. Due to festival season, the shopkeeper allows a further discount of 5%. Find the selling price of the article.

Solution: M.P. = ₹ 5400, Discount = 15%

∴ SP = ₹ 5400 – 15% of ₹ 5400

$$= ₹\ 5400 - ₹\ \frac{15}{100} \times 5400$$

= ₹ (5400 – 810) = ₹ 4590

Festival discount = 5%

∴ Net selling Price = ₹ 4590 – 5% of ₹ 4590

$$= ₹\ 4590 - ₹\ \frac{5}{100} \times 4590$$

= ₹ (4590 – 229.50)

= ₹ 4360.50

∴ Net selling price of article = ₹ 4360.50.

Example 8.28: A retailer buys books from a wholesaler at the rate of ₹ 300 per book and marked them at ₹ 400 each. He allows some discount and gets a profit of 30% on the cost price. What percent discount does he allow to his customers?

Solution: C.P. of one book = ₹ 300

M.P. = ₹ 400

Profit = 30%

$$\therefore \text{ S.P.} = \frac{\text{C.P.}(100+\text{Profit}\%)}{100} = ₹\ \frac{300 \times 130}{100} = ₹\ 390$$

$$\therefore \text{ Discount offered} = ₹\ (400 - 390) = ₹\ 10$$

$$\text{Discount } \% = \frac{10}{400} \times 100 = 2.5\%$$

CHECK YOUR PROGRESS 8.4

1. A shirt with marked price ₹ 375/- is sold at a discount of 15%. Find its net selling price.
2. A pair of socks marked at ₹ 60 is being offered for ₹ 48. Find the discount percent being offered.
3. A washing machine is sold at a discount of 10% on its marked price. A further discount of 5% is offered for cash payment. Find the selling price of the washing machine if its marked price is ₹ 18000.
4. A man pays ₹ 2100 for a machine listed at ₹ 2800. Find the rate of discount offered.
5. The list price of a table fan is ₹ 840 and it is available to a retailer at a discount of 25%. For how much should the retailer sell it to earn a profit of 15%.
6. A shopkeeper marks his goods 50% more than their cost price and allows a discount of 40%, find his gain or loss percent.
7. A dealer buys a table listed at ₹ 2500 and gets a discount of 28%. He spends ₹ 100 on transportation and sells it at a profit of 15%. Find the selling price of the table.
8. A retailer buys shirts from a manufacturer at the rate of ₹ 175 per shirt and marked them at ₹ 250 each. He allows some discount and earns a profit of 28% on the cost price. What percent discount does he allow to his customers?

8.5.3 Simple Interest

When a person has to borrow some money as a loan from his friends. relatives, bank etc. he promises to return it after a specified time period along with some extra money for using the money of the lender.

The money borrowed is called the **Principal**, usually denoted by P, and the extra money paid is called the **Interest**, usually denoted by I.

The total money paid back, that is, the sum of Principal and the Interest is called the **Amount**, and is usually denoted by A.

Thus, $A = P + I$

The interest is mostly expresed as a rate percent per year (per annum).

Interest depends on, how much money (P) has been borrowed and the duraton of time (T) for which it is used. Interest is calculated according to a mutually agreed rate percent, per annum (R). [i.e. $R = r\,\% = \frac{r}{100}$]

Thus, Interest = (Principal) × (Rate % per annum) × time

$$\therefore\ I = P \times R \times T$$

Interest calculated as above, is called simple interest. Let us take some examples, involving simple interest.

Example 8.29: Find the simple interest in each of the following cases

	P	R	T
(a)	₹ 8000	5%	2 yrs
(b)	₹ 20,000	15%	$1\frac{1}{2}$ yrs

Solution: (a) $I = P.\ R.\ T.$

$$= \text{₹}\left[8000 \times \frac{5}{100} \times 2\right] = \text{₹}\ 800$$

(b) $I = \text{₹}\left[20000 \times \frac{15}{100} \times \frac{3}{2}\right] = \text{₹}\ 4500$

Example 8.30: Find at what rate of simple interest per annum will ₹ 5000 amount to ₹ 6050 in 3 years.

Solution: Here A = ₹ 6050, P = ₹ 5000, T = 3 yrs

$$\therefore\ I = \text{₹}\ (6050 - 5000) = \text{₹}\ 1050$$

$I = P \times R \times T$ or $r\% = \frac{I}{P \times T}$ $\therefore\ r = \frac{I \times 100}{P \times T}$

$r = \frac{1050 \times 100}{5000 \times 3} = 7$ $\therefore\ R = 7\%$

Example 8.31: A sum amounts to ₹ 4875 at $12\frac{1}{2}$ % simple interest per annum after 4 years. Find the sum.

Solution: Here A = ₹ 4875, $R = 12\frac{1}{2}\% = \frac{25}{2}\%$, T = 4 yrs

$$I = P \times R \times T$$

$$I = ₹\left(P \times \frac{25}{200} \times 4\right) = ₹\ \frac{P}{2}$$

$$\therefore\ A = ₹\left(P + \frac{P}{2}\right) = ₹\ \frac{3P}{2}$$

Thus, $\frac{3P}{2}$ = ₹ 4875 or 3P = ₹ 9750 or P = ₹ 3250

Example 8.32: In how many years will a sum of ₹ 2000 yield an interest (Simple) of ₹ 560 at the rate of 14% per annum?

Solution: Here P = ₹ 2000, I = ₹ 560 R = 14%

$$I = P \times R \times T \text{ or } 560 = 2000 \times \frac{14}{100} \times T$$

$$\therefore T = \frac{560 \times 100}{2000 \times 14} = 2 \text{ years}$$

Thus, in 2 years, a sum of ₹ 2000 will yield an interest of ₹ 560 at 14% per annum.

Example 8.33: A certain sum of money at simple interest amounts to ₹ 1300 in 4 years and to ₹ 1525 in 7 years. Find the sum and rate percent.

Solution: Here $1300 = \frac{P \times R \times 4}{100} + P$ ...(i)

and $1525 = \frac{P \times R \times 7}{100} + P$ ...(ii)

Subtracting (i) from (ii) $225 = \frac{P \times R \times 3}{100}$ or $\frac{P \times R}{100} = 75$

Putting in (i) we get

$$1300 = 75 \times 4 + P \text{ or } P = ₹\ (1300 - 300) = ₹\ 1000$$

Again, we have $\frac{P \times R}{100} = 75$ or $R = \frac{75 \times 100}{P} = \frac{75 \times 100}{1000} = 7.5\%$

$\therefore$ Principal = ₹ 1000 and rate = 7.5%

Alternatively:

Amount after 4 years = ₹ 1300

Amount after 7 years = ₹ 1525

$\therefore$ Interest for 3 years = ₹ [1525 – 1300] = ₹ 225

$\therefore$ Interest for 1 year = ₹ $\frac{225}{3}$ = ₹ 75

$\therefore$ 1300 = P + Interest for 4 yrs = P + 4 × 75 or P = ₹ (1300 – 300) = ₹ 1000

$$R = \frac{75 \times 100}{1000 \times 1} = 7.5\%$$

Example 8.34: A certain sum of money doubles itself in 10 years. In how many years will it become $2\frac{1}{2}$ times at the same rate of simple interest.

Solution: Let P = ₹ 100, T = 10 yrs, A = ₹ 200, $\therefore$ I = ₹ 100

$$\therefore 100 = \frac{100 \times R \times 10}{100} \text{ or } R = 10\%$$

Now P = ₹ 100, R = 10% and A = ₹ 250 $\therefore$ I = ₹ 150

$$\therefore 150 = 100 \times \frac{10}{100} \times T \text{ or } T = 15 \text{ yrs}$$

Thus, in 15 yrs, the sum will become $2\frac{1}{2}$ times.

Example 8.35: Out of ₹ 70,000 to invest for one year, a man invests ₹ 30,000 at 4% and ₹ 20,000 at 3% per annum simple interest. At what rate percent, should he lend the remaining money, so that he gets 5% interest on the total amount he has?

Solution: Interest on total amount at 5% for one year

$$= ₹\ 70{,}000 \times \frac{5}{100} \times 1 = ₹\ 3500$$

Interest on ₹ 30,000 at 4% for 1 year $= ₹\ 30000 \times \frac{4}{100} \times 1$

$= ₹\ 1200$

$$\text{Interest on ₹ 20,000 at 3\% for 1 year} = ₹\ 20000 \times \frac{3}{100} \times 1$$

$$= ₹\ 600$$

$$\therefore \text{ Interest on remaining ₹ 20,000 for 1 yr} = ₹\ [3500 - 1200 - 600]$$

$$= ₹\ 1700$$

$$\therefore 1700 = 20000 \times \frac{R}{100} \times 1 \text{ or } R = \frac{1700 \times 100}{20000} = 8.5\%$$

∴ The remaining amount should be invested at 8.5% per annum.

CHECK YOUR PROGRESS 8.5

1. Rama borrowed ₹ 14000 from her friend at 8% per annum simple interest. She returned the money after 2 years. How much did she pay back altogether?
2. Ramesh deposited ₹ 15600 in a financial company, which pays simple interest at 8% per annum. Find the interest he will receive at the end of 3 years.
3. Naveen lent ₹ 25000 to his two friends. He gave ₹ 10,000 at 10% per annum to one of his friend and the remaining to other at 12% per annum. How much interest did he receive after 2 years.
4. Shalini deposited ₹ 29000 in a finance company for 3 years and received ₹ 38570 in all. What was the rate of simple interest per annum?
5. In how much time will simple interest on a sum of money be $\frac{2}{5}$ th of the sum, at the rate of 10% per annum.
6. At what rate of interest will simple interest be half the principal in 5 years.
7. A sum of money amounts to ₹ 1265 in 3 years and to ₹ 1430 in 6 years, at simple interest. Find the sum and the rate percent.
8. Out of ₹ 75000 to invest for one year, a man invested ₹ 30000 at 5% per annum and ₹ 24000 at 4% per annum. At what percent per annum, should he invest the remaining money to get 6% interest on the whole money.
9. A certain sum of money doubles itself in 8 years. In how much time will it become 4 times of itself at the same rate of interest?
10. In which case, is the interest earned more:
    (a) ₹ 5000 deposited for 5 years at 4% per annum, or
    (b) ₹ 4000 deposited for 6 years at 5% per annum?

8.5.4 Compound Interest

In the previous section, you have studied about simple interest. When the interest is calculated on the Principal for the entire period of loan, the interest is called simple interest and is given by

$$I = P \times R \times T$$

But if this interest is due (not paid) after the decided time period, then it becomes a part of the principal and so is added to the principal for the next time preiod, and the interest is calculated for the next time period on this new principal. Interest calculated, this way is called compound interest.

The time period after which the interest is added to the principal for the next time period is called the **Conversion Period.**

The conversion period may be one year, six months or three months and the interest is said to compounded, annually, semi-annually or quarterly, respectively. Let us take an example:

Example 8.36: Find the compound interest on a sum of Rs. 2000, for two years when the interest is compounded annually at 10% per annum.

Solution: Here P = ₹ 2000 and R = 10%

∴ Interest for the first conversion time period (i.e. first year)

$$= ₹\ 2000 \times \frac{10}{100} \times 1 = ₹\ 200$$

∴ Principal for the second year (or 2nd conversion period)

$$= ₹\ (2000 + 200) = ₹\ 2200$$

∴ Interest for the 2nd time period $= ₹\ 2200 \times \frac{10}{100} \times 1 = ₹\ 220$

∴ Amount payable at the end of two years = ₹ (2200 + 220)

$$= ₹\ 2420$$

∴ Total interest paid at the end of two years = ₹ (2420 – 2000)

$$= ₹\ 420$$

or [₹ (200 + 220) = ₹ 420]

∴ Compound interest = ₹ 420

Thus, for calculating the compound interest, the interest due after every coversion period is added to the principal and then interest is calculated for the next period.

8.5.4.1 Formula for Compound Interest

Let a sum P be borrowed for n years at the rate of r% per annum, then

$$\text{Interest for the first year} = P \times \frac{r}{100} \times 1 = \frac{Pr}{100}$$

$$\text{Amount after one year} = \text{Principal for 2nd year} = P + \frac{Pr}{100}$$

$$= P\left(1 + \frac{r}{100}\right)$$

$$\text{Interest for 2nd year} = P\left(1 + \frac{r}{100}\right) \times \frac{r}{100} \times 1 = \frac{Pr}{100} \cdot \left(1 + \frac{r}{100}\right)$$

$$\text{Amount after 2 years} = P\left(1 + \frac{r}{100}\right) + \frac{Pr}{100}\left(1 + \frac{r}{100}\right) = P\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right)$$

$$= P\left(1 + \frac{r}{100}\right)^2$$

$$\text{Similarly, amount after 3 years} = P\left(1 + \frac{r}{100}\right)^3 \text{ and so on.}$$

$$\text{Amount after n years} = P\left(1 + \frac{r}{100}\right)^n$$

Thus, if A represents the amount and R represents r% or $\frac{r}{100}$, then

$$A = P(1 + R)^n = P\left(1 + \frac{r}{100}\right)^n$$

$$\text{and compound interest} = A - P = P(1 + R)^n - P$$

$$= P[(1 + R)^n - 1] \text{ or } P\left[\left(1 + \frac{r}{100}\right)^n - 1\right]$$

Note: Simple interest and compound interest are equal for first year (first conversion period)

Example 8.37: Calculate the compound interest on ₹ 20,000 for 3 years at 5% per annum, when the interest is compounded annually.

Solution: Here P = ₹ 20,000, R = 5% and n =3

$$\therefore \text{CI} = P[(1 + R)^n - 1]$$

$$= ₹\ 20000\left[\left(1 + \frac{5}{100}\right)^3 - 1\right]$$

$$= ₹\ \left[\left(\frac{21}{20}\right)^3 - 1\right] = ₹\ 20000 \times \left[\frac{9261 - 8000}{8000}\right]$$

$$= ₹\ 3152.50$$

Example 8.38: Calculate the compound interest on ₹ 20,000 for $1\frac{1}{2}$ years at the rate of 10% per annum, when the interest is compounded semi-annually.

Solution: Here P = ₹ 20,000, R = 10% per annum

= 5% per half year

and $n = 1\frac{1}{2}$ yrs = 3 half years

$$\therefore \text{CI} = P[(1+R)^n - 1] = ₹\ 20{,}000\left[\left(1+\frac{5}{100}\right)^3 - 1\right]$$

$$= ₹\ 20{,}000 \times \left[\frac{9261}{8000} - 1\right] = ₹\ 3152.50$$

Example 8.39: Calculate the compound interest on ₹ 20,000 for 9 months at the rate of 4% per annum, when the interest is compounded quarterly.

Solution: Here P = ₹ 20,000, R = 4% per annum

= 1% per quarter of year

and n = 3/4 yrs = 3 quarters

$$\therefore \text{CI} = P[(1+R)^n - 1] = ₹\ 20{,}000\left[\left(1+\frac{1}{100}\right)^3 - 1\right]$$

$$= ₹\ 20{,}000 \times \left[\left(\frac{101}{100}\right)^3 - 1\right] = ₹\ \frac{20000 \times 30301}{100 \times 100 \times 100}$$

$$= ₹\ 606.02$$

Example 8.40: calculate the amount and compound interest on ₹ 12000 for $1\frac{1}{2}$ years at the rate of 10% per annum compounded annually.

Solution: Here P = ₹ 12000, R = 10% and $n = 1\frac{1}{2}$ years

Since interest is compounded, annually, so, amount at the end of 1 year is given by

$$A = P\left(1+\frac{R}{100}\right)^1 = ₹\ 12000\times\left(1+\frac{10}{100}\right)^1$$

$$= ₹\ 12000\times\frac{11}{10} = ₹\ 13200$$

$\therefore$ Principal for next 6 months = ₹ 13200

$$\text{and Rate R} = \frac{10}{2}\% = 5\%$$

$$\therefore\ A = ₹\ 13200\left(1+\frac{5}{100}\right)^1 = ₹\ 13200\times\frac{21}{20}$$

$$= ₹\ 13860$$

$\therefore$ Amount after $1\frac{1}{2}$ years = ₹ 13860

Compound interest = ₹ [13860 – 12000]

= ₹ 1860

Note: We can calculate the amount for $1\frac{1}{2}$ yrs as

$$A = ₹\ 12000\left(1+\frac{10}{100}\right)^1\left(1+\frac{5}{100}\right)^1$$

Example 8.41: At what rate percent per annum, will a sum of ₹ 15,625 become ₹ 17576 in three years, when the interest is compounded annually?

Solution: Here A = ₹ 17576, P = ₹ 15,625 and n = 3

Let R = r% per annum

$$\therefore 17576 = 15625\left(1+\frac{r}{100}\right)^3$$

$$\therefore \left(1+\frac{r}{100}\right)^3 = \frac{17576}{15625} = \left(\frac{26}{25}\right)^3$$

$$\therefore \left(1+\frac{r}{100}\right) = \frac{26}{25} \text{ or } \frac{r}{100} = \frac{26}{25} - 1 = \frac{1}{25}$$

or $$r = \frac{100}{25} = 4$$

$\therefore$ Rate = 4% per annum.

Example 8.42: In how much time will a sum of ₹ 8000 amount to ₹ 9261 at 10% per annum, compounded semi-annually?

Solution: Here A= ₹ 9261, P = ₹ 8000 and n = x semi yrs

R = 10% per annum = 5% semi annually

$$\therefore 9261 = 8000\left(1 + \frac{5}{100}\right)^x$$

$$\text{or } \frac{9261}{8000} = \left(\frac{21}{20}\right)^x \text{ or } \left(\frac{21}{20}\right)^3 = \left(\frac{21}{20}\right)^x \quad \therefore x = 3$$

$$\therefore \text{ Time} = 3 \text{ half years} = 1\frac{1}{2} \text{ years}$$

Example 8.43: Find the difference between simple interest and compound interest for $1\frac{1}{2}$ years at 4% per annum, for a sum of ₹ 24000, when the interest is compounded semi-annually..

Solution: Here P = ₹ 24000, R = 4% per annum

$$T = \frac{3}{2} \text{ years} \qquad R = 2\% \text{ per semi-annually}$$

$$n = 1\frac{1}{2} \text{ years} = \frac{3}{2} \text{ years} = 3 \text{ semi years}$$

$$\text{Simple Interest} = P \times R \times T \quad = ₹\ 24000 \times \frac{4}{100} \times \frac{3}{2}$$

$$= ₹\ 1440.$$

$$\text{For compound interest, } A = P\left[\left(1 + \frac{R}{100}\right)^n\right]$$

$$A = ₹\ 24000\left[\left(1 + \frac{2}{100}\right)^3\right]$$

$$A = ₹\ 24000\left[\left(\frac{51}{50}\right)^3\right] = ₹\ 24000\left[\frac{51}{50}\times\frac{51}{50}\times\frac{51}{50}\right]$$

$$= ₹\ \frac{24\times51\times51\times51}{125} = ₹\ 25468.99 \text{ or } ₹\ 25469$$

$\therefore$ CI = ₹ [25469 – 24000] = ₹ 1469

Difference = CI – SI = ₹ [1469 – 1440]

= ₹ 29

Example 8.44: A sum of money is invested at compound interest for $1\frac{1}{2}$ year at 4% compounded annually. If the interests were compounded semi-annually, it would have fetched ₹ 20.40 more than in the previous case. Find the sum.

Solution: Let the sum be ₹ x.

Here R = 4% annually, or 2% semi-annually

$$T = 1\frac{1}{2} \text{ yrs or 3 semi years}$$

In first case

$$A = ₹\ x\left[1+\frac{4}{100}\right]^1\left[1+\frac{2}{100}\right]^1$$

$$= ₹\ x\left(\frac{26}{25}\right)\left(\frac{51}{50}\right) = ₹\ \frac{1326\,x}{1250}$$

In 2nd case

$$A = ₹\ x\left(1+\frac{2}{100}\right)^3 = ₹\ x\left(\frac{51}{50}\right)^3$$

$$= ₹\ \frac{132651}{125000}$$

$$\therefore \text{Difference} = ₹\ \left[\frac{132651}{125000}x - \frac{1326}{1250}x\right]$$

$$= ₹\ \frac{51x}{125000}$$

$$\therefore \frac{51x}{125000} = \frac{2040}{100} \text{ or } x = ₹ \frac{2040}{100} \times \frac{125000}{51} = ₹\,5000$$

$$\therefore \text{Sum} = ₹\,50{,}000$$

CHECK YOUR PROGRESS 8.6

1. Calculate the compound interest on ₹ 15625 for 3 years at 4% per annum, compounded annually.
2. Calculate the compound interest on ₹ 15625 for $1\frac{1}{2}$ years at 8% per annum, compounded semi-annually.
3. Calculate the compound interest on ₹ 16000 for 9 months at 20% per annum, compounded quarterly
4. Find the sum of money which will amount to ₹ 27783 in 3 years at 5% per annum, the interest being compounded annually.
5. Find the difference between simple interest and compound interest for 3 years at 10% per annum, when the interest is compounded annually on ₹ 30,000.
6. The difference between simple interest and compound interest for a certain sum of money at 8% per annum for $1\frac{1}{2}$ years, when the interest is compounded half-yearly is ₹ 228. Find the sum.
7. A sum of money is invested at compound interest for 9 months at 20% per annum, when the interest is compounded half yearly. If the interest were compounded quarterly, it would have fetched ₹ 210 more than in the previous case. Find the sum.
8. A sum of ₹ 15625 amounts to ₹ 17576 at 8% per annum, compounded semi-annually. Find the time.
9. Find the rate at which ₹ 4000 will give ₹ 630.50 as compound interest in 9 months, interest being compounded quarterly.
10. A sum of money becomes ₹ 17640 in two years and ₹ 18522 in 3 years at the same rate of interest, compounded annually. Find the sum and the rate of interest per annum.

8.5.5 Rate of Growth and Depreciation

In our daily life, we come across the terms like growth of population, plants, viruses etc and depreciation in the value of articles like machinery, crops, motor cycles etc.

The problems of growth and depreciation can be solved using the formula of compound interest derived in the previous section.

If V_o is the value of an article in the beginning and V_n is its value after 'n' conversion periods and the rate of growth/depreciation for the period be denoted by r%, then we can write

$$V_n = V_o\left(1+\frac{r}{100}\right)^n \text{ in case of growth, and}$$

$$V_n = V_o\left(1-\frac{r}{100}\right)^n \text{ in case of depreciation.}$$

If the rate of growth/depreciation varies for each conversion period, then

$$V_n = V_o\left(1+\frac{r_1}{100}\right)\left(1+\frac{r_2}{100}\right)\left(1+\frac{r_3}{100}\right)\text{.... for growth, and}$$

$$V_n = V_o\left(1-\frac{r_1}{100}\right)\left(1-\frac{r_2}{100}\right)\left(1-\frac{r_3}{100}\right)\text{.... for depreciation.}$$

Let us take some examples to illustrate the above concepts.

Example 8.45: The population of a city is 9765625. What will be its population after 3 years, if the rate of growth of population is 4% per year?

Solution: Here $V_o = 9765625$, $r = 4\%$ and $n = 3$

$$\therefore\ V_3 = 9765625\left[1+\frac{4}{100}\right]^3$$

$$= 9765625\times\left(\frac{26}{25}\right)^3$$

$$= 10985000.$$

Hence, the population of that city after 3 years will be = 10985000.

Example 8.46: The cost of a car was ₹ 3,50,000 in January 2005. If the rate of depreciation is 15% for the first year and 10% for the subsequent years, find its value after 3 years.

Solution: Here $V_o = ₹\ 3,50,000$

$r_1 = 15\%$, $r_2 = 10\%$ and $r_3 = 10\%$

$$\therefore V_3 = V_o\left(1-\frac{r_1}{100}\right)\left(1-\frac{r_2}{100}\right)\left(1-\frac{r_3}{100}\right)$$

$$= ₹\ 350000\left(1-\frac{15}{100}\right)\left(1-\frac{10}{100}\right)\left(1-\frac{10}{100}\right)$$

$$= ₹\ 350000\times\frac{17}{20}\times\frac{9}{10}\times\frac{9}{10} = ₹\ 2,40,975/-$$

$\therefore$ The value of car after 3 years = ₹ 240975.

Example 8.47: A plant gains its height at the rate of 2% per month of what was its height in the beginning of the month. If its height was 1.2 m in the beginning of January 2008, find its height in the beginning of April 2008, correct upto 3 places of decimal.

Solution: Here $V_o = 1.2$ m, $r = 2\%$, $n = 3$

$$\therefore V_3 = V_o\left(1+\frac{r}{100}\right)^n$$

$$= 1.2\left(1+\frac{2}{100}\right)^3 = 1.2\left(\frac{51}{50}\right)^3 = 1.2734 \text{ m}$$

$$= 1.273 \text{ m}$$

Hence, height of plant in the beginning of April = 1.273 m.

Example 8.48: The virus of a culture decreases at the rate of 5% per hour due to a medicine. If the virus count in the culture at 11.00 AM was 2.3×10^7, find the virus count at 1.00 PM on the same day.

Solution: $V_o = 2.3 \times 10^7$, $r = 5\%$, $n = 2$

$$V_2 = 2.3\times10^7\left(1-\frac{5}{100}\right)^2 = 2.3\times10^7\times(0.95)^2$$

$$= 2.076 \times 10^7$$

Hence, the virus count at 1.00 PM is 2.076×10^7.

CHECK YOUR PROGRESS 8.7

1. The population of a town is 281250. What will be its population after 3 years, if the rate of growth of population is 4% per year?
2. The cost of a car was ₹ 4,36,000 in January 2005. Its value depreciates at the rate of 15% in the first year and then at the rate of 10% in the subsequent years. Find the value of the car in January 2008.

3. The cost of machinery is ₹ 360000 today. In the first year the value depreciates by 12% and subsequently, the value depreciates by 8% each year. By how much, the value of machinery has depreciated at the end of 3 years?
4. The application of manure increases the output of a crop by 10% in the first year, 5% in the second year and 4% in the third year. If the production of crop in the year 2005 was 3.5 tons per hectare, find the production of crop per hectare in 2008.
5. The virus of a culture decreases at the rate of 4% per hour due to a medicine. If the virus count in the culture at 9.00 AM was 3.5×10^8, find the virus count at 11.00 AM on the same day.
6. Three years back, the population of a village was 50000. After that, in the first year, the rate of growth of population was 5%. In the second year, due to some epidemic, the population decreased by 10% and in the third year, the population growth rate was noticed as 4%. Find the population of the town now.

LET US SUM UP

- Percent means 'per hundred'.
- Percents can be written as fractions as well as decimals and vice-versa.
- To write a percent as a fraction, we drop the % sign and divide the number by 100.
- To write a fraction as a percent, we multiply the fraction by 100, simplify it and suffix the % sign.
- To determine the specific percent of a number or quantity, we change the percent to a fraction or a decimal and then multiply.
- When the selling price is more than the cost price of the goods, there is a profit (or gain).
- When the selling price is less than the cost price of the goods, there is a loss.

  Profit (Gain) = S.P. – C.P. ; Loss = C.P. – S.P.

  $$\text{Gain\%} = \frac{\text{Gain}}{\text{C.P.}} \times 100 \quad ; \quad \text{Loss\%} = \frac{\text{Loss}}{\text{C.P.}} \times 100$$

  $$\text{S.P.} = \frac{100 + \text{Gain\%}}{100} \times \text{C.P.} \quad ; \quad \text{S.P.} = \frac{100 - \text{Loss\%}}{100} \times \text{C.P.}$$

- The simple interest (I.) on a principal (P) at the rate of R% for a time T years, is calculated, using the formula

  $$\text{I.} = \text{P} \times \text{R} \times \text{T}$$

- Discount is a reduction in the list price of goods.
- Discount is always calculated on the marked price of the goods
- (Marked price – discount), gives the price, which a customer has to pay while buying an article.
- Two or more successive discounts are said to form a discount series.
- A discount series can be reduced to a single discount.
- Sales tax is charged on the sale price of goods.
- An instalment plan enables a person to buy costlier goods.
- In the case of compound interest

  Amount (A) = P (1 + R)^n, where P is the Principal, R = rate% and n = time.
- Compound interest is greater than simple interest, except for the first conversion period.
- If V_o is the value of an article in the beginning and V_n is its value after 'n' conversion periods and 'r' be the rate of growth/depreciation per period, then

$$V_n = V_o\left(1+\frac{r}{100}\right)^n \text{ in case of growth, and}$$

$$V_n = V_o\left(1-\frac{r}{100}\right)^n \text{ in case of depreciation.}$$

- If the rate of growth/depreciation varies for each conversion period, then

$$V_n = V_o\left(1+\frac{r_1}{100}\right)\left(1+\frac{r_2}{100}\right)\left(1+\frac{r_3}{100}\right)\text{.... for growth, and}$$

$$V_n = V_o\left(1-\frac{r_1}{100}\right)\left(1-\frac{r_2}{100}\right)\left(1-\frac{r_3}{100}\right)\text{.... for depreciation.}$$

TERMINAL EXERCISE

1. Write each of the following as a percent

   (a) $\frac{9}{20}$ (b) $\frac{7}{10}$ (c) 0.34 (d) 0.06

2. Write each of the following as a decimal:

   (a) 36% (b) 410% (c) 2% (d) 0.35%

3. Write each of the following as fraction:

   (a) 0.12% (b) 2.5% (c) 25.5% (d) 255%

4. Find each of the following:

   (a) 23% of 500 (b) 2.5% of 800 (c) 0.4% of 1000 (d) 115% of 400

5. What percent of 700 is 294?

6. By what percent is 60 more than 45?

7. What number increased by 10% of itself is 352?

8. Find the number whose 15% is 270.

9. What number decreased by 7% of itself is 16.74?

10. If three fourth of the students of a class wear glasses, what percent of the students of the class do not wear glasses?

11. There are 20 eggs in a fridge and 6 of them are brown. What percent of eggs are not brown?

12. 44% of the students of a class are girls. If the number of girls is 6 less than the number of boys, how many students are there in the class?

13. During an election, 70% of the population voted. If 70,000 people cast their votes, what is the population of the town?

14. A man donated 5% of his monthly income to a charity and deposited 12% of the rest in a Bank. If he has Rs. 11704 left with him, what is his monthly income?

15. Ratan stores has a sale of ₹ 12000 on Saturday, while Seema stores had a sale of ₹ 15000 on that day. Next day, they had respective sales of ₹ 15000 and ₹ 17500. Which store showed more improvement in Sales?

16. A candidate has to secure 45% marks in aggregate of three papers of 100 marks each to get through. He got 35% marks in the first paper and 50% marks in the second paper. At least how many marks should he get in third paper to pass the examination?

17. The price of sugar rises by 25%. By how much percent should a householder reduce his consumption of sugar, so as not to increase his expenditure on sugar?

18. By selling 90 ball pens for ₹ 160, a person loses 20%. How many ball pens should he sell for Rs. 96, so as to have a gain of 20%?

19. A vendor bought bananas at 6 for 5 rupees and sold them at 4 for 3 rupees. Find his gain or loss percent.

20. A man bought two consignments of eggs, first at ₹ 18 per dozen and an equal number at ₹ 20 per dozen. He sold the mixed egges at ₹ 23.75 per dozen. Find his gain percent.

21. A man sells an article at a gain of 10%. If he had bought it for 10% less and sold it for ₹ 10 more, he would have gained 25%. Find the cost price of the article.

22. A pair of socks is marked at ₹ 80 and is being offered at ₹ 64. Find the discount percent being offered.

23. A dealer buys a table listed at ₹ 1800 and gets a discount of 25%. He spends ₹ 150 on transportation and sells it at a profit of 10%. Find the selling price of the table.

24. A T.V. set was purchased by paying ₹ 18750. If the discount offered by the dealer was 25%, what was the marked price of the TV set?

25. A certain sum of money was deposited for 5 years. Simple interest at the rate of 12% was paid. Calculate the sum deposited if the simple interest received by the depositor is ₹ 1200.

26. Simple intrest on a sum of money is $\frac{1}{3}$rd of the sum itself and the number of years is thrice the rate percent. Find the rate of interest.

27. In what time will ₹ 2700 yield the same interest at 4% per annum as ₹ 2250 in 4 years at 3% per annum?

28. The difference between simple interest on a sum of money for 3 years and for 2 years at 10% per annum is ₹ 300. Find the sum.

29. Find the sum which when invested at 4% per annum for 3 years will becomes ₹ 70304, when the interest is compounded annually.

30. the difference between compound interest and simple interest at 10% per annum in 2 years (compounded annually) is ₹ 50. Find the sum.

31. A sum of money becomes ₹ 18522 in three years and ₹ 19448.10 in 4 years at the same rate of interest, compounded annually. Find the sum and the rate of interest per annum.

32. Find the sum of money which will amount to ₹ 26460 in six months at 20% per annum, when the interest is compounded quarterly.

33. At what rate percent per annum will a sum of ₹ 12000 amount to ₹ 15972 in three years, when the interest is compounded annually?

34. The price of a scooter depreciates at the rate of 20% in the first year, 15% in the second year and 10% afterwards, what will be the value of a scooter now costing ₹ 25000, after 3 years.

35. The population of a village was 20,000, two years ago. It increased by 10% during first year but decreased by 10% in the second year. Find the population at the end of 2 years.

ANSWERS TO CHECK YOUR PROGRESS

8.1

1. (a) 48% (b) 45% (c) $41\frac{2}{3}\%$ (d) 40% (e) 20%
(f) 30% (g) 36% (h) 126% (i) 288% (j) 98.48%

2. (a) $\frac{53}{100}$ (b) $\frac{17}{20}$ (c) $\frac{27}{160}$ (d) $\frac{137}{4000}$ (e) $\frac{1}{16}$
(f) $\frac{7}{10}$ (g) $\frac{63}{400}$ (h) $\frac{1}{40000}$ (i) $\frac{947}{2000}$ (j) $\frac{21}{4000}$

3. (a) 97% (b) 73.5% (c) 3% (d) 207% (e) 80%
(f) 175% (g) 2.5% (h) 325.75% (i) 15.2% (j) 300.15%

4. (a) 0.72 (b) 0.41 (c) 0.04 (d) 1.25 (e) 0.09
(f) 4.1 (g) 3.5 (h) 1.025 (i) 0.00025 (j) 0.1025

5. 50% 6. 90% 7. 6.25% 8. 47.5% 9. 30%
10. 5%

8.2

1. (a) 200 (b) 564
2. Rs. 2625 3. 175, 100, 125, 100 4. 25%
5. 56.25% 6. 500 7. 36 minutes
8. 6000 9. Rs. 40, Rs. 32 10. 4% decrease
11. B 12. 10.8%
13. Rs. 13200, Rs. 12000, Rs. 10000

8.3

1. $33\frac{1}{3}\%$ profit 2. 10% 3. 25% 4. 120
5. Rs. 2576 6. 21 7. 4% loss 8. Rs. 1108.80
9. 12% gain 10. 15% gain

8.4

1. Rs. 318.75 2. 20% 3. Rs. 15390 4. 25%
5. Rs. 724.50 6. 10% loss 7. Rs. 2185 8. 10.4%

8.5

1. Rs. 16240 2. Rs. 3744 3. Rs. 5600 4. 11%

5. 4 years 6. 10% 7. Rs. 1100, 5% 8. $9\frac{5}{7}\%$

9. 24 years 10. b

8.6

1. Rs. 1951 2. Rs. 1951 3. Rs. 2522 4. Rs. 24000

5. Rs. 630 6. Rs. 46875 7. Rs. 80000 8. $1\frac{1}{2}$ years

9. 20% 10. Rs. 1600, 5%

8.7

1. 316368 2. Rs. 300186 3. Rs. 291456

4. 4.2042 tons/hectare 5. 3.2256×10^{8} 6. 49140

ANSWERS TO TERMINAL EXERCISE

1. (a) 45% (b) 70% (c) 34% (d) 6%

2. (a) 0.36 (b) 4.10 (c) 0.02 (d) 0.0035

3. (a) $\frac{3}{2500}$ (b) $\frac{1}{40}$ (c) $\frac{51}{200}$ (d) $\frac{51}{20}$

4. (a) 115 (b) 20 (c) 4 (d) 460

5. 42% 6. 25% 7. 320 8. 1800

9. 18 10. 25% 11. 70% 12. 50

13. 1 Lakh 14. Rs. 14000 15. Ratan Stores 16. 50

17. 20% 18. 36 19. 60% gain 20. 25%

21. Rs. 400 22. 20% 23. Rs. 1650 24. Rs. 25000

25. Rs. 2000 26. $3\frac{1}{3}\%$ 27. $2\frac{1}{2}$ years 28. Rs. 3000

29. Rs. 62500 30. Rs. 5000 31. Rs. 16000, 5% 32. Rs. 24000

33. 10% 34. Rs. 13500 35. 19800

9

INSTALMENT BUYING

You must have seen advertisements like, “Pay just ₹ 500 and take home a color TV, rest in easy instalments”, or “buy a car of your choice by paying ₹ 50,000 and the balance in easy instalments”. Such plans attract customers, specially the common man, who could not buy some costly articles like car, scooter, fridge, colour TV, etc. due to cash constraints. Under these plans, a fixed amount is paid at the time of purchase and the rest of the amount is to be paid in instalments, which may be monthly, quarterly, half yearly or yearly, as per the agreement signed between the customer and the seller.

Instalment purchase scheme, thus, enables a person to buy costly goods, on convenient terms of payment. Under this scheme, the customer, after making a partial payment in the beginning, takes away the article for use after signing the agreement to pay the balance amount in instalments. Such a scheme also encourages the buyer to save at regular intervals, so as to pay the instalments.

In this lesson, we shall study different types of instalment plans and shall find out how much easy they are, by calculating the interest charged under these plans.

OBJECTIVES

After studying this lesson, you will be able to

- *explain the advantages/disadvantages of buying a commodity under instalment plan;*
- *determine the amount of each instalment, when goods are purchased under instalment plan at a given rate of interest (simple interest);*
- *determine the rate of interest when the amount of each (equal) instalment and the number of instalments is given;*
- *determine the amount of each instalment under instalment plan when compound interest is charged yearly, half yearly or quarterly;*
- *solve problems pertaining to instalment plan.*

EXPECTED BACKGROUND KNOWLEDGE

- Simple interest and compound interest.
- Calculation of interest when the interest is calculated yearly, half yearly, quarterly or monthly..

9.1 INSTALMENT BUYING SCHEME-SOME DEFINITIONS

Cash Price: The cash price of an article is the amount which a customer has to pay in full for the article at the time of purchase.

Cash Down Payment: The amount to be paid (in cash) under an instalment plan at the time of purchase of a commodity, is called the **cash down payment**. It is the partial payment made by the customer at the time of signing the agreement and taking away the article for use.

Instalments: It is the amount which is paid by the customer at regular intervals towards the remaining part of the selling price of the article.

Interest under the Instalment Plan: In an instalment plan only part payment of the total cost is paid by the customer at the time of purchase. The remaining part of cost is paid on subsequent dates; and therefore the seller charges some extra amount for deferred payments. This extra amount is actually the interest charged on the amount of money which the customer ows to the seller at different times of payment of instalments.

9.2 TO FIND THE INTEREST IN AN INSTALMENT PLAN

Let us solve a few examples to illustrate the process.

Example 9.1: A Television set is sold for ₹ 20000 cash or for ₹ 6000 as cash down payment followed by ₹ 16800 after six months. Find the rate of interest charged under the instalment plan.

Solution: The cash price of the television = ₹ 20000

Cash down payment = ₹ 6000

Balance to be paid = ₹ 14000

$\therefore$ The present value of Rs. 16800 to be paid after 6 months = Rs. 14000

If the rate of interest per annum under instalment plan is r%, then

$$14000 + 14000 \times \frac{r}{100} \times \frac{6}{12} = 16800$$

$$\text{or } \frac{7r}{10} = 28 \quad \text{i.e., } r = 40, \text{ i.e. rate} = 40\%$$

Example 9.2: A table fan is sold for ₹ 450 cash or ₹ 210 cash down payment followed by two monthly instalments of ₹ 125 each. Find the rate of interest charged under the instalment plan.

Solution: Cash price of the table fan = ₹ 450

Cash down payment = ₹ 210

Balance to be paid = ₹ (450 – 210) = ₹ 240

Let the rate of interest charged under instalment plan be r% p.a. then

₹ 240 at the end of two months will become $= \text{₹}\left(240 + 240 \times \frac{r}{100} \times \frac{2}{12}\right)$

$$= \text{₹}\left(240 + \frac{2r}{5}\right) \qquad \text{...(i)}$$

₹ 125 paid after 1 month will amount to (after another 1 month)

$$= \text{₹}\ 125 + 125 \times \frac{r}{100} \times \frac{1}{12} = Rs.\left(125 + \frac{5r}{48}\right) \qquad \text{...(ii)}$$

Amount for ₹ 125 paid after two months = ₹ 125 ...(iii)

$$\therefore 240 + \frac{2r}{5} = 125 + \frac{5r}{48} + 125 \text{ i.e., } \left(\frac{2}{5} - \frac{5}{48}\right) r = 10$$

$$\Rightarrow r = \frac{2400}{71} = 33.8 \text{ (approx)}$$

Hence, rate of interest = 33.8%

Alternative method:

Cash price of the fan = ₹ 450

Cash down payment = ₹ 210

Payment in 2 instalments = ₹ (125 × 2) = ₹ 250

Total amount paid under instalment plan = ₹ (210 + 250)

= ₹ 460

∴ Interest paid = ₹ (460 – 450) = ₹ 10

The Principal for the first month = ₹ (450 – 210) = ₹ 240

Principal for the 2nd month = ₹ (240 – 125) = ₹ 115

$\therefore$ Total Principal (for 1 month) = ₹ (240 +115) = ₹ 355

Thus we have

$$355 \times \frac{r}{100} \times \frac{1}{12} = 10, \text{ or, } r = \frac{10 \times 100 \times 12}{355}$$

$$= \frac{2400}{71} \approx 33.8$$

Hence, rate of interest = 33.8% p.a.

Example 9.3: A microwave oven is available for ₹ 9600 cash or for ₹ 4000 cash down payment and 3 monthly instalments of ₹ 2000 each. Find the rate of interest charged under the instalment plan.

Solution: Cash price of microwave oven = ₹ 9600

Cash down payment = ₹ 4000

Payment in 3 instalments = ₹ (3 × 2000) = ₹ 6000

Total amount paid under instalment plan = ₹ (4000 + 6000)

= ₹ 10000

$\therefore$ Interest paid = ₹ (10000 – 9600) = ₹ 400

Principal for 1st month = ₹ (9600 – 4000) = ₹ 5600

Principal for 2nd month = ₹ (5600 – 2000) = ₹ 3600

Principal for 3rd month = ₹ (3600 – 2000) = ₹ 1600

$\therefore$ Total Principal (for 1 month) = ₹ (5600 + 3600 + 1600)

= ₹ 10800

Thus, we have

$$10800 \times \frac{r}{100} \times \frac{1}{12} = 400 \Rightarrow 9r = 400 \text{ or } r = \frac{400}{9} \approx 44.4\%$$

So, rate of interest charged = 44.4%

Example 9.4: A computer is sold for ₹ 30,000 cash or ₹ 18000 cash down payment and 6 monthly instalments of ₹ 2150 each. Find the rate of interest charged under the instalment plan.

Solution: Cash price of the computer = ₹ 30000

Cash down payment = ₹ 18000

Payment in 6 instalments = ₹ (6 × 2150) = ₹ 12900

$\therefore$ Total amount paid under instalment plan = ₹ (18000 + 12900)

= ₹ 30900

∴ Interest paid = ₹ (30900 – 30000) = ₹ 900

Principal for 1st month = ₹ (30000 – 18000) = ₹ 12000

Principal for 2nd month = ₹ (12000 – 2150) = ₹ 9850

Principal for 3rd month = ₹ (9850 – 2150) = ₹ 7700

Principal for 4th month = ₹ (7700 – 2150) = ₹ 5550

Principal for 5th month = ₹ (5550 – 2150) = ₹ 3400

Principal for 6th month = ₹ (3400 – 2150) = ₹ 1250

∴ Total Principal for one month = ₹ (12000 + 9850 + 7700 + 5550 + 3400 + 1250)

= ₹ 39750

∴ We have

$$39750 \times \frac{r}{100} \times \frac{1}{12} = 900 \Rightarrow r = \frac{900 \times 12 \times 100}{39750} = \frac{1440}{53}$$

$$= 27.17\%$$

Thus, the rate of interest = 27.17% per annum.

Note: In Examples 2 to 4, observe that the Principal for the last month is less than the amount of the instalment. If interest is added to the last Principal, the sum will be equal to the amount of monthly instalment.

CHECK YOUR PROGRESS 9.1

1. A table is sold for ₹ 2000 cash or for ₹ 600 as cash down payment, followed by ₹ 1500 paid after 2 months. Find the rate of interest charged under the instalment plan.
2. A cycle is sold for ₹ 2700 cash or ₹ 600 as cash down payment, followed by 3 monthly instalments of ₹ 750 each. Find the rate of interest charged under the instalment plan.
3. A T.V. set is available for ₹ 21000 cash or for ₹ 4000 cash down payment and 6 equal monthly instalments of ₹ 3000 each. Calculate the rate of interest charged under the instalment plan.
4. Anil purchased a computer monitor priced at ₹ 6800 cash, under the instalment plan by making a cash down payment of ₹ 2000 and 5 monthly instalments of ₹ 1000 each. Find the rate of interest charged under the instalment plan.
5. A scooter can be purchased for ₹ 28000 cash or for ₹ 7400 as cash down payment followed by 4 equal monthly instalments of ₹ 5200 each. Find the rate of interest charged under instalment plan.

6. An air conditioner is sold for ₹ 20,000 cash or ₹ 12000 cash down payment followed by 4 monthly instalments of ₹ 2200 each. Find the rate of interest under the instalment plan correct upto one decimal place.
7. An article is available for ₹ 25000 cash or 20% cash down payment followed by 6 monthly instalments of ₹ 3750 each. Calculate the rate of interest charged under the instalment plan.

9.3 TO FIND THE AMOUNT OF INSTALMENT

Now, let us think the problem with the shopkeeper's angle. A shopkeeper purchases an article at some price and wants to offer an instalment plan to his customers, as he knows that more items can be sold in this way. Now he wishes to charge interest at a particular rate and wants to decide the cash down payment, the amount of equal instalments and the number of instalments.

Let us take some examples to illustrate the process.

Example 9.5: A ceiling fan is marked at ₹ 1940 cash or for ₹ 420 cash down payment followed by three equal monthly instalments. If the rate of interest charged under the instalment plan is 16% per annum, find the monthly instalment.

Solution: Cash price of ceiling fan = ₹ 1940

Cash down payment = ₹ 420

Let each instalment = ₹ x

$\therefore$ Amount paid in instalment plan = ₹ [420 + 3x]

$\therefore$ Interest paid = ₹ (420 + 3x – 1940) = ₹ (3x – 1520)

The buyer owes to the seller for first month = ₹ 1520

The buyer owes to the seller for 2nd month = ₹ (1520 – x)

The buyer owes to the seller for 3rd month = ₹ (1520 – 2x)

$\therefore$ Totol principal for one month = ₹ [4560 – 3x]

Rate of interest = 16%

$$\therefore (3x - 1520) = (4560 - 3x)\frac{16}{100}\cdot\frac{1}{12}$$

$$25(3x - 1520) = (1520 - x)$$

i.e., $76x = 39520$

or $x = 520$

So, the amount of each instalment = ₹ 520

Example 9.6: A computer is available for ₹ 34000 cash or ₹ 20000 cash down payment together with 5 equal monthly instalments. If the rate of interest charged under the instalment plan is 30% per annum, calculate the amount of each instalment.

Solution: Cash price = ₹ 34000

Cash down payment = ₹ 20000

Balance to be paid in 5 equal instalments = ₹ 14000

Let each instalment be ₹ *x*

So, interest charged under instalment plan = ₹ (5x – 14000)

The buyer owes to the seller for

1st month	2nd month	3rd month	4th month	5th month
₹ 14000	₹ (14000 –x)	₹ (14000 –2x)	₹ (14000 –3x)	₹ (14000 –4x)

Therefore, total principal for one month = ₹ [70000 – 10x]

So, $(5x - 14000) = (70000 - 10x) \times \frac{30}{100} \times \frac{1}{12}$

40 (5x – 14000) = 10(7000 – x)

20x – 56000 = 7000 – x

or 21x = 63000

or x = 3000

Thus, the amount of each instalment = ₹ 3000

Example 9.7: The cost of a washing machine is ₹ 12000. The company asks for ₹ 5200 in advance and the rest to be paid in equal monthly instalments. The rate of interest to be charged is 12% per annum. If a customer can pay ₹ 1400 each month, then how many instalments he will have to pay?

Solution: Let number of instalments be 'n'

Cash price of washing machine = ₹ 12000

Price under instalment plan = ₹ (5200 + 1400*n*)

∴ Interest charged = ₹ (5200 + 1400*n* – 12000)

= ₹ (1400*n* – 6800)

Principal owed each month is

First month = ₹ 6800

2nd month = ₹ 5400

3rd month = ₹ 4000

4th month = ₹ 2600

5th month = ₹ 1200

6th month = nil

Total for one month = ₹ 20000

So, $$20000\times\frac{12}{100}\times\frac{1}{12}=(1400n-6800)$$

$1400\,n = 7000$ i.e. n = 5

Thus, the number of instalments = 5

CHECK YOUR PROGRESS 9.2

1. A scooter is available for ₹ 30000 cash or for ₹ 15000 cash down payment and 4 equal monthly instalments. If the rate of interest charged under the instalment plan is $33\frac{1}{3}\%$, find the amount of each instalment.

2. A microwave oven is available for ₹ 9600 cash or for ₹ 4000 cash down payment and 3 equal monthly instalments. If the rate of interest charged is $22\frac{2}{9}\%$ per annum, find the amount of each instalment.

3. An article is sold for ₹ 5000 cash or for ₹1500 cash down payment followed by 5 equal monthly instalments. If the rate of interest charged is 18% p.a., compute the amount of each monthly instalment.

4. An article is sold for ₹ 500 cash or ₹ 150 cash down payment followed by 5 equal monthly instalments. If the rate of interest charged is 18% per annum, compute the monthly instalment.

9.4 TO FIND CASH PRICE

Let us now take problems where we are to find the cash price of an article when in the instalment scheme, amount of each equal instalment, the rate of interest, the number of instalments and the amount of cashdown payment, are given.

Example 9.8: A bicycle is sold for ₹ 500 cash down payment and ₹ 610 after one month. If the rate of interest being charged is 20% p.a., find the cash price of the bicycle.

Solution: Cash down payment = ₹ 500

Amount of instalment paid after one month = ₹ 610

Rate of interest = 20%

Thus we have to find present value (i.e. Principal) of Rs. 610 paid after one month.

So, $$610 = \left[(\text{Principal}) \times \frac{20}{100} \times \frac{1}{12} + \text{Principal} \right]$$

$$\Rightarrow 610 = \text{Principal}\left(1 + \frac{20}{1200}\right) \text{ or Principal} = ₹\ \frac{610 \times 1200}{1220}$$

$$= ₹\ 600$$

$\therefore$ The cash price of bicycle = ₹ (500 + 600) = ₹ 1100

Example 9.9: A camera is sold for ₹ 2500 as cash down payment and ₹ 2100 after 3 months. If the rate of interest charged is 20% p.a., find the cash price of the camera.

Solution: Cash down payment = ₹ 2500

Instalment paid after 3 months = ₹ 2100

Rate of interest = 20% p.a.

So, Principal amount for ₹ 2100

$$= ₹\ \frac{2100 \times 100}{100 + 20 \times \frac{3}{12}} = ₹\ \frac{2100 \times 1200}{1260}$$

$$= ₹\ 2000$$

Therefore, cash price = ₹ (2500 + 2000) = ₹ 4500

Alternative Method:

Let cash price be ₹ x.

Cash down payment = ₹ 2500

Instalment paid = ₹ 2100

$\therefore$ Interest = ₹ (4600 – x)

Principal for the instalment = ₹ (x – 2500)

$$\therefore (4600 - x) = (x - 2500) \times \frac{3}{12} \times \frac{20}{100} = \frac{x - 2500}{20}$$

20(4600 – x) = x – 2500

or 21x = 92000 + 2500

or $21x = 94500$

or $x = 4500$

Hence, cash price = ₹ 4500

Example 9.10: A mixi was purchased by paying ₹ 360 as cash down payment followed by three equal monthly instalments of ₹ 390 each. If the rate of interest charged under instalment plan is 16% p.a., find the cash price of the mixi.

Solution: Let the cash price of the mixi be ₹ x

Cash down payment = ₹ 360

Amount paid in 3 instalments = ₹ (3 × 390) = ₹ 1170

Total paid = ₹ (360 + 1170) = ₹ 1530

∴ Interest = ₹ (1530 – x)

Principal for 1st month = ₹ (x – 360)

Principal for 2nd month = ₹ (x – 360 – 390) = ₹ (x – 750)

Principal for 3rd month = ₹ (x – 750 – 390) = ₹ (x – 1140)

Total principal for one month = ₹ [x – 360 + x – 750 + x – 1140]

= ₹ [3x – 2250]

So,
$$(1530 - x) = (3x - 2250) \times \frac{1}{12} \times \frac{16}{100} = \frac{(x - 750)}{25}$$

$25(1530 - x) = x - 750$

or $26x = 38250 + 750 = 39000$

or $$x = \frac{39000}{26} = 1500$$

Thus, the cash price of mixi = ₹ 1500

CHECK YOUR PROGRESS 9.3

1. A table was purchased by paying a cash down payment of ₹ 750 followed by ₹ 436 after a period of 6 months. If the rate of interest charged is 18% p.a., what is the cash price of the table?
2. A refrigerator was purchased for a cashdown payment of ₹ 7000 followed by a sum of ₹ 3180 after 3 months. If the rate of interest charged is 24% p.a., find the cash price of the refrigerator.

3. A cooking range is available for ₹ 520 cash down payment followed by 4 equal monthly instalments of ₹ 520 each. If the rate of interest charged is 25% per annum, find the cash price of the cooking range.
4. A ceiling fan was purchased for ₹ 210 as cash down payment followed by three equal instalments of ₹ 260 each. If the rate of interest charged under the instalment plan is 16% p.a., then find the cash price of the ceiling fan.
5. An electrical oven was purchased for ₹ 1500 cash down payment, followed by five equal monthly instalments of ₹ 440 each. If the rate of interest charged per annum under the instalment plan is 24%, find the cash price of the oven

9.5 PROBLEMS INVOLVING COMPOUND INTEREST

In instalment buying which involved monthly instalments with the total time period being less than a year, simple interest was used.

Sometimes the individuals take long-term loans, for purposes like, buying a house, a car or setting up a factory etc. In that case, the instalments are to be paid annually for a long period and therefore involves the use of compound interest. Even in instalment buying for a period less than a year, sometimes the seller charges compound interest when the instalments are semi annually or quarterly.

Now, we shall take some problems involving compound interest.

Example 9.11: A refrigerator is available for ₹ 12000 cash or ₹ 3600 cash down payment along with 2 equal half yearly instalments. If the dealer charges an interest of 20% p.a. compounded semi-annually, under the instalment plan, find the amount of each instalment.

Solution: Cash price of refrigerator = ₹ 12000

Cash down payment = ₹ 3600

Balance = ₹ 8400

Rate of interest = 20% p.a. or 10% semi-annually

Let each monthly instalment be ₹ x, then we shall find the present value (or the Principal) for each instalment.

Let P_1, P_2 be the present values of first, 2nd conversion period respectively.

$$\therefore \quad x = P_1\left(1+\frac{10}{100}\right)^1 \text{ and } x = P_2\left(1+\frac{10}{100}\right)^2$$

$$\text{Therefore, } P_1 = \frac{10}{11}x \text{ and } P_2 = \left(\frac{10}{11}\right)^2 x$$

$$\text{Thus, we have, } \frac{10}{11}x + \frac{100}{121}x = 8400$$

or $$x = \frac{8400 \times 121}{210} = 4840$$

Thus, the amount of each instalment = ₹ 4840.

Example 9.12: A washing machine was available for ₹ 15000 cash but was purchased under an instalment plan after paying ₹ 2250 as cash down payment followed by two equal half yearly instalments. If interest charged was 8% per annum compounded semi-annually, find the value of each instalment.

Solution: Cash price of the washing machine = ₹ 15000

Cash down payment = ₹ 2250

Balance to be paid = ₹ [15000 – 2250] = ₹ 12750

Rate of interest = 8% p.a. = 4% semi-annually

Let each instalment be ₹ x (semi-annually) and

P_1, P_2 be the present values respectively of the two instalments, then

$$\therefore\ x = P_1\left(1+\frac{4}{100}\right)^1 \text{ and } x = P_2\left(1+\frac{4}{100}\right)^2$$

This gives $P_1 = \frac{25}{26}x$ and $P_2 = \left(\frac{25}{26}\right)^2 x$

Hence, $12750 = \frac{25}{26}x + \left(\frac{25}{26}\right)^2 x = \frac{25}{26}x\left(1+\frac{25}{26}\right) = \frac{25}{26}.\frac{51}{26}x$

$$\Rightarrow x = 12750 \times \frac{26}{25} \times \frac{26}{51} = 6760$$

Thus, each instalment = ₹ 6760.

Example 9.13: A juicer is available for ₹ 3500 cash but was sold under instalment plan where the purchaser agreed to pay ₹ 1500 cash down and 3 equal quarterly instalments. If the dealer charges interest at 12% p.a. compounded quarterly, find the amount of each instalment to the nearest rupee.

Solution: Cash price of the juicer = ₹ 3500

Cash down payment = ₹ 1500

Balance to be paid = ₹ (3500 – 1500) = ₹ 2000

Rate of interest = 12% p.a. = $\frac{12}{4}$ = 3% quarterly

Let the amount of each instalment be Rs. x and P_1, P_2, P_3 respectively be their present values, then

$$x = P_1\left(1+\frac{3}{100}\right), \quad x = P_2\left(1+\frac{3}{100}\right)^2 \text{ and } x = P_3\left(1+\frac{3}{100}\right)^3$$

$$P_1 = \frac{100}{103}x, \quad P_2 = \left(\frac{100}{103}\right)^2 x \text{ and } P_3 = \left(\frac{100}{103}\right)^3 x$$

$$\frac{100}{103}x + \left(\frac{100}{103}\right)^2 x + \left(\frac{100}{103}\right)^3 x = 2000 \Rightarrow \frac{100}{103}x\left[1+\frac{100}{103}+\left(\frac{100}{103}\right)^2\right] = 2000$$

$$x = 2000\times\frac{103}{100}\times\frac{(103)^2}{30909} = ₹\ 707$$

$\therefore$ Each instalment = ₹ 707

Example 9.14: A television set is sold for ₹ 7110 cash down payment along with 2 equal monthly instalments of ₹ 5581.50 each. If the dealer charges interest at 20% p.a. compounded monthly under the instalment plan, find the cash price of the television set.

Solution: Cash down payment = ₹ 7110

Amount of each monthly instalment = ₹ 5581.50 = ₹ $\frac{11163}{2}$

Rate of interest = 20% p.a. = $\frac{20}{12}$ monthly

Let P_1, P_2 be the Principals for 1st and 2nd instalment respectively

$$\frac{11163}{2} = P_1\left(1+\frac{20}{1200}\right) \text{ and } \frac{11163}{2} = P_2\left(1+\frac{20}{1200}\right)^2$$

This gives $P_1 = \frac{11163}{2}\times\frac{60}{61} = \text{Rs.}5490$ and $P_2 = \frac{11163}{2}\times\frac{60}{61}\times\frac{60}{61} = \text{Rs.}5400$

Thus, cash Price = ₹ [7110 + 5490 + 5400] = ₹ 18000

Example 9.15: A dealer offeres a micro-oven for ₹ 5800 cash. A customer agrees to pay ₹ 1800 cash down and 3 equal annual instalments. If the dealer charges interest at 12% p.a. compounded annually, what is the amount of each instalment.

Solution: Cash price of the micro-oven = ₹ 5800

Cash down payment = ₹ 1800

Balance to be paid = ₹ 4000

Rate of interest = 12% p.a. compounded annually

$\therefore$ Let Rs. x be the amount of each instalment and P_1, P_2, P_3 be the principals for each instalment respectively.

$$\therefore\ x = P_1\left(1+\frac{12}{100}\right),\quad x = P_2\left(1+\frac{12}{100}\right)^2 \text{ and } x = P_3\left(1+\frac{12}{100}\right)^3$$

$$\Rightarrow\ P_1 = \frac{25}{28}x,\ P_2 = \left(\frac{25}{28}\right)^2 x \text{ and } P_3 = \left(\frac{25}{28}\right)^3 x$$

$$\therefore\ \frac{25}{28}x + \left(\frac{25}{28}\right)^2 x + \left(\frac{25}{28}\right)^3 x = 4000$$

$$\text{or}\qquad \frac{25}{28}x\left(1+\frac{25}{28}+\frac{625}{784}\right) = 4000$$

$$\text{or}\qquad x = 4000\times\frac{28}{25}\times\frac{784}{2109} = ₹\ 1665.40$$

Hence each instalment = ₹ 1665.40

Example 9.16: A flat is available for ₹ 1600000 cash or ₹ 585500 cash down payment and three equal half yearly instalments. If the interest charged is 16% per annum compounded half yearly, calculate the value of each instalment. Find also the total interest charged.

Solution: Cash price of the flat = ₹ 1600000

Cash down payment = ₹ 585500

Balance to be paid = ₹ 1014500

Rate of interest = 16% per annum = 8% semi annually

Let the amount of each instalment be ₹ x and Let P_1, P_2 and P_3 be the Principals for each instalment respectively.

$$\text{So, } x = P_1\left(1+\frac{8}{100}\right) \text{ or } x = P_1\left(\frac{27}{25}\right) \text{ or } P_1 = x\left(\frac{25}{27}\right)$$

$$\text{Similarly, } P_2 = x\left(\frac{25}{27}\right)^2 \text{ and } P_3 = x\left(\frac{25}{27}\right)^3$$

$$\therefore\ P_1 + P_2 + P_3 = 1014500$$

$$x\left(\frac{25}{27}\right)+x\left(\frac{25}{27}\right)^2+x\left(\frac{25}{27}\right)^3=1014500$$

$$x\left(\frac{25}{27}\right)\left[1+\frac{25}{27}+\left(\frac{25}{27}\right)^2\right]=1014500$$

$$x.\frac{25}{27}.\frac{2029}{729}=1014500$$

$$\text{x}=\frac{1014500\times27\times729}{25\times2029}$$

$$= ₹\ 393660$$

Interest paid = ₹ [393660 × 3 – 1014500]

= ₹ [1180980 – 1014500]

= ₹ 166480.

CHECK YOUR PROGRESS 9.4

1. A bicycle is available for ₹ 1661 cash or by paying ₹ 400 cash down and balance in three equal half yearly instalments. If the interest charged is 10% per annum compounded semi-annually, find the instalment.
2. A washing machine is available for ₹ 15000 cash or ₹ 2000 cash down with two equal half yearly instalments. If the rate of interest charged is 16% per annum compounded half yearly, find the instalment.
3. Kamal purchased a computer in instalment plan by paying ₹ 5612.50 cash down followed by three equal quarterly instalments of ₹ 8788 each. If the rate of interest charged was 16% per annum, compounded quarterly, find the cash price of the computer. Also find the total interest charged.
4. A car was available for ₹ 70000 cash or by paying ₹ 21200 cash down along with three equal annual instalments. If the dealer charges interest of 25% per annum, compounded annually, find the amount of each instalment.
5. A microwave oven was purchased by paying a cash down payment of ₹ 2800 along with 2 equal annual instalments of ₹ 2420 each. If the rate of interest charged under the instalment plan was 10% p.a. compounded annually, find the cash price of the article.

LET US SUM UP

- Under an instalment scheme, the customer, after making a partial payment in the beginning takes away the article for use, after signing the agreement to pay the balance amount in instalments.
- Under instalment plan, the buyer pays some extra amount, which is **interest** on the deferred payments.
- Instalment scheme encourages the buyer to save at regular intervals, so as to pay the instalments.
- The price at which the article is available, if full payment is made in cash, is called the **cash price** of the article.
- The partial payment made at the time of purchase under instalment plan is called **Cash down payment.**
- The payments, which the buyer has to make at regular intervals, are called instalments.

TERMINAL EXERCISE

1. A sewing machine is available for ₹ 2600 cash payment or under an instalment plan for ₹ 1000 cash down payment and 3 equal monthly instalments of ₹ 550 each. Find the rate of interest charged under the instalment plan.
2. Anil purchased a typewriter priced at ₹ 8000 cash payment under the instalment plan by making a cashdown payment of ₹ 3200 and 5 equal monthly instalments of ₹ 1000 each. Find the rate of interest charged under the instalment plan.
3. A table is sold for ₹ 2000 cash or ₹ 500 as cash payment followed by 4 equal monthly instalments of ₹ 400 each. Find the rate of interest charged under the instalment plan.
4. A T.V. set has a cash price of ₹ 7500 or ₹ 2000 as cash down payment followed by 6 monthly instalments of ₹ 1000 each. Find the rate of interest charged under instalment plan.
5. An article is available for ₹ 7000 cash or for ₹ 1900 cash down payment and six equal monthly instalments. If the rate of interest charged is $2\frac{1}{2}$% per month, determine each instalment.
6. An article is sold for ₹ 1000 cash or Rs. 650 cash down payment followed by 5 equal monthly instalments. If the rate of interest charged is 18% per annum, compute the monthly instalment.

7. The selling price of a washing machine is ₹ 14000. The company asks for ₹ 7200 in advance and the rest to be paid in equal monthly instalments of ₹ 1400 each. If the rate of interest is 12% per annum, find the number of instalments.
8. A scooter is available for ₹ 30000 cash or for ₹ 15000 cash down payment and 4 equal monthly instalments. If the rate of interest charged under the instalment plan is $33\frac{1}{3}$%, find the amount of each instalment.
9. A plot of land is available for ₹ 200000 cash or ₹ 100000 cash down payment and 5 monthly instalments of ₹ 21000 each. Find the rate of interest charged under the instalment plan.
10. A steel almirah is marked for ₹ 3575 cash or ₹ 1600 as cash down payment and ₹ 420 per month for 5 months. Find the rate of interest under the instalment plan.
11. A watch is sold for ₹ 1000 cash or for ₹ 300 cash down payment followed by 5 equal monthly instalments. If the rate of interest charged is 18% p.a., compute the monthly instalment.
12. A computer is available for ₹ 34000 cash or ₹ 20000 cash down payment, together with 5 equal monthly instalments. If the rate of interest charged under instalment plan is 30% per annum, calculate the amount of each instalment.
13. Rita purchased a washing machine for ₹ 4000 cash down payment and 4 equal monthly instalments. The washing machine was also available for ₹ 15000 cash payment. If the rate of interest charged under the instalment plan is 18% per annum, find the amount of each instalment.
14. A ceiling fan is marked at ₹ 970 cash or ₹ 210 cash down payment followed by three equal monthly instalments. If the rate of interest charged under the instalment plan is 16% p.a., find the monthly instalment.
15. A watch is available for ₹ 970 cash or for ₹ 350 as cash down payment followed by 3 equal monthly instalments. If the rate of interest is 24% per annum, find the monthly instalment.
16. A DVD player was purchsed by the customer with a cash down payment of ₹ 2750 and agreed to pay 3 equal half yearly instalments of ₹ 331 each. If the interest charged was 20% p.a. compounded half yearly, then find the cash price of the DVD player.
17. A flat can be purchased for ₹ 200000 cash from a housing society or on the terms that ₹ 67600 be paid in the beginning as cash down payment followed by three equal half yearly instalments. If the society charges interest at the rate of 20% per annum compounded semi-annually. If the flat is purchased under instalment plan, find each instalment.
18. A scooter was sold by a shopkeeper for cash down payment of ₹ 11000 alongwith 2 equal annual instalments of ₹ 6250 each. If the rate of interest charged was 25% per annum compounded annually, find the cash price of the scooter.

19. A computer is available for ₹ 78600 cash or for ₹ 25640 cash down payment and three equal quarterly instalments. If the dealer charges interest at the rate of 20% per annum compounded quarterly, find the value of each instalment.

20. A builder announces sale of flats each for ₹ 3000000 cash or ₹ 1031600 cash down payment and three equal quarterly instalments. If the rate of interest charged is 10% per annum compounded quarterly, compute the value of each instalment under the instalment scheme. Also find the total interest.

ANSWERS TO CHECK YOUR PROGRESS

9.1

1. 42.87%   2. $44\frac{4}{9}$ 3. $21\frac{1}{19}\%$ 4. $17\frac{1}{7}\%$ 5. 4.69%

6. 51.1%   7. 47.06%

9.2

1. ₹ 4000   2. $\frac{200}{9}$ 3. ₹ 775.77

4. ₹ 1934.55   4. ₹ 77.6 approx.

9.3

1. ₹ 1150   2. ₹ 10,000   3. ₹ 2500

4. ₹ 970   5. ₹ 3580

9.4

1. ₹ 463.05   2. ₹ 7290   3. ₹ 30,000, ₹ 1976.50

4. ₹ 25000   5. ₹ 7000

ANSWERS TO TERMINAL EXERCISE

1. $19\frac{1}{21}\%$ 2. $17\frac{1}{7}\%$ 3. $33\frac{1}{3}$ 4. $33\frac{1}{3}$

5. ₹ 920   6. ₹ 63.35   7. 5   8. ₹ 4000

9. 20.7%   10. 26.43%   11. ₹ 146.12   12. ₹ 3000

13. ₹ 2850.86   14. ₹ 366 (Approx)   15. ₹ 220   16. ₹ 6060

17. ₹ 53240   18. ₹ 20,000   19. ₹ 19448

20. ₹ 689210, ₹ 99230

Secondary Course Mathematics

Practice Work-Commercial Mathematics

Maximum Marks: 25 **Time : 45 Minutes**

Instructions:

1. Answer all the questions on a separate sheet of paper.
2. Give the following informations on your answer sheet

   Name

   Enrolment number

   Subject

   Topic of practice work

   Address
3. Get your practice work checked by the subject teacher at your study centre so that you get positive feedback about your performance.

1. By selling a school bag to a customer for ₹ 660, a shopkeeper makes a profit of 10%. The cost price (in rupees) of the school bag is 1

   (A) 625 (B) 600

   (C) 575 (D) 550

2. A customer purchases a radio set for ₹ 5400 after getting 10% discount on its list price. The list price of the radio set is 1

   (A) ₹ 5050 (B) ₹ 5800

   (C) ₹ 5950 (D) ₹ 6000

3. List price of a book is ₹ 300. A student purchases the book for ₹ 234. Percentage of discount is 1

   (A) 25 (B) 24

   (C) 22 (D) 20

4. The ratio (in simplest form) of 35 cm to 2 m is 1

   (A) 35: 2 (B) 35:200

   (C) 7:40 (D) 40:7

5. The difference in simple and compound interest for ₹ 2000 at 10% per annum in 2 years, compounded annually is 1

   (A) ₹ 20 (B) ₹ 200

   (C) ₹ 400 (D) ₹ 0

6. Determine the value of k if 20 : k :: 25 : 450. 2

7. If 120 is reduced to 96, what is the percentage reduction? 2

8. If the cost price of 15 articles is the same as the selling price of 12 articles, find the gain or loss percent in the transaction. 2

9. Find the single discount equivalent to the discount series of 20%, 15% and 10%. 2

10. Find the the sum of money which will amount to ₹ 26010 in six months at the rate of 8% per annum, when interest is compounded quarterly. 2

11. A sewing machine is available for ₹ 2600 cash or under instalment plan for ₹ 1000 cash down payment followed by 3 monthly instalments of ₹ 550 each. Find the rate of interest charged under the instalment plan. 4

12. A tree gains its height at the rate of 2% of what it was in the beginning of the month. If its height was 1.5 m in the beginning of January 2010, find the height at the end of April 2010.

10

LINES AND ANGLES

Observe the top of your desk or table. Now move your hand on the top of your table. It gives an idea of a plane. Its edges give an idea of a line, its corner, that of a point and the edges meeting at a corner give an idea of an angle.

OBJECTIVES

After studying this lesson, you will be able to

- *illustrate the concepts of point, line, plane, parallel lines and interesecting lines;*
- *recognise pairs of angles made by a transversal with two or more lines;*
- *verify that when a ray stands on a line, the sum of two angles so formed is* 180^0*;*
- *verify that when two lines intersect, vertically opposite angles are equal;*
- *verify that if a transversal intersects two parallel lines then corresponding angles in each pair are equal;*
- *verify that if a transversal intersects two parallel lines then*

  (a) alternate angles in each pair are equal

  (b) interior angles on the same side of the transversal are supplementary;
- *prove that the sum of angles of a triangle is* 180^0
- *verify that the exterior angle of a triangle is equal to the sum of two interior opposite angles; and*
- *explain the concept of locus and exemplify it through daily life situations.*
- *find the locus of a point equidistent from (a) two given points, (b) two intersecting lines.*
- *solve problems based on starred result and direct numerical problems based on unstarred results given in the curriculum.*

EXPECTED BACKGROUND KNOWLEDGE

- point, line, plane, intersecting lines, rays and angles.
- parrallel lines

10.1 POINT, LINE AND ANGLE

In earlier classes, you have studied about a point, a line, a plane and an angle. Let us quickly recall these concepts.

Point : If we press the tip of a pen or pencil on a piece of paper, we get a fine dot, which is called a point.

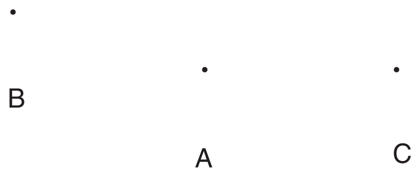


Fig. 10.1

A point is used to show the location and is represented by capital letters A, B, C etc.

10.1.1 Line

Now mark two points A and B on your note book. Join them with the help of a ruler or a scale and extend it on both sides. This gives us a straight line or simply a line.

Fig. 10.2

In geometry, a line is extended infinitely on both sides and is marked with arrows to give this idea. A line is named using any two points on it, viz, AB or by a single small letter *l*, *m* etc. (See fig. 10.3)

Fig. 10.3

The part of the line between two points A and B is called a line segment and will be named AB.

Observe that a line segment is the shortest path between two points A and B. (See Fig. 10.4)

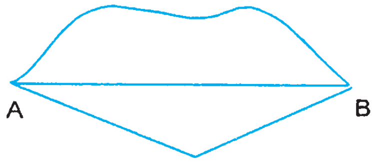


Fig. 10.4

10.1.2 Ray

If we mark a point X and draw a line, starting from it extending infinitely in one direction only, then we get a ray XY.

Fig. 10.5

X is called the initial point of the ray XY.

10.1.3 Plane

If we move our palm on the top of a table, we get an idea of a plane.

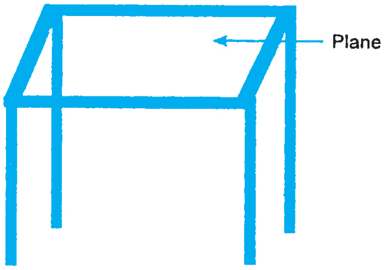


Fig. 10.6

Similarly, floor of a room also gives the idea of part of a plane.

Plane also extends infintely lengthwise and breadthwise.

Mark a point A on a sheet of paper.

How many lines can you draw passing though this point? As many as you wish.

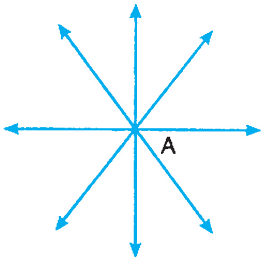


Fig. 10.7

In fact, we can draw an infinite number of lines through a point.

Take another point B, at some distance from A. We can again draw an infinite number of lines passing through B.

Fig. 10.8

Out of these lines, how many pass through both the points A and B? Out of all the lines passing through A, only one passes through B. Thus, only one line passes through both the points A and B. We conclude that **one and only one line can be drawn passing through two given points.**

Now we take three points in plane.

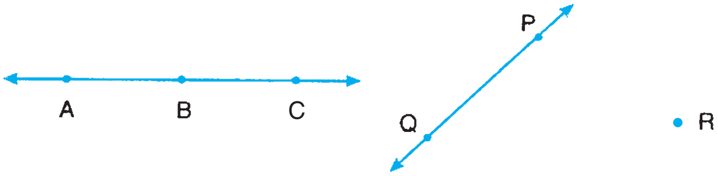


Fig. 10.9

We observe that a line may or may not pass through the three given points.

If a line can pass through three or more points, then these points are said to be **collinear.** For example the points A, B and C in the Fig. 10.9 are collinear points.

If a line **can not** be drawn passing through all three points (or more points), then they are said to be **non-collinear.** For example points P, Q and R, in the Fig. 10.9, are non-collinear points.

Since two points always lie on a line, we talk of collinear points only when their number is three or more.

Let us now take two distinct lines AB and CD in a plane.

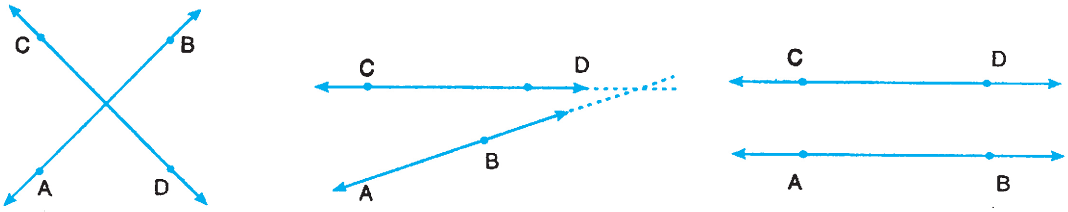


Fig. 10.10

How many points can they have in common? We observe that these lines can have. either (i) one point in common as in Fig. 10.10 (a) and (b). [In such a case they are called intersecting lines] or (ii) no points in common as in Fig. 10.10 (c). In such a case they are called **parrallel lines**.

Now observe three (or more) distinct lines in plane.

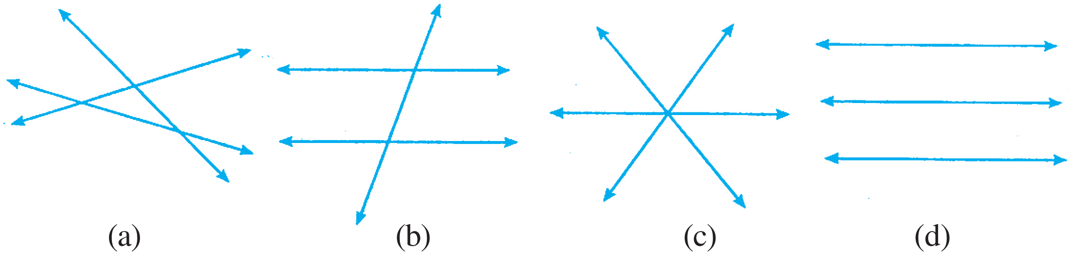


Fig. 10.11

What are the possibilities ?

(i) They may interest in more than one point as in Fig. 10.11 (a) and 10.11 (b).

or (ii) They may intesect in one point only as in Fig. 10.11 (c). In such a case they are called concurrent lines.

or (iii) They may be non intersecting lines parrallel to each other as in Fig. 10.11 |(d).

10.1.4 Angle

Mark a point O and draw two rays OA and OB starting from O. The figure we get is called an angle. Thus, an angle is a figure consisting of two rays starting from a common point.

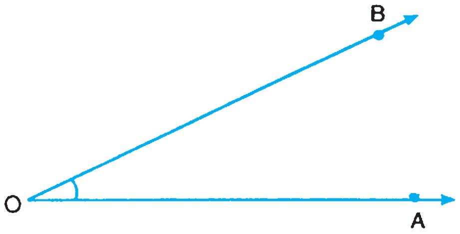


Fig. 10.11(A)

This angle may be named as angle AOB or angle BOA or simply angle O; and is written as $\angle AOB$ or $\angle BOA$ or $\angle O$. [see Fig. 10.11A]

An angle is measured in degrees. If we take any point O and draw two rays starting from it in opposite directions then the measure of this angle is taken to be 180^0 degrees, written as 180^0.

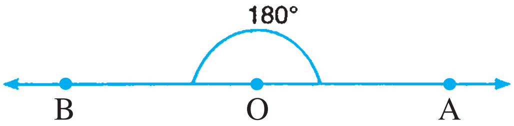


Fig. 10.12

This measure divided into 180 equal parts is called one degree (written as 1^o).

Angle obtained by two opposite rays is called a **straight angle**.

An angle of 90^0 is called a **right angle**, for example ∠BOA or ∠BOC is a right angle in Fig. 10.13.

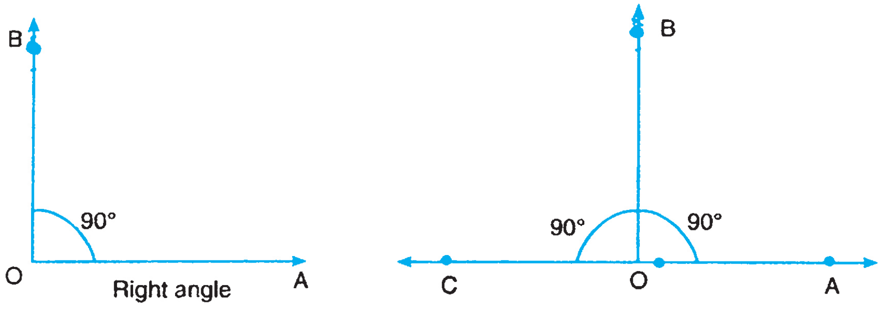
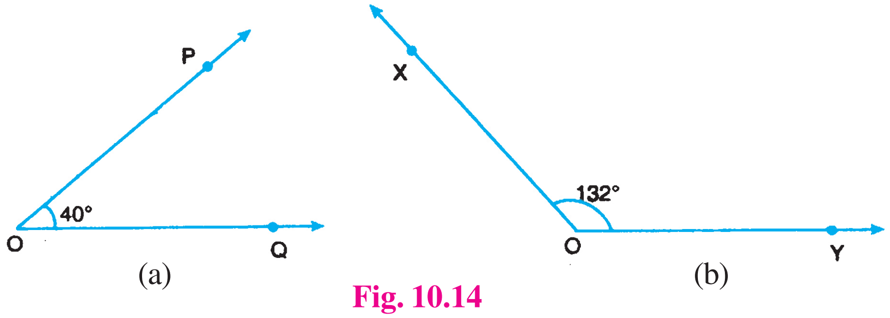


Fig. 10.13

Two lines or rays making a right angle with each other are called **perpendicular lines**. In Fig. 10.13 we can say OA is perpendicular to OB or vice-versa.

An angle less than 90^0 is called an **acute angle**. For example ∠POQ is an acute angle in Fig. 10.14(a).

An angle greater than 90^0 but less than 180^0 is called an **obtuse angle**. For example, ∠XOY is an obtuse angle in Fig. 10.14(b).



(a) (b)

Fig. 10.14

10.2 PAIRS OF ANGLES

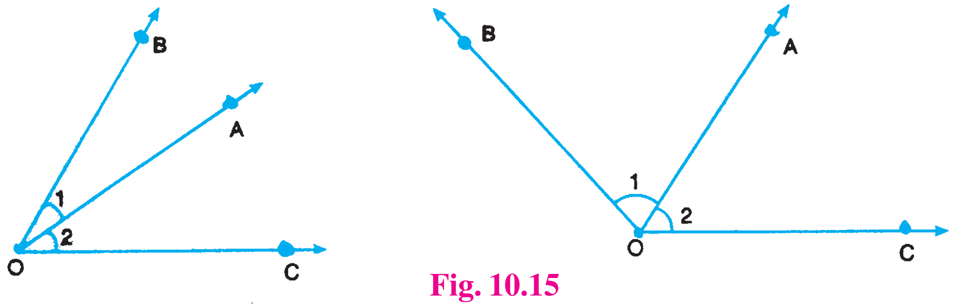
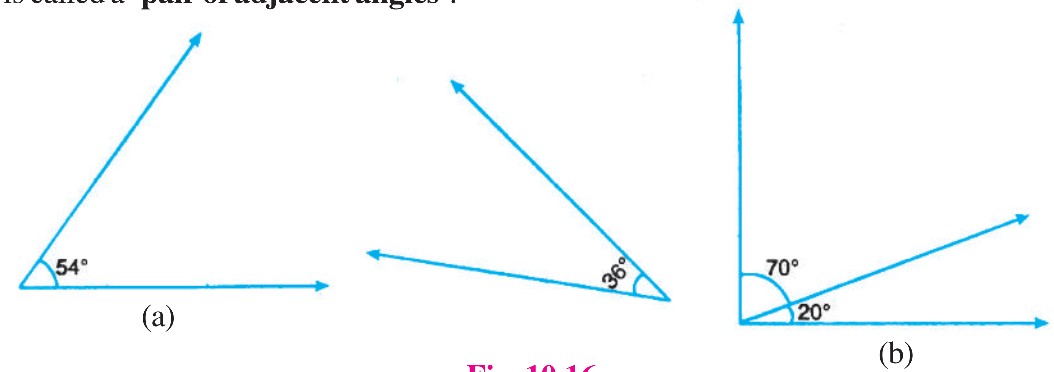


Fig. 10.15

Observe the two angles $\angle 1$ and $\angle 2$ in each of the figures in Fig. 10.15. Each pair has a common vertex O and a common side OA in between OB and OC. Such a pair of angles is called a '**pair of adjacent angles**'.

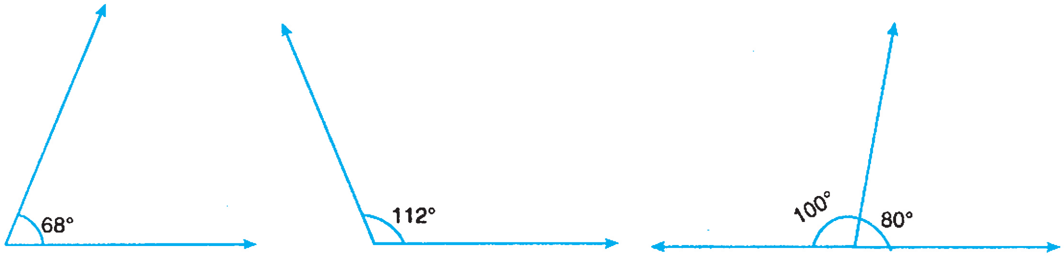


(a) (b)

Fig. 10.16

Observe the angles in each pair in Fig. 10.16[(a) and (b)]. They add up to make a total of 90°.

A pair of angles, whose sum is 90°, is called a pair of **complementary angles**. Each angle is called the **complement** of the other.



(a) (b)

Fig. 10.17

Again observe the angles in each pair in Fig. 10.17[(a) and (b)].

These add up to make a total of 180°.

A pair of angles whose sum is 180^0, is called a pair of supplementary angles.

Each such angle is called the **supplement** of the other.

Draw a line AB. From a point C on it draw a ray CD making two angles $\angle X$ and $\angle Y$.

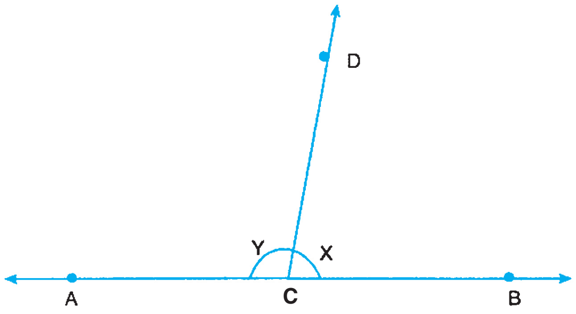


Fig. 10.18

If we measure ∠X and ∠Y and add, we will always find the sum to be 180^0, whatever be the position of the ray CD. We conclude

If a ray stands on a line then the sum of the two adjacent angles so formed is 180^o.

The pair of angles so formed as in Fig. 10.18 is called a **linear pair** of angles.

Note that they also make a pair of supplementary angles.

Draw two intersecting lines AB and CD, intersecting each other at O.

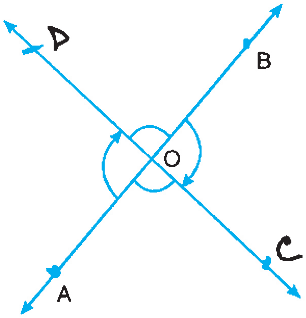


Fig. 10.19

∠AOC and ∠DOB are angles opposite to each other. These make a pair of **vertically oppposite angles**. Measure them. You will always find that

∠AOC = ∠DOB.

∠AOD and ∠BOC is another pair of vertically opposite angles. On measuring, you will again find that

∠AOD = ∠BOC

We conclude :

If two lines intersect each other, the pair of vertically opposite angles are equal.

An activity for you.

Attach two strips with a nail or a pin as shown in the figure.

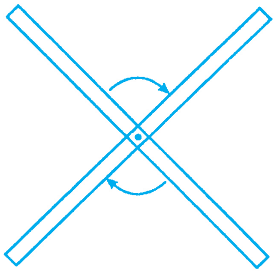


Fig. 10.20

Rotate one of the strips, keeping the other in position and observe that the pairs of vertically opposite angles thus formed are always equal.

A line which intersects two or more lines at distinct points is called a **transversal**. For example line *l* in Fig. 10.21 is a transversal.

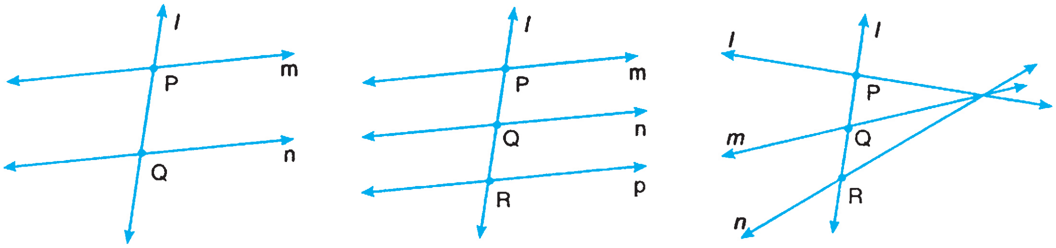


Fig. 10.21

When a transversal intersects two lines, eight angles are formed.

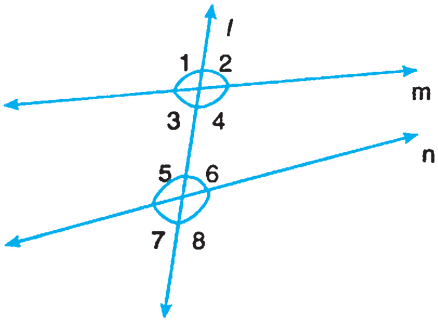


Fig. 10.22

These angles in pairs are very important in the study of properties of parallel lines. Some of the useful pairs are as follows :

(a) $\angle 1$ and $\angle 5$ is a pair of corresponding angles. $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$ and $\angle 4$ and $\angle 8$ are other pairs of corresponding angles.

(b) $\angle 3$ and $\angle 6$ is a pair of alternate angles. $\angle 4$ and $\angle 5$ is another pair of alternate angles.

(c) $\angle 3$ and $\angle 5$ is a pair of interior angles on the same side of the transversal.

$\angle 4$ and $\angle 6$ is another pair of interior angles.

In Fig. 10.22 above, lines m and n are not parallel; as such, there may not exist any relation between the angles of any of the above pairs. However, when lines are parallel, there are some very useful relations in these pairs, which we study in the following:

When a transversal intersects two parallel lines, eight angles are formed, whatever be the position of parallel lines or the transversal.

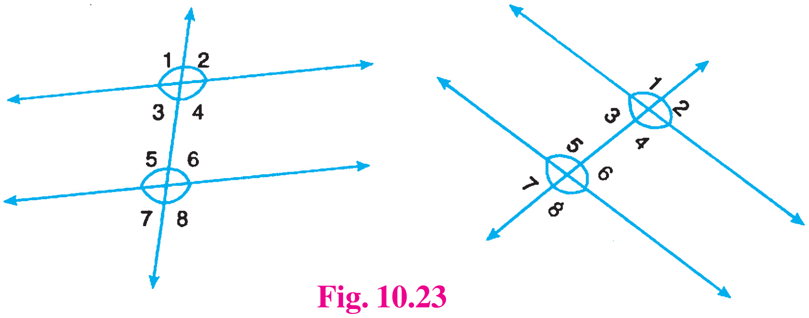


Fig. 10.23

If we measure the angles, we shall alwys find that

$$\angle 1 = \angle 5, \qquad \angle 2 = \angle 6, \qquad \angle 3 = \angle 7 \text{ and } \angle 4 = \angle 8$$

that is, angles in each pair of corresponding angles are equal.

Also $\angle 3 = \angle 6$ and $\angle 4 = \angle 5$

that is, angles in each pair of alternate angle are equal.

Also, $\angle 3 + \angle 5 = 180^\circ$ and $\angle 4 + \angle 6 = 180^\circ$.

Hence we conclude :

When a transversal intersects two parallel lines, then angles in

(i) each pair of corresponding angles are equal

(ii) each pair of alternate angles are equal

(iii) each pair of interior angles on the same side of the transversal are supplementary,

You may also verify the truth of these results by drawing a pair of parallel lines (using parallel edges of your scale) and a transversal and measuring angles in each of these pairs.

Converse of each of these results is also true. To verify the truth of the first converse, we draw a line AB and mark two points C and D on it.

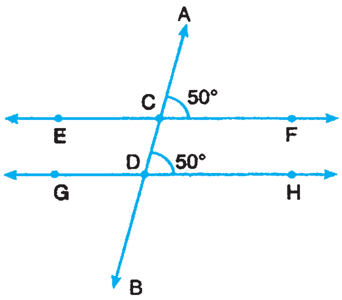


Fig. 10.24

At C and D, we construct two angles ACF and CDH equal to each other, say 50°, as shown in Fig. 10.24. On producing EF and GH on either side, we shall find that they do not intersect each other, that is, they are parallel.

In a similar way, we can verify the truth of the other two converses.

Hence we conclude that

When a transversal inersects two lines in such a way that angles in

(i) any pair of corresponding angles are equal

or (ii) any pair of alternate angles are equal

or (iii) any pair of interior angles on the same side of transversal are supplementary then the two lines are parallel.

Example 10.1 : Choose the correct answwer out of the alternative options in the following multiple choice questions.

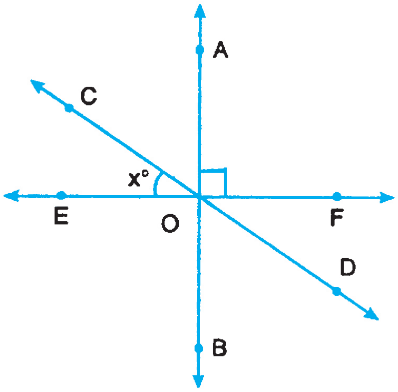


Fig. 10.25

(i) In Fig. 10.25, ∠FOD and ∠BOD are

(A) supplementary angles (B) complementary angles

(C) vertically opposite angles (D) a linear pair of angles **Ans.** (B)

(ii) In Fig. 10.25, ∠COE and ∠BOE are

(A) complementary angles (B) supplementary angles

(C) a linear pair (D) adjacent angles **Ans.** (D)

(iii) In Fig. 10.25, ∠BOD is equal to

(A) x^o (B) $(90 + x)^o$

(C) $(90 - x)^o$ (D) $(180 - x)^o$ **Ans** (C)

(iv) An angle is 4 times its supplement; the angle is

(A) 39^o (B) 72^o

(C) 108^o (D) 144^o **Ans** (D)

(v) What value of x will make ACB a straight angle in Fig. 10.26

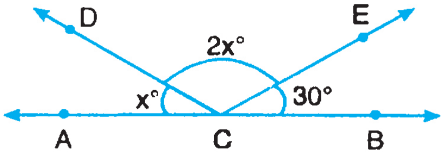


Fig. 10.26

(A) 30° (B) 40°

(C) 50° (D) 60° **Ans** (C)

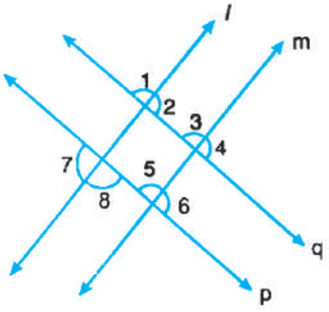


Fig. 10.27

In the above figure, *l* is parallel to m and p is parallel to q.

(vi) ∠3 and ∠5 form a pair of

(A) Alternate angles (B) interior angles

(C) vertically opposite (D) corresponding angles **Ans** (D)

(vii) In Fig. 10.27, if ∠1 = 80°, then ∠6 is equal to

(A) 80° (B) 90°

(C) 100° (D) 110° **Ans** (C)

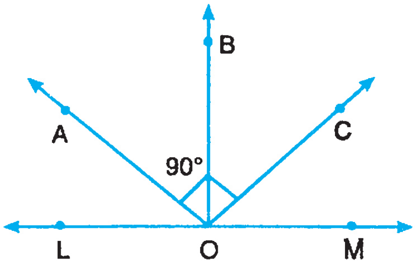


Fig. 10.28

(viii) In Fig. 10.28, OA bisects ∠LOB, OC bisects ∠MOB and ∠AOC = 90^0. Show that the points L, O and M are collinear.

Solution : $\angle BOL = 2\ \angle BOA$ ...(i)

and $\angle BOM = 2\ \angle BOC$ ...(ii)

Adding (i) and (ii), $\angle BOL + \angle BOM = 2\ \angle BOA + 2\angle BOC$

$\therefore\ \angle LOM = 2[\angle BOA + \angle BOC]$

$= 2 \times 90^o$

$= 180^o$ = a straight angle

$\therefore$ L, O and M are collinear.

CHECK YOUR PROGRESS 10.1.

1. Choose the correct answer out of the given alternatives in the following multiple choice questions :

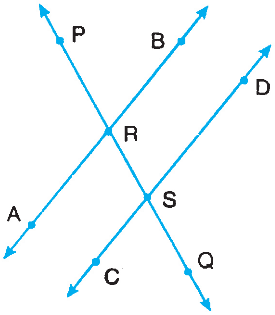


Fig. 10.29

In Fig. 10.29, AB || CD and PQ intersects them at R and S respectively.

(i) $\angle ARS$ and $\angle BRS$ form

(A) a pair of alternate angles

(B) a linear pair

(C) a pair of corresponding angles

(D) a pair of vertically opposite angles

(ii) $\angle ARS$ and $\angle RSD$ form a pair of

(A) Alternate angles (B) Vertically opposite angles

(C) Corresponding angles (D) Interior angles

(iii) If $\angle PRB = 60^o$, then $\angle QSC$ is

(A) 120^o (B) 60^o

(C) 30° (D) 90°

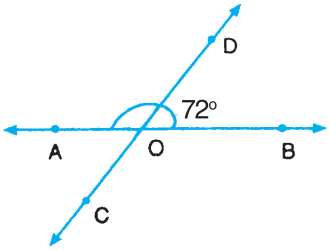


Fig. 10.30

(iv) In Fig. 10.30 above, AB and CD intersect at O. ∠COB is equal to

(A) 36° (B) 72°

(C) 108° (D) 144°

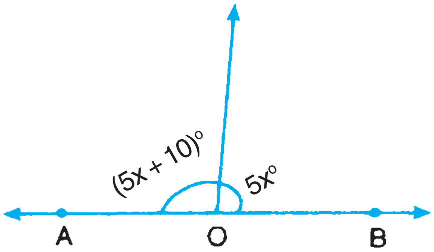


Fig. 10.31

2. In Fig. 10.31 above, AB is a straight line. Find x

3. In Fig. 10.32 below, *l* is parallel to m. Find angles 1 to 7.

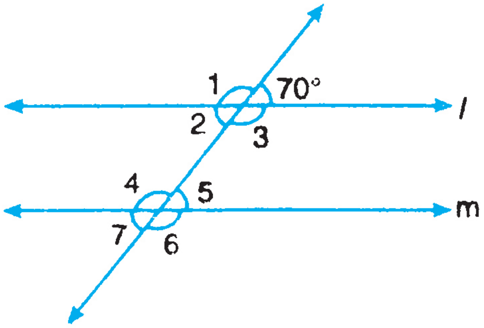


Fig. 10.32

10.3 TRIANGLE, ITS TYPES AND PROPERTIES

Triangle is the simplest polygon of all the closed figures formed in a plane by three line segments.

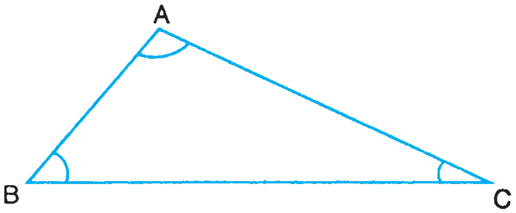


Fig. 10.33

It is a closed figure formed by three line segments having six elements, namely three **angles** (i) ∠ABC or ∠B (ii) ∠ACB or ∠C (iii) ∠CAB or ∠A and three **sides** : (iv) AB (v) BC (vi) CA

It is named as Δ ABC or Δ BAC or Δ CBA and read as triangle ABC or triangle BAC or triangle CBA.

10.3.1 Types of Triangles

Triangles can be classified into different types in two ways.

(a) On the basis of sides

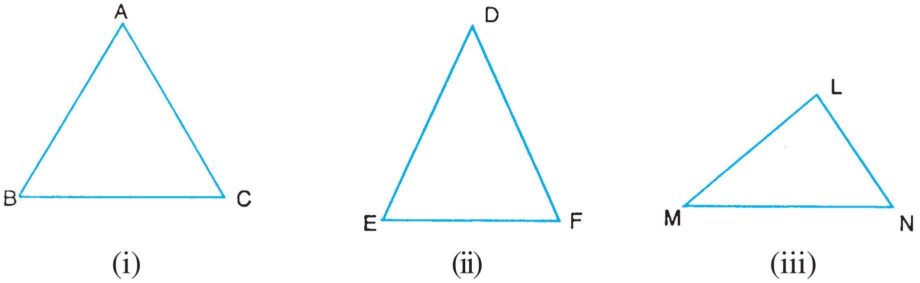


Fig. 10.34

(i) **Equilateral triangle :** a triangle in which all the three sides are equal is called an equilateral trangle. [Δ ABC in Fig. 10.34(i)]

(ii) **Isosceles triangle :** A triangle in which two sides are equal is called an isosceles triangle. [ΔDEF in Fig. 10.34(ii)]

(iii) **Scalene triangle :** A triangle in which all sides are of different lengths, is called a sclene triangle [Δ LMN in Fig. 10.34(iii)]

(b) **On the basis of angles :**

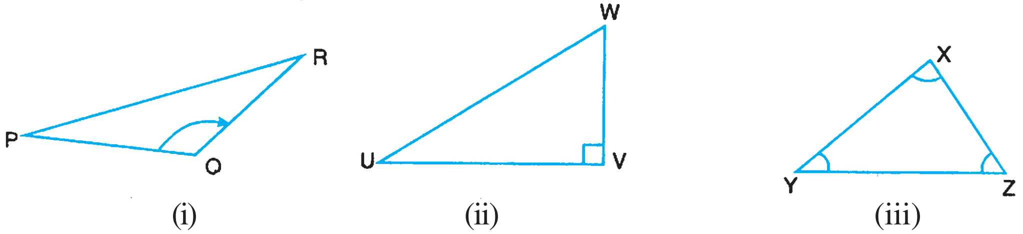


Fig. 10.35

(i) **Obtuse angled triangle :** A triangle in which one of the angles is an obtuse angle is called an **obtuse angled triangle** or simply obtuse triangle [Δ PQR is Fig. 10.35(i)]

(ii) **Right angled triangle :** A triangle in which one of the angles is a right angle is called a **right angled triangle** or right triangle. [Δ UVW in Fig. 10.35(ii)]

(iii) **Acute angled triangle :** A triangle in which all the three angles are acute is called an **acute angled triangle** or acute triangle [Δ XYZ in Fig. 10.35(iii)

Now we shall study some important properties of angles of a triangle.

10.3.2 Angle Sum Property of a Triangle

We draw two triangles and measure their angles.

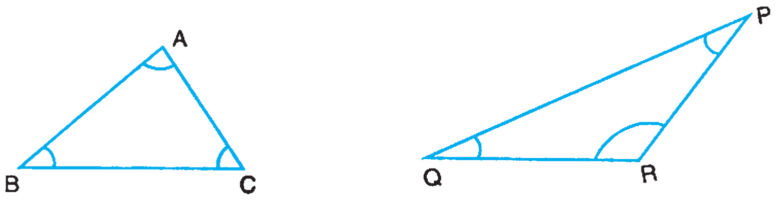


Fig. 10.36

In Fig. 10.36 (a), $\angle A = 80^o$, $\angle B = 40^o$ and $\angle C = 60^o$

$\therefore \quad \angle A + \angle B + \angle C = 80^o + 40^o + 60^o = 180^o$

In Fig. 10.36(b), $\angle P = 30^o$, $\angle Q = 40^o$, $\angle R = 110^o$

$\therefore \quad \angle P + \angle Q + \angle R = 30^o + 40^o + 110^o = 180^o$

What do you observe? Sum of the angles of triangle in each case in 180^0.

We will prove this result in a logical way naming it as a theorem.

Theorem : The sum of the three angles of triangle is 180^o.

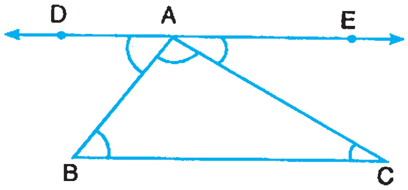


Fig. 10.37

Given : A triangle ABC

To Prove : $\angle A + \angle B + \angle C = 180^0$

Construction : Through A, draw a line DE parallel to BC.

Proof : Since DE is parallel to BC and AB is a transversal.

∴ $\angle B = \angle DAB$ (Pair of alternate angles)

Similarly $\angle C = \angle EAC$ (Pair of alternate angles)

∴ $\angle B + \angle C = \angle DAB + \angle EAC$ ...(1)

Now adding ∠A to both sides of (1)

$$\angle A + \angle B + \angle C = \angle A + \angle DAB + \angle EAC$$

$$= 180^o \quad \text{(Angles making a straight angle)}$$

10.3.3 Exterior Angles of a Triangle

Let us produce the side BC of ΔABC to a point D.

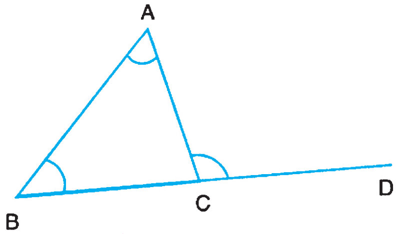


Fig. 10.38

In Fig. 10.39, observe that there are six exterior angles of the ΔABC, namely ∠1, ∠2, ∠3, ∠4, ∠5 and ∠6.

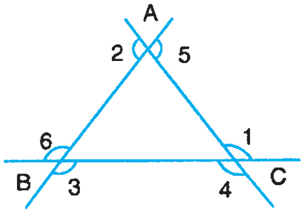


Fig. 10.39

In Fig. 10.38, ∠ACD so obtained is called an exterior angle of the ΔABC. Thus,

The angle formed by a side of the triangle produced and another side of the triangle is called an exterior angle of the triangle.

Corresponding to an exterior angle of a triangle, there are two interior opposite angles.

Interior opposite angles are the angles of the triangle not forming a linear pair with the given exterior angle.

For example in Fig. 10.38, ∠A and ∠B are the two interior opposite angles corresponding to the exterior angle ACD of ΔABC. We measure these angles.

$\angle A = 60^o$

$\angle B = 50^o$

and $\angle ACD = 110^o$

We observe that $\angle ACD = \angle A + \angle B$.

This observation is true in general.

Thus, we may conclude :

An exterior angle of a triangle is equal to the sum of the two interior opposite angles.

Examples 10.3 : Choose the correct answer out of the given alternatives in the following multiple choice questions:

(i) Which of the following can be the angles of a triangle?

(A) 65^o, 45^o and 80^o (B) 90^o, 30^o and 61^o

(C) 60^o, 60^o and 59^o (D) 60^o, 60^o and 60^o. **Ans** (D)

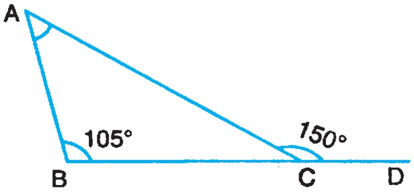


Fig. 10.40

(ii) In Fig. 10.40 $\angle A$ is equal to

(A) 30^o (B) 35^o

(C) 45^o (D) 75^o **Ans** (C)

(iii) In a triangle, one angle is twice the other and the third angle is 60^0. Then the largest angle is

(A) 60^o (B) 80^o

(C) 100^o (D) 120^o **Ans** (B)

Example 10.4:

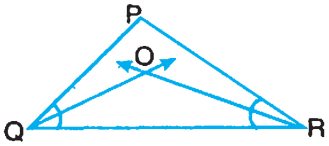


Fig. 10.41

In Fig. 10.41, bisctors of $\angle PQR$ and $\angle PRQ$ intersect each other at O. Prove that

$\angle QOR = 90^o + \frac{1}{2}\angle P$.

Solution : $\angle QOR = 180^o - \frac{1}{2}[\angle PQR + \angle PRQ)]$

$$= 180^o - \frac{1}{2}(\angle PQR + \angle PRQ)$$

$$= 180^o - \frac{1}{2}(180^o \ - \angle P)$$

$$= 180^o - 90^o + \frac{1}{2}\angle P = 90^o + \frac{1}{2}\angle P$$

CHECK YOUR PROGRESS 10.2

1. Choose the correct answer out of given alternatives in the following multiple choice questions:

   (i) A triangle can have

   (A) Two right angles (B) Two obtuse angles

   (C) At the most two acute angles (D) All three acute angles

   (ii) In a right triangle, one exterior angles is 120^0, The smallest angle of the triangles is

   (A) 20^o (B) 30^0

   (C) 40^o (D) 60^0

   (iii)

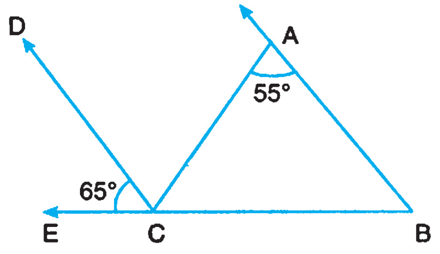


Fig. 10.42

In Fig. 10.42, CD is parallel to BA. ∠ACB is equal to

(A) 55^o (B) 60^o

(C) 65^o (D) 70^o

2. The angles of a triangle are in the ratio 2 : 3 : 5, find the three angles.

3. Prove that the sum of the four angles of a quadrilateral is 360^o.

4. In Fig. 10.43, ABCD is a trapezium such that AB||DC. Find ∠D and ∠C and verify that sum of the four angles is 360°.

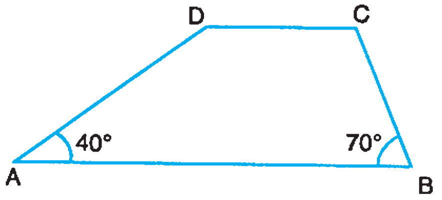


Fig. 10.43

5. Prove that if one angle of a triangle is equal to the sum of the other two angles, then it is a right triangle.

6. In Fig. 10.44, ABC is triangle such that ∠ABC = ∠ACB. Find the angles of the triangle.

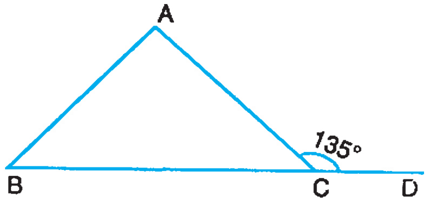


Fig. 10.44

10.4 LOCUS

During the game of cricket, when a player hits the ball, it describes a path, before being caught or touching the ground.

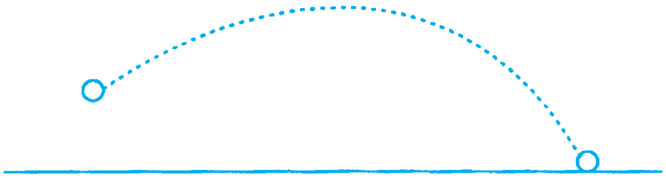


Fig. 10.44

The path described is called Locus.

A figure in geometry is a result of the path traced by a point (or a very small particle) moving under certain conditions.

For example:

(1) Given two parallel lines *l* and m, also a point P between them equidistant from both the lines.

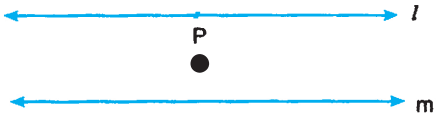


Fig. 10.45

If the particle moves so that it is equidistant from both the lines, what will be its path?

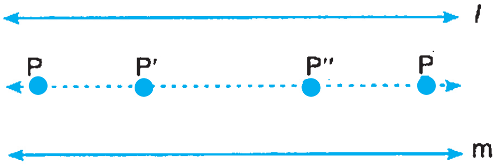


Fig. 10.46

The path traced by P will be a line parallel to both the lines and exactly in the middle of them as in Fig. 10.46.

(2) Given a fixed point O and a point P at a fixed distance *d*.

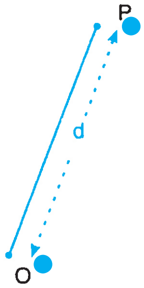


Fig. 10.47

If the point P moves in a plane so that it is always at a constant distance *d* from the fixed point O, what will be its path?

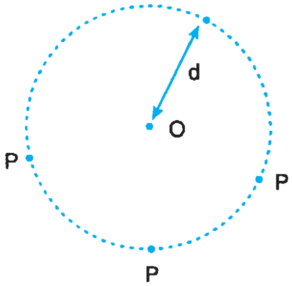


Fig. 10.48

The path of the moving point P will be a circle as shown in Fig. 10.48.

(3) Place a small piece of chalk stick or a pebble on top of a table. Strike it hard with a pencil or a stick so that it leaves the table with a certain speed and observe its path after it leaves the table.

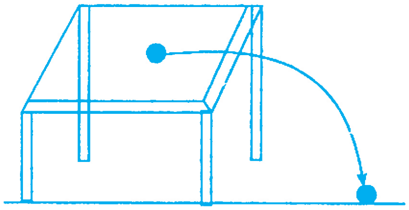


Fig. 10.49

The path traced by the pebble will be a curve (part of what is known as a parabola) as shown in Fig. 10.49.

Thus, locus of a point moving under certain conditions is the path or the geometrical figure, every point of which satisfies the given conditon(s).

10.4.1 Locus of a point equidistant from two given points

Let A and B be the two given points.

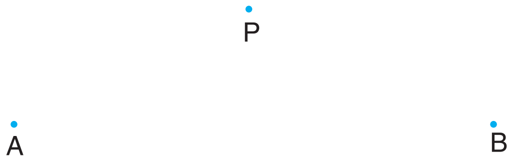


Fig. 10.50

We have to find the locus of a point P such that PA = PB.

Joint AB. Mark the mind point of AB as M. Clearly, M is a point which is equidistant from A and B. Mark another point P using compasses such that PA = PB. Join PM and extend it on both sides. Using a pair of divider or a scale, it can easily be verified that every point on PM is equidistant from the points A and B. Also, if we take any other point Q not lying on line PM, then QA ≠ QB.

Also $\angle AMP = \angle BMP = 90^o$

That is, PM is the perpendicular bisector of AB.

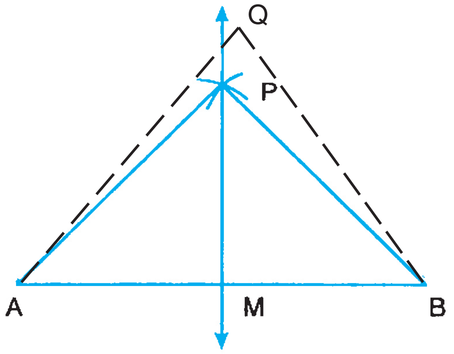


Fig. 10.51

Thus, we may conclude the following:

The locus of a point equidistant from two given poitns is the perpendicular bisector of the line segment joining the two points.

Activity for you :

Mark two points A and B on a sheet of paper and join them. Fold the paper along mid-point of AB so that A coincides with B. Make a crease along the line of fold. This crease is a straight line. This is the locus of the point equidistant from the given points A and B. It can be easily checked that very point on it is equidistant from A and B.

10.4.2 Locus of a point equidistant from two lines intersecting at O

Let AB and CD be two given lines intersecting at O.

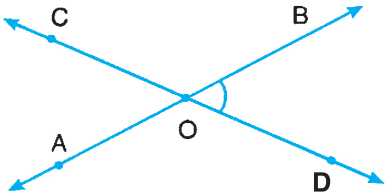


Fig. 10.52

We have to find the locus of a point P which is equidistant from both AB and CD.

Draw bisectors of ∠BOD and ∠BOC.

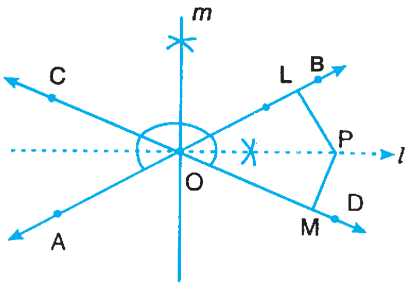


Fig. 10.53

If we take any point P on any bisector *l* or m, we will find perpendicular distances PL and PM of P from the lines AB and CD are equal.

that is, PL = PM

If we take any other point, say Q, not lying on any bisector *l* or m, then QL will not be equal to QM.

Thus, we may conclude :

The locus of a point equidistant from two intersecting lines is the pair of lines, bisecting the angles formed by the given lines.

Activity for you :

Draw two lines AB and CD intersecting at O, on a sheet of paper. Fold the paper through O so that AO falls on CO and OD falls on OB and mark the crease along the fold. Take a piont P on this crease which is the bisector of ∠BOD and check using a set square that

PL = PM

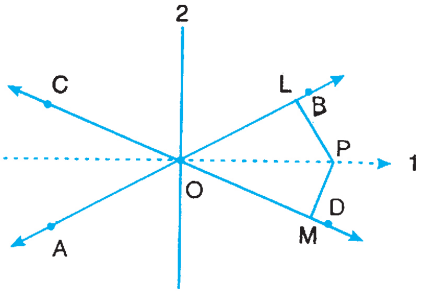


Fig. 10.54

In a similar way find the other bisector by folding again and getting crease 2. Any point on this crease 2 is also equidistant from both the lines.

Example 10.5 : Find the locus of the centre of a circle passing through two given points.

Solution : Let the two given points be A and B. We have to find the position or positions of centre O of a circle passing through A and B.

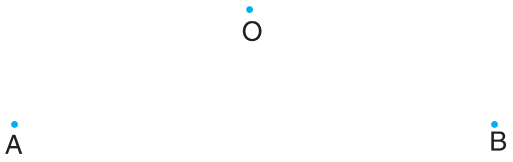


Fig. 10.55

Point O must be equidistant from both the points A and B. As we have already learnt, the locus of the point O will be the perpendicular bisector of AB.

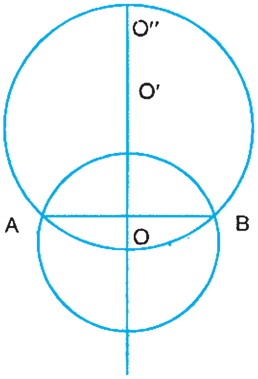


Fig. 10.56

CHECK YOU PROGRESS 10.3

1. Find the locus of the centre of a circle passing through three given points A, B and C which are non-collinear.
2. There are two villages certain distance apart. A well is to be dug so that it is equidistant from the two villages such that its distance from each village is not more than the distance between the two villages. Representing the villages by points A and B and the well by point P. show in a diagram the locus of the point P.
3. Two straight roads AB and CD are intersecting at a point O. An observation post is to be constructred at a distance of 1 km from O and equidistant from the roads AB and CD. Show in a diagram the possible locations of the post.
4. Find the locus of a point which is always at a distance 5 cm from a given line AB.

LET US SUM UP

- A line extends to inifinity on both sides and a line segment is only a part of it between two points.
- Two distinct lines in a plane may either be intersecting or parallel.
- If three or more lines intersect in one point only then they are called cocurrent lines.
- Two rays starting from a common point form an angle.
- A pair of angles, whose sum is 90^0 is called a pair of complementary angles.
- A pair of angles whose sum is 180^0 is called a pair of supplementary angles.
- If a ray stands on a line then the sum of the two adjacent angles, so formed is 180^0
- If two lines intersect each other the pairs of vertically opposite angles are equal
- When a transversal intersects two parallel lines, then

  (i) corresponding angles in a pair are equal.

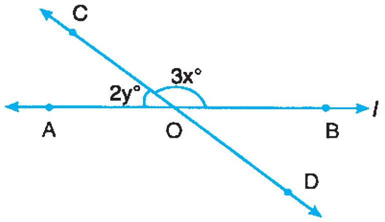
  (ii) alternate angles are equal.

  (iii) interior angles on the same side of the transversal are supplementary.
- The sum of the angles of a triangle is 180^0
- An exterior angle of a triangle to equal to the sum of the two interior opposite angles
- Locus of a point equidistant from two given points is the perpendicular bisector of the line segment joing the points.

- The locus of a point equidistant from the intersecting lines is the pair of lines, bisecting the angle formed by the given lines.

TERMINAL EXERCISE

1. In Fig. 10.57, if x = 42, then determine (a) y (b) ∠AOD



2.

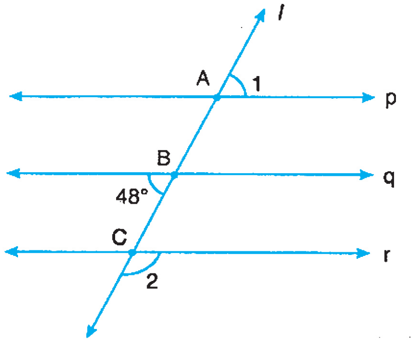


Fig. 10.58

In the above figure p, q and r are parallel lines intersected by a transversal l at A, B and C respectively. Find ∠1 and ∠2.

3. The sum of two angles of a triangle is equal to its third angle. Find the third angle. What type of triangle is it?

4.

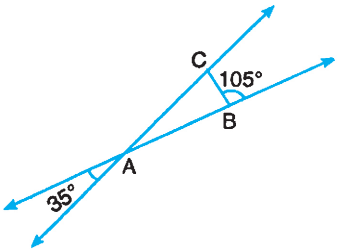


Fig. 10.59

In Fig. 10.59, sides of Δ ABC have been produced as shown. Find the angles of the triangle.

5.

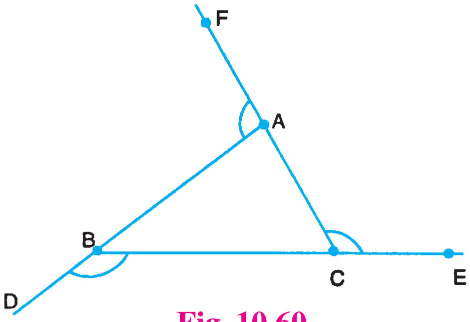


Fig. 10.60

In Fig. 10.60, sides AB, BC and CA of the triangle ABC have been produced as shown. Show that the sum of the exterior angles so formed is 360°.

6.

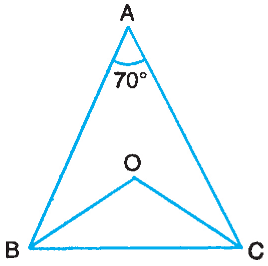


Fig. 10.61

In Fig. 10.61 ABC is a triangle in which bisectors of $\angle B$ and $\angle C$ meet at O. Show that $\angle BOC = 125°$.

7.

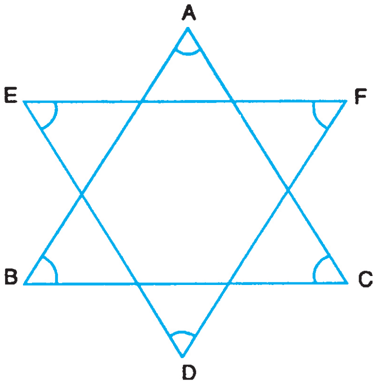


Fig. 10.62

In Fig. 10.62 above, find the sum of the angles, $\angle A$, $\angle F$, $\angle C$, $\angle D$, $\angle B$ and $\angle E$.

8.

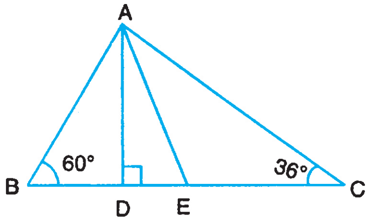


Fig. 10.63

In Fig. 10.63 in Δ ABC, AD is perpendicular to BC and AE is bisector of ∠BAC. Find ∠DAE,

9.

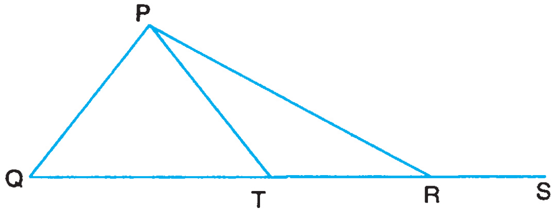


Fig. 10.64

In Fig. 10.64 above, in Δ PQR, PT is bisector of ∠P and QR is produced to S. Show that ∠PQR + ∠PRS = 2 ∠PTR.

10. Prove that the sum of the (interior) angles of a pentagon is 540^0.

11. Find the locus of a point equidistant from two parallel lines *l* and m at a distance of 5 cm from each other.

12. Find the locus of a point equidistant from points A and B and also equidistant from rays AB and AC of Fig. 10.65.

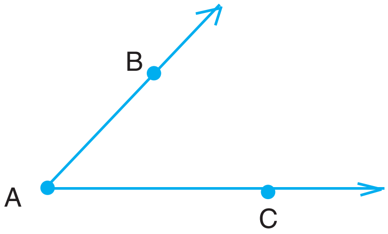


Fig. 10.65

ANSWERS TO CHECK YOUR PROGRESS

10.1

1. (i) (B) (ii) (A) (iii) (B) (iv) (C)

2. $x = 17^0$.

3. ∠1 = ∠3 = ∠4 = ∠6 = 110°

and ∠2 = ∠5 = ∠7 = 70°.

10.2

1. (i) (D) (ii) (B) (iii) (B)

2. 36°, 54° and 90° 4. ∠D = 140° and ∠C = 110°

6. ∠ABC = 45°, ∠ACB = 45° and ∠A = 90°

10.3

1. Only a point, which is the point of intersection of perpendicular bisectors of AB and BC.

2. Let the villages be A and B, then locus will be the line segment PQ, perpendicular bisector of AB such that

   AP = BP = QA = QB = AB

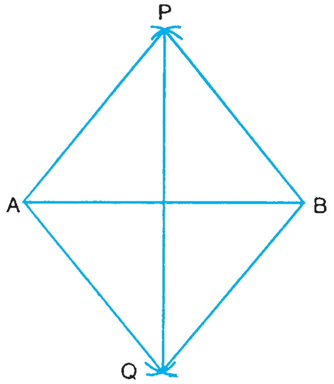


Fig. 10.65

3. Possible locations will be four points two points P and Q on the bisector of ∠AOC and two points R and S on the bisector of ∠BOC.

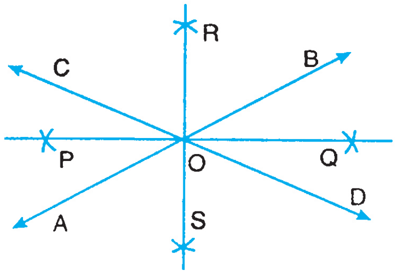


Fig. 10.66

4. Two on either side of AB and lines parallel to AB at a distance of 5 cm from AB.

ANSWERS TO TERMINAL EXERCISE

1. (a) y = 27 (b) = 126°
2. ∠1 = 48° and ∠2 = 132°
3. Third angle = 90°, Right triangle
4. ∠A = 35°, ∠B = 75° ∠C = 70°
7. 360°
8. 12°
11. A line parallel to locus *l* and m at a distance of 2.5 cm from each.
12. Point of intersection of the perpendicular bisector of AB and bisector of ∠BAC.

11

CONGRUENCE OF TRIANGLES

You might have observed that leaves of different trees have different shapes, but leaves of the same tree have almost the same shape. Although they may differ in size. The geometrical figures which have same shape and same size are called congruent figures and the property is called congruency.

In this lesson you will study congruence of two triangles, some relations between their sides and angles in details.

OBJECTIVES

After studying this lesson, you will be able to

- *verify and explain whether two given figures are congruent or not.*
- *state the criteria for congruency of two triangles and apply them in solving problems.*
- *prove that angles opposite to equal sides of a triangle are equal.*
- *prove that sides opposite to equal angles of a triangle are equal.*
- *prove that if two sides of triangle are unequal, then the longer side has the greater angle opposite to it.*
- *state and verify inequalities in a triangle involving sides and angles.*
- *solve problems based on the above results.*

EXPECTED BACKGROUND KNOWLEDGE

- Recognition of plane geometrical figures
- Equality of lines and angles
- Types of angles
- Angle sum property of a triangle
- Paper cutting and folding.

11.1 CONCEPT OF CONGRUENCE

In our daily life you observe various figures and objects. These figures or objects can be categorised in terms of their shapes and sizes in the following manner.

(i) Figures, which have different shapes and sizes as shown in Fig. 11.1

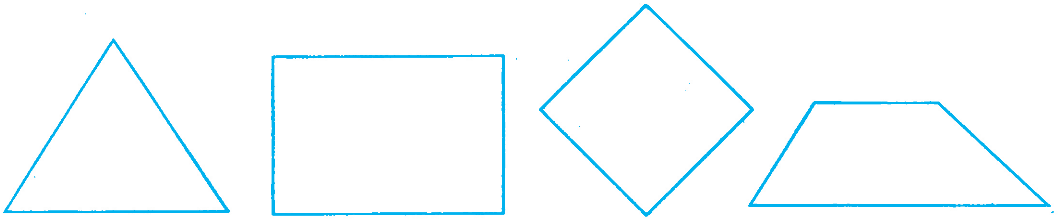


Fig. 11.1

(ii) Objcts, which have same shpaes but different sizes as shown in Fig. 11.2

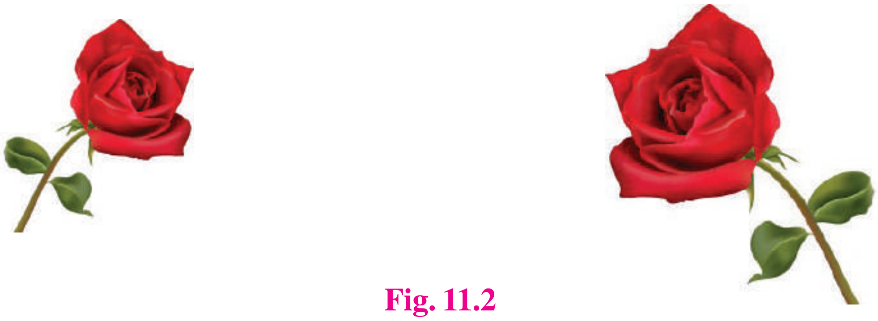


Fig. 11.2

(iii) Two one-rupee coins.

Fig. 11.3

(iv) Two postage stamps on post cards

Fig. 11.4

(v) Two photo prints of same size from the same negative.

Fig. 11.5

We will deal with the figures which have same shapes and same sizes.

Two figures, which have the same shape and same size are called congruent figures and this property is called congruence.

11.1.1. Activity

Take a sheet of paper, fold it in the middle and keep a carbon (paper) between the two folds. Now draw a figure of a leaf or a flower or any object which you like, on the upper part of the sheet. You will get a carbon copy of it on the sheet below.

The figure you drew and its carbon copy are of the same shape and same size. Thus, these are congruent figures. Observe a butterfly folding its two wings. These appear to be one.

11.1.2 Criteria for Congruence of Some Figures

Congruent figures, when palced one over another, exactly coincide with one another or cover each other. In other words, two figures will be congruent, if parts of one figure are equal to the corresponding parts of the other. For example :

(1) Two line - segments are congruent, when they are of equal length.

Fig. 11.6

(2) Two squares are congruent if their sides are equal.

Fig. 11.7

(3) Two circles are congruent, if their radii are equal, implying their circumferences are also equal.

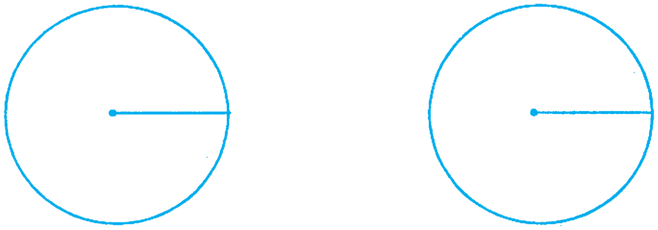


Fig. 11.8

11.2 CONGRUENCE OF TRIANGLES

Triangle is a basic rectilinear figure in geometry, having minimum number of sides. As such congruence of triangles plays a very important role in proving many useful results. Hence this needs a detailed study.

Two triangles are congruent, if all the sides and all the angles of one are equal to the corresponding sides and angles of other.

For example, in triangles PQR and XYZ in Fig. 11.9

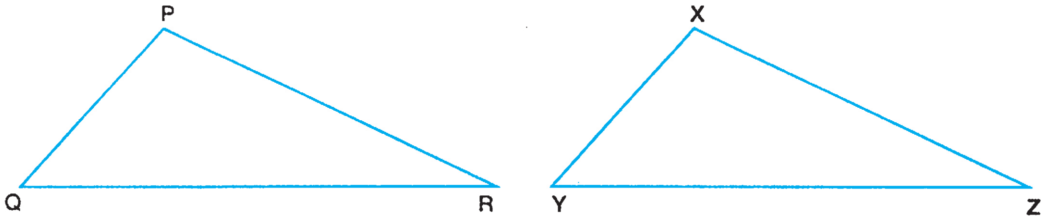


Fig. 11.9

$$PQ = XY,\ PR = XZ,\ QR = YZ$$

$$\angle P = \angle X,\ \angle Q = \angle Y \text{ and } \angle R = \angle Z$$

Thus we can say

Δ PQR is congruent to Δ XYZ and we write

$$\Delta PQR \cong \Delta XYZ$$

Relation of congruence between two triangles is always written with corresponding or matching parts in proper order.

Here $\Delta PQR \cong \Delta XYZ$

also means P corresponds to X, Q corresponds to Y and R corresponds to Z.

This congruence may also be written as Δ QRP $\cong \Delta$ YZX whichmeans, Q corresponds to Y, R corresponds to Z and P corresponds to X. It also means corresponding parts, (elements) are equal, namely

$$QR = YZ,\ RP = ZX,\ QP = YX,\ \angle Q = \angle Y,\ \angle R = \angle Z$$

and $\angle P = \angle X$

This congruence may also be written as

$$\Delta\, RPQ \cong \Delta\, ZXY$$

but NOT as $\Delta\, PQR \cong \Delta\, YZX.$

Or NOT as $\Delta\, PQR \cong \Delta\, ZXY.$

11. 3 CRITERIA FOR CONGRUENCE OF TRIANGLES

In order to prove, whether two triangles are congruent or not, we need to know that all the six parts of one triangle are equal to the corresponding parts of the other triangle. We shall now learn that it is possible to prove the congruence of two triangles, even if we are able to know the equality of three of their corresponding parts.

Consider the triangle ABC in Fig. 11.10

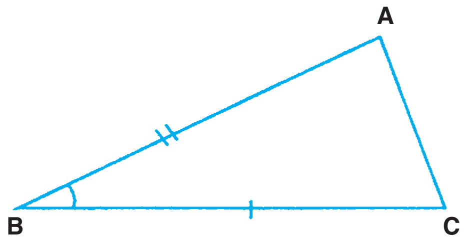


Fig. 11.10

Construct another triangle PQR such that QR = BC, $\angle Q = \angle B$ and PQ = AB. (See Fig. 11.11)

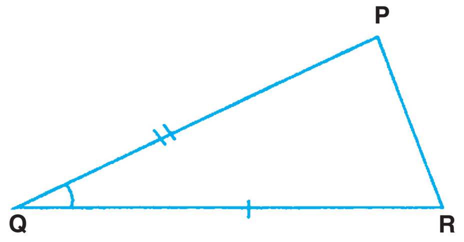


Fig. 11.11

If we trace or cut out triangle ABC and place it over triangle PQR. we will observe that one covers the other exactly. Thus, we may say that they are congruent.

Alternatively we can also measure the remaining parts, and observe that

$$AC = PR, \angle A = \angle P \text{ and } \angle C = \angle R$$

showing that $\Delta\ PQR \cong \Delta\ ABC$.

It should be noted here that in constructing $\Delta\, PQR$ congruent to $\Delta\, ABC$ we used only two parts of sides $PQ = AB$, $QR = BC$ and the included angle between them $\angle Q = \angle B$.

This means that equality of these three corresponding parts results in congruent triangles. Thus we have

Criterion 1 : If any two sides and the included angle of one triangle are equal to the corresponding sides and the included angle of the other triangle, the two triangles are congruent.

This criterion is referred to as SAS (Side Angle Side).

Again, consider $\Delta\, ABC$ in Fig. 11.12

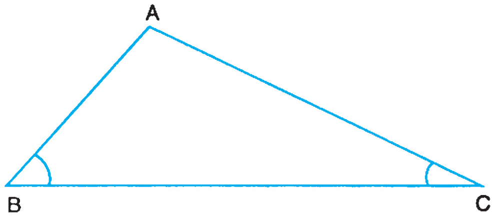


Fig. 11.12

Construct another $\Delta\, PQR$ such that, $QR = BC$, $\angle Q = \angle B$ and $\angle R = \angle C$. (See Fig. 11.13)

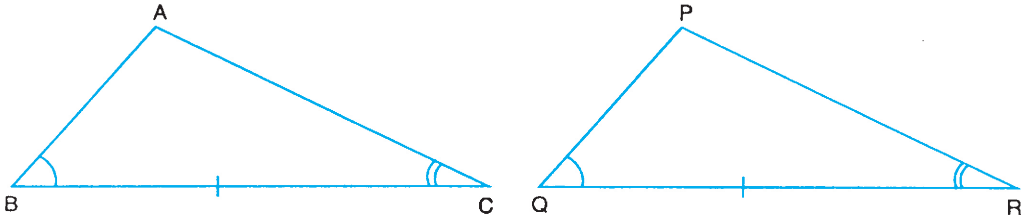


Fig. 11.13

By superimposition or by measuring the remaining corresponding parts, we observe that $\angle P = \angle A$, $PQ = AB$ and $PR = AC$ establishing that $\Delta\, PQR \cong \Delta\, ABC$, which again means that equality of the three corresponding parts (two angles and the inluded side) of two triangles results in congruent triangles.

We also know that the sum of the three angles of a triangle is 180^o, as such if two angles of one triangle are equal to the corresponding angles of another triangle, then the third angles will also be equal. Thus instead of included side we may have any pair of corresponding sides equal. Thus we have

Criterion 2 : If any two angles and one side of a triangle are equal to corresponding angles and the side of another triangle, then the two triangles are congruent.

This criterion is referred to as ASA or AAS (Angle Side Angle or Angle Angle Side)

11.3.1 Activity

In order to explore another criterion we again take a triangle ABC (See Fig. 11.14)

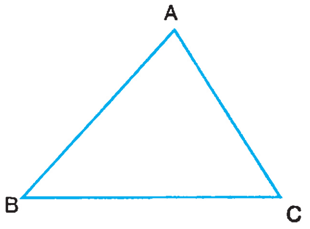


Fig. 11.14

Now take three thin sticks equal in lengths to sides AB, BC and CA of Δ ABC. Place them in any order to form Δ PQR or Δ P′Q′R′ near the Δ ABC (Fig. 11.15)

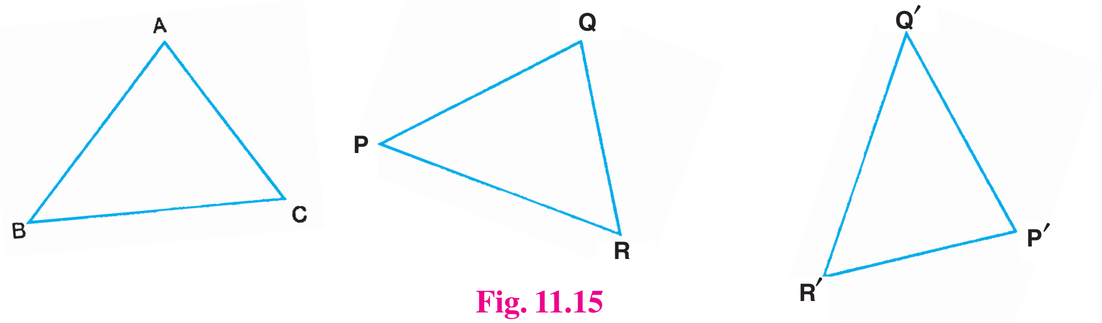


Fig. 11.15

By measuring the corresponding angles. we find that, $\angle P = \angle P' = \angle A$, $\angle Q = \angle Q' = \angle B$ and $\angle R = \angle R' = \angle C$, establishing that

$$\Delta\, PQR \;\cong\; \Delta\, P'Q'R' \cong \Delta\, ABC$$

which means that equality of the three corresponding sides of two triangles results in congruent triangles. Thus we have

Criterion 3 : If the three sides of one trianle are equal to the corresponding sides of another triangle, then the two triangles are congruent.

This is referred to as SSS (Side, Side, Side), criterion.

Similarly, we can establish one more criterion which will be applicable for two right trangles only.

Criterion 4 : If the hypotenuse and a side of one right triangle are respectively equal to the hypotenuse and a side of another right triangle, then the two triangles are congruent.

This criterion is referred to as RHS (Right Angle Hypotenuse Side).

Using these criteria we can easily prove, knowing three corresponding parts only, whether two triangles are congruent and establish the equality of remaining corresponding parts.

Example 11.1 : In which of the following criteria, two given triangles are **NOT** congruent.

(a) All corresponding sides are equal

(b) All corresponding angles are equal

(c) All corresponding sides and their included angles are equal

(d) All corresponding angles and any pair of corresponding sides are equal.

Ans. (b)

Example 11.2 : Two rectilinear figures are congruent if they have

(a) All corresponding sides equal

(b) All corresponding angles equal

(c) The same area

(d) All corresponding angles and all corresponding sides equal.

Ans. (d)

Example 11.3 : In Fig. 11.16, PX and QY are perpendicular to PQ and PX = QY. Show that AX = AY.

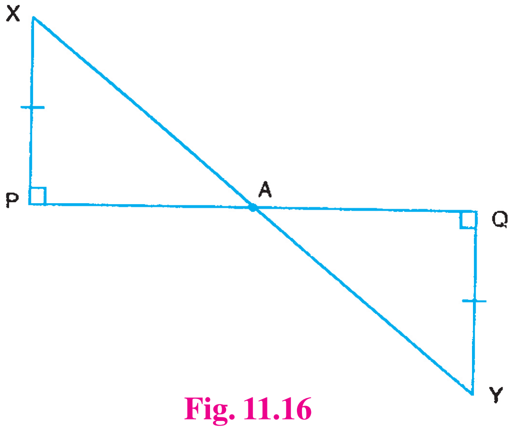


Fig. 11.16

Solution :

In Δ PAX and Δ QAY,

∠XPA = ∠YQA	(Each is 90^o)
∠PAX = ∠QAY	(Vertically opposite angles)

and PX = QY

$\therefore\ \Delta\ PAX \cong \Delta\ QAY$ (AAS)

$\therefore$ AX = AY.

Example 11.4 : In Fig. 11.17, Δ ABC is right triangle in which $\angle B = 90^0$ and D is the mid point of AC.

Prove that $BD = \frac{1}{2}$ AC.

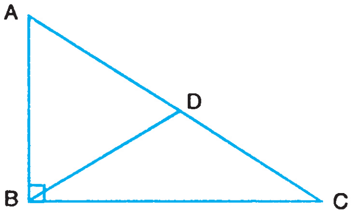


Fig. 11.17

Solution : Produce BD to E such that BD = DE. Join CE

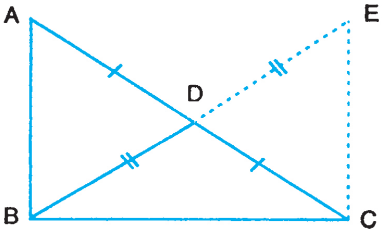


Fig.. 11.18

In Δ ADB and Δ CDE,

AD = CD (D being and point of AC)

DB = DE (By construction)

and $\angle ADB = \angle CDE$ (Vertically opposite angles)

$\therefore$ $\Delta\ ADB \cong \Delta\ CDE$ (i)

$\therefore$ AB = EC

Also $\angle DAB = \angle DCE$

But they make a pair of alternate angles

$\therefore$ AB is parallel to EC

$\therefore$ $\angle ABC + \angle ECB = 180^0$ (Pair of interior angles)

$\therefore\ \angle 90^0 + \angle ECB = 180^0$

$\therefore\ \angle ECB = 180^0 - 90^0 = 90^0$

Now in Δ ABC and Δ ECB,

$AB = EC$ (From (i) above)

$BC = BC$ (Common)

and $\angle ABC = \angle ECB$ (Each 90^0)

$\therefore\ \Delta ABC \cong \Delta ECB$

$\therefore\ AC = EB$

But $BD = \frac{1}{2} EB$

$\therefore\ BD = \frac{1}{2} AC$

CHECK YOUR PROGRESS 11.1

1. In Δ ABC (Fig. 11.19) if $\angle B = \angle C$ and $AD \perp BC$, then $\Delta ABD \cong \Delta ACD$ by the criterion.

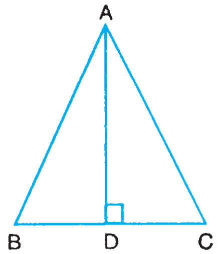


Fig. 11.19

(a) RHS (b) ASA

(c) SAS (d) SSS

2. In Fig. 11.20, $\Delta ABC \cong \Delta PQR$. This congruence may also be written as

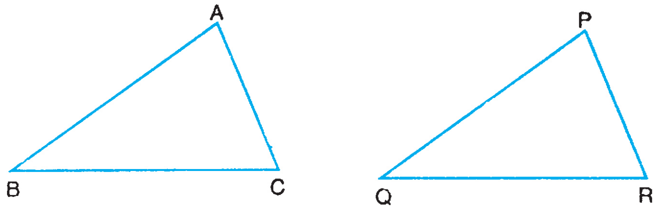


Fig. 11.20

(a) Δ BAC $\cong \Delta$ RPQ (b) Δ BAC $\cong \Delta$ QPR

(c) Δ BAC $\cong \Delta$ RQP (d) Δ BAC $\cong \Delta$ PRQ.

3. In order that two given triangles are congruent, along with equality of two corresponding angles we must know the equality of :

(a) No corresponding side

(b) Minimum one corresponding side

(c) Minimum two corresponding sides

(d) All the three corresponding sides

4. Two triangles are congruent if ....

(a) All three corresponding angles are equal

(b) Two angles and a side of one are equal to two angles and a side of the other.

(c) Two angles and a side of one are equal to two angles and the corresponding side of the other.

(d) One angle and two sides of one are equal to one angle and two sides of the other.

5. In Fig. 11.21, $\angle B = \angle C$ and AB = AC. Prove that Δ ABE $\cong \Delta$ ACD. Hence show that CD = BE.

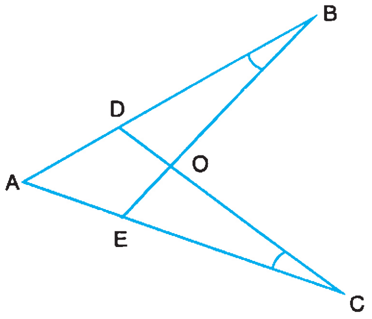


Fig. 11.21

6. In Fig. 11.22, AB is parallel to CD. If O is the mid-point of BC, show that it is also the mid-point of AD.

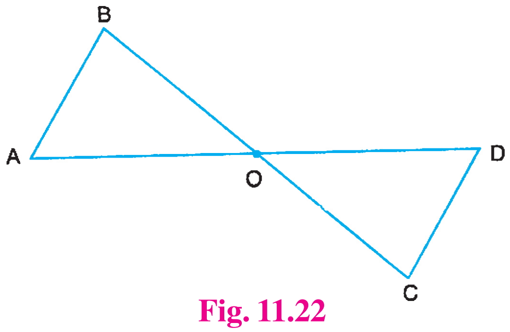


Fig. 11.22

7. In Δ ABC (Fig. 11.23), AD is ⊥ BC, BE is ⊥ AC and AD = BE. Prove that AE = BD.

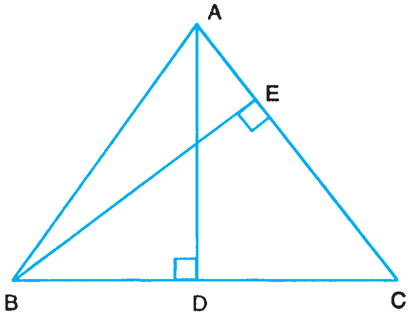


Fig. 11.23

8. From Fig. 11.24, show that the triangles are congruent and make pairs of equal angles.

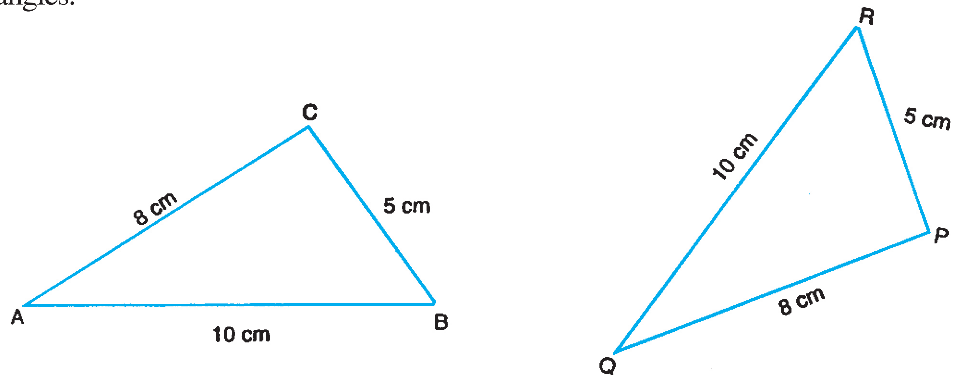


Fig. 11.24

11.4 ANGLES OPPOSITE TO EQUAL SIDES OF A TRIANGLE AND VICE VERSA

Using the criteria for congruence of triangles, we shall now prove some important theorems.

Theorem : The angles opposite to equal sides of a triangle are equal.

Given : A triangle ABC in which AB = AC.

To prove : $\angle B = \angle C$.

Construction : Draw bisector of $\angle BAC$ meeting BC at D.

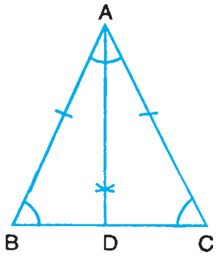


Fig. 11.25

Proof : In Δ ABD and Δ ACD,

AB = AC (Given)

$\angle BAD = \angle CAD$ (By construction)

and AD = AD (Common)

$\Delta\, ABD \cong \Delta\, ACD$ (SAS)

Hence $\angle B = \angle C$ (Corresponding parts of congruent triangles)

The converse of the above theorem is also true. We prove it as a theorem.

11.4.1 The sides opposite to equal angles of a triangle are equal

Given : A triangle ABC in which $\angle B = \angle C$

To prove : $AB = AC$

Construction : Draw bisector of $\angle BAC$ meeting BC at D.

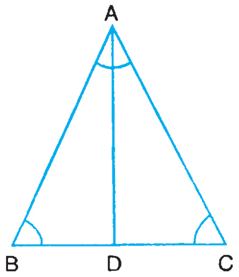


Fig. 11.26

Proof : In Δ ABD and Δ ACD,

$\angle B = \angle C$ (Given)

$\angle BAD = \angle CAD$ (By construction)

and $AD = AD$ (Common)

$\Delta\, ABD \cong \Delta\, ACD$ (SAS)

Hence $AB = AC$ (c.p.c.t)

Hence the theorem.

Example 11.5 : Prove that the three angles of an equilateral triangle are equal.

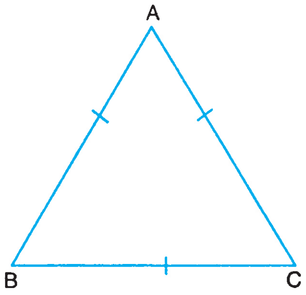


Fig. 11.27

Solution :

Given : An equilateral Δ ABC

To prove : $\angle A = \angle B = \angle C$

Proof : $AB = AC$ (Given)

$\therefore$ $\angle C = \angle B$ (Angles opposite equal sides) ...(i)

Also $AC = BC$ (Given)

$\therefore$ $\angle B = \angle A$ ...(ii)

From (i) and (ii),

$$\angle A = \angle B = \angle C$$

Hence the result.

Example 11.6 : ABC is an isosceles triangle in which $AB = AC$

(Fig. 11.28), If $BD \perp AC$ and $CE \perp AB$, proe that $BD = CE$.

Solution : In ΔBDC and ΔCEB

$\angle BDC = \angle CEB$ (Measure of each is 90°)

$\angle DCB = \angle EBC$ (Angles opposite equal sides of a triangle)

and $BC = CB$ (Common)

$\therefore$ $\Delta BDC \cong \Delta CEB$ (AAS)

Hence $BD = CE$ (c.p.c.t.)

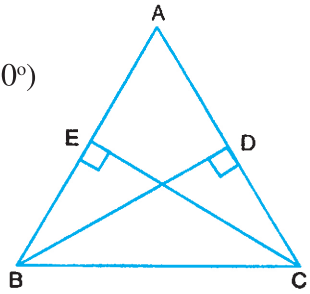


Fig. 11.28

This result can be stated in the following manner:

Perpendiculars (altitudes) drawn to equal sides, from opposite vertices of an isosceles triangle are equal.

The result can be extended to an equilateral triangle after which we can say that all the three altitudes of an equilateral triangle are equal.

Example : 11.7 : In ΔABC (Fig. 11.29), D and E are mid-points of AC and AB respectively.

If $AB = AC$, then prove that $BD = CE$.

Solution : $BE = \frac{1}{2} AB$

and $CD = \frac{1}{2} AC$

$\therefore$ $BE = CD$ ...(i)

In ΔBEC and ΔCDB,

$BE = CD$ [By (i)]

$BC = CB$ (Common)

and $\angle EBC = \angle DCB$ ($\because$ $AB = AC$)

$\therefore$ $\Delta BEC \cong \Delta CDB$ (SAS)

Hence, $CE = BD$ (c.p.c.t)

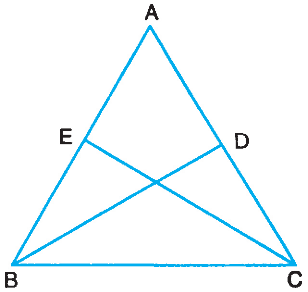


Fig. 11.29

Example 11.8 : In ΔABC (Fig. 11.30) $AB = AC$ and $\angle DAC = 124°$; find the angles of the triangle.

Solution $\angle BAC = 180° - 124° = 56°$

$\angle B = \angle C$

(Angles opposite to equal sides of a triangle)

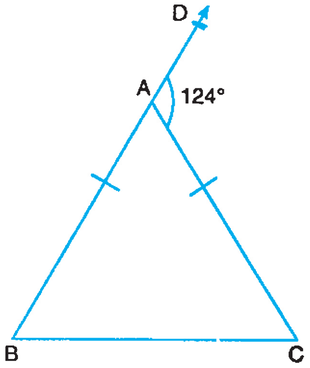


Fig. 11.30

Also $\angle B + \angle C = 124^{o}$

$$\angle B = \angle C = \frac{124^{0}}{2} = 62^{0}$$

Hence $\angle A = 56^{o}$, $\angle B = 62^{o}$, and $\angle C = 62^{o}$

CHECK YOUR PROGRESS 11.2

1. In Fig. 11.31, PQ = PR and SQ = SR. Prove that $\angle PQS = \angle PRS$.

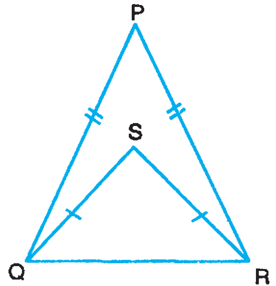


Fig. 11.31

2. Prove that ΔABC is an isosceles triangle, if the altitude AD bisects the base BC (Fig. 11.32).

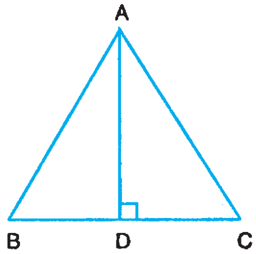


Fig. 11.32

3. If the line *l* in Fig. 11.33 is parallel to the base BC of the isosceles ΔABC, find the angles of the triangle.

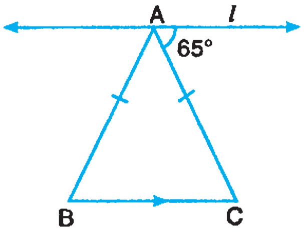


Fig. 11.33

4. ΔABC is an isosceles triangle such that AB = AC. Side BA is produced to a point D such that AB = AD. Prove that $\angle BCD$ is a right angle.

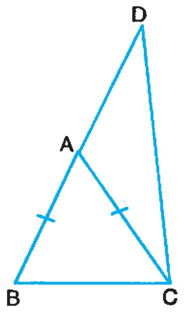


Fig. 11.34

5. In Fig. 11.35. D is the mid point of BC and perpendiculars DF and DE to sides AB and AC respectively are equal in length. Prove that ΔABC is an isosceles triangle.

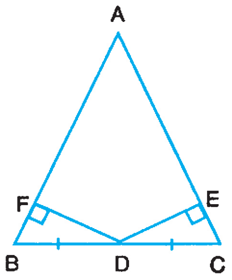


Fig. 11.35

6. In Fig. 11.36, PQ = PR, QS and RT are the angle bisectors of ∠Q and ∠R respectively. Prove that QS = RT.

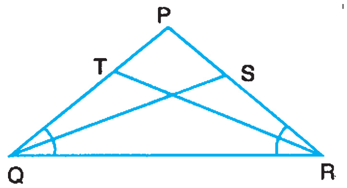


Fig. 11.36

7. ΔPQR and ΔSQR are isosceles triangles on the same base QR (Fig. 11.37). Prove that ∠PQS = ∠PRS.

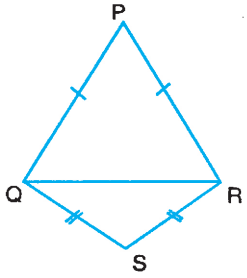


Fig. 11.37

8. In ΔABC, AB = AC (Fig. 11.38). P is a point in the interior of the triangle such that $\angle ABP = \angle ACP$. Prove that AP bisects ∠BAC.

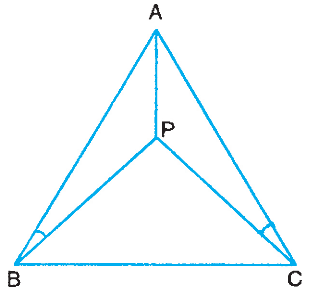


Fig. 11.38

11.5 INEQUALITIES IN A TRIANGLE

We have learnt the relationship between sides and angles of a triangle when they are equal. We shall now study some relations among sides and angles of a triangle, when they are unequal.

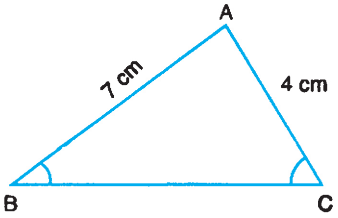


Fig. 11.39

In Fig. 11.39, triangle ABC has side AB longer than the side AC. Measure ∠B and ∠C. You will find that these angles are not equal and ∠C is greater than ∠B. If you repeat this experiment, you will always find that this observation is true. This can be proved easily, as follows.

11.5.1 Theorem

If two sides of a triangle are unequal, then the longer side has the greater angle opposite to it.

Given. A triangle ABC in which AB > AC.

To prove. ∠ACB > ∠ABC

Construction. Make a point D on the side AB such that AD = AC and join DC.

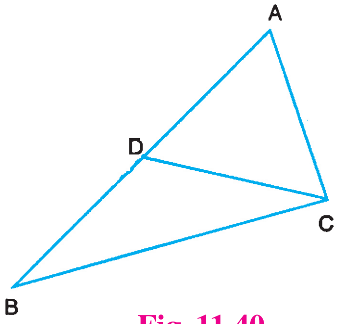


Fig. 11.40

Proof: In ΔACD,

AD = AC

∴ ∠ACD = ∠ADC (Angles opposite equal sides)

But $\angle ADC > \angle ABC$

(Exterior angle of a triangle is greater than opposite interior angle)

Again $\angle ACB > \angle ACD$ (Point D lies in the interior of the $\angle ACB$).

$\therefore$ $\angle ACB > \angle ABC$

What can we say about the converse of this theorem. Let us examine.

In ΔABC, (Fig. 11.41) compare $\angle C$ and $\angle B$. It is clear that $\angle C$ is greater than $\angle B$. Now compare sides AB and AC opposite to these angles by measuring them. We observe that AB is longer than AC.

Again compare $\angle C$ and $\angle A$ and measure sides AB and BC opposite to these angles. We observe that $\angle C > \angle A$ and AB > BC; i.e. side opposite to greater angle is longer.

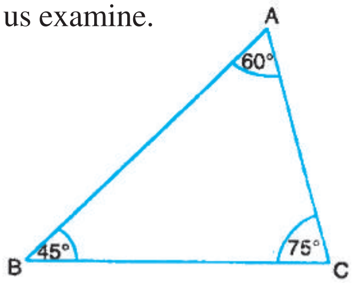


Fig. 11.41

Comparing $\angle A$ and $\angle B$, we observe a similar result. $\angle A > \angle B$ and BC > AC; i.e. side opposite to greater angle is longer.

You can also verify this property by drawing any type of triangle, a right triangle or an obtuse triangle.

Measure any pair of angles in a triangle. Compare them and then compare the sides opposite to them by measurement. You will find the above result always true, which we state as a property.

In a triangle, the greater angle has longer side opposite to it.

Observe that in a triangle if one angle is right or an obtuse then the side opposite to that angle is the longest.

You have already learnt the relationship among the three angles of a triangle i.e., the sum of the three angles of a triangle is 180°. We shall now study whether the three sides of a triangle are related in some way.

Draw a triangle ABC.

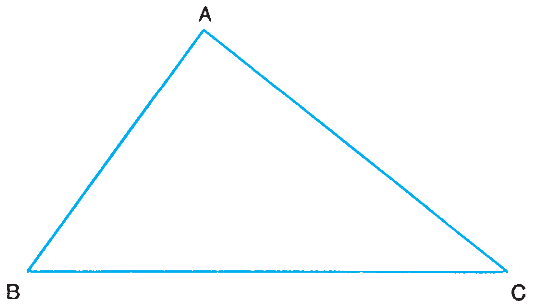


Fig. 11.42

Measure its three sides AB, BC and CA.

Now find the sum of different pairs AB+BC, BC+CA, and CA+AB separately and compare each sum of a pair with the third side, we observe that

(i) $AB + BC > CA$

(ii) $BC + CA > AB$ and

(iii) $CA + AB > BC$

Thus we conclude that

Sum of any two sides of a triangle is greater than the third side.

ACTIVITY

Fix three nails P, Q and R on a wooden board or any surface.

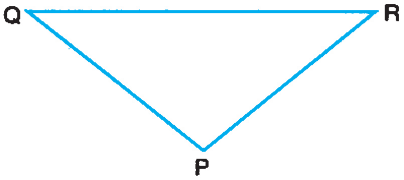


Fig. 11.43

Take a piece of thread equal in length to QR and another piece of thread equal in length (QP + PR). Compare the two lengths, you will find that the length corresponding to (QP + PR) > the length corresponding to QR confirming the above property.

Example 11.9 : In which of the following four cases, is construction of a triangle possible from the given measurements

(a) 5 cm, 8 cm and 3 cm

(b) 14 cm, 6 cm and 7 cm

(c) 3.5 cm, 2.5 cm and 5.2 cm

(d) 20 cm, 25 cm and 48 cm.

Solution. In (a) $5 + 3 \ngtr 8$, in (b) $6 + 7 \ngtr 14$

in (c) $3.5 + 2.5 > 5.2$, $3.5 + 5.2 > 2.5$ and $2.5 + 5.2 > 3.5$ and

in (d) $20 + 25 \ngtr 48$.

Ans. (c)

Example 11.10 : In Fig. 11.44, AD is a median of ΔABC. Prove that $AB + AC > 2AD$.

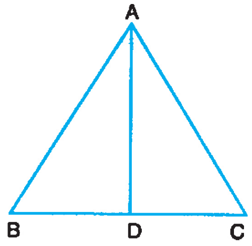


Fig. 11.44

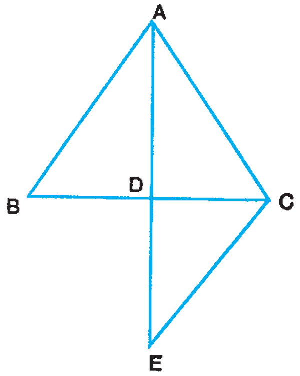


Fig. 11.45

Solution: Produce AD to E such that AD = DE and join C to E.

Consider ΔABD and ΔECD

Here, $BD = CD$

$\angle ADB = \angle EDC$

and $AD = ED$

$\therefore \quad \Delta ABC \cong \Delta ECD$

$\therefore \quad AB = EC$

Now in ΔACE,

$EC + AC > AE$

or $AB + AC > 2AD$ $\quad (\therefore\ AD = ED \Rightarrow AE = 2AD)$

CHECK YOUR PROGRESS 11.3

1. PQRS is a quadrilateral in which diagonals PR and QS intersect at O. Prove that $PQ + QR + RS + SP > PR + QS$.
2. In triangle ABC, AB = 5.7 cm, BC = 6.2 cm and CA = 4.8 cm. Name the greatest and the smallest angle.
3. In Fig. 11.46, if $\angle CBD > \angle BCE$ then prove that $AB > AC$.

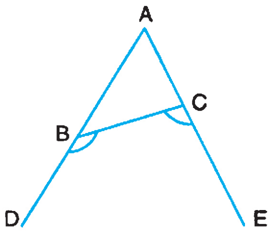


Fig. 11.46

4. In Fig. 11.47, D is any point on the base BC of a ΔABC. If AB > AC then prove that AB > AD.

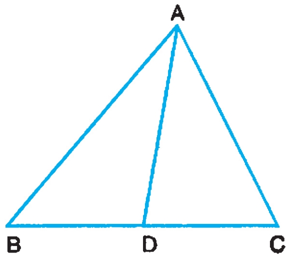


Fig. 11.47

5. Prove that the sum of the three sides of triangle is greater than the sum of its three medians.

(Use Example 11.10)

6. In Fig. 11.48, if AB = AD then prove that BC > CD.

[Hint : ∠ADB = ∠ABD].

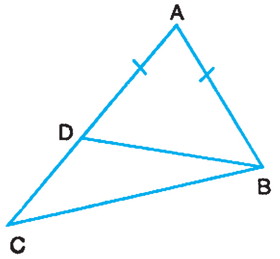


Fig. 11.48

7. In Fig. 11.49, AB is parallel to CD. If ∠A > ∠B then prove that BC > AD.

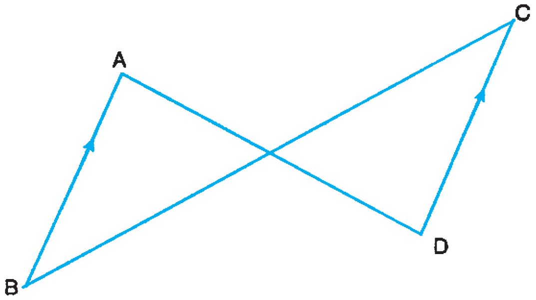


Fig. 11.49

LET US SUM UP

- Figures which have the same shape and same size are called congruent figures.
- Two congruent figures, when placed one over the other completely cover each other. All parts of one figure are equal to the corresponding parts of the other figure.

- To prove that two triangles are congruent we need to know the equality of only three corresponding parts. These corresponding parts must satisfy one of the four criteria.
  (i) SAS (ii) ASA or AAS
  (iii) SSS (iv) RHS
- Angles opposite to equal sides of a triangle are equal.
- Sides opposite to equal angles of a triangle are equal.
- If two sides of a triangle are unequal, then the longer side has the greater angle opposite to it.
- In a triangle, the greater angle has the longer side opposite to it.
- Sum of any two sides of a triangle is greater than the third side.

TERMINAL EXERCISE

1. Two lines AB and CD bisect each other at O. Prove that CA = BD (Fig. 11.50)

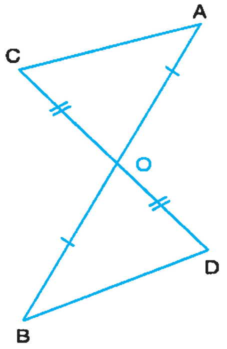


Fig. 11.50

2. In a ΔABC, if the median AD is perpendicular to the base BC then prove that the triangle is an isosceles triangle.

3. In Fig. 11.51, ΔABC and ΔCDE are such that BC = CE and AB = DE. If ∠B = 60°, ∠ACE = 30° and ∠D = 90°, then prove that the two triangles are congruent.

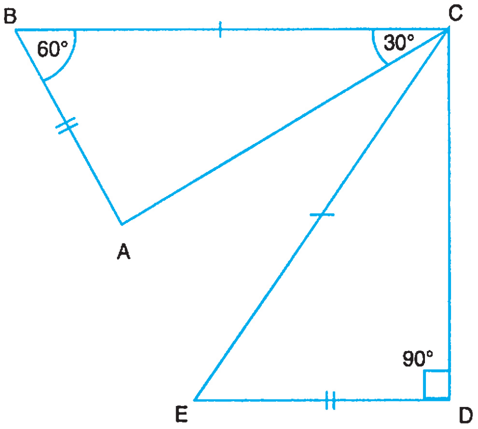


Fig. 11.51

4. In Fig. 11.52 two sides AB and BC and the altitude AD of ΔABC are respectively equal to the sides PQ and QR and the altitude PS, Prove that $\Delta ABC \cong \Delta PQR$.

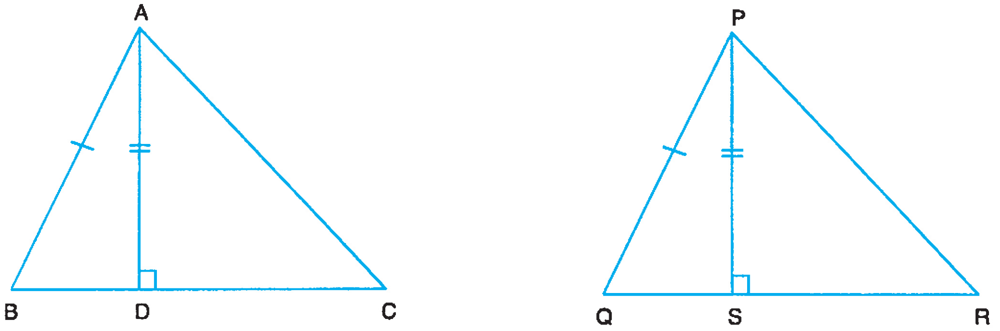


Fig. 11.52

5. In a right triangle, one of the acute angles is 30^o. Prove that the hypotenuse is twice the side opposite to the angle of 30^o.

6. Line segments AB and CD intersect each other at O such that O is the midpoint of AB. If AC is parallel to DB then prove that O is also the mid piont of CD.

7. In Fig. 11.53, AB is the longest side and DC is the shortest side of a quadrilateral ABCD. Prove that $\angle C > \angle A$ and $\angle D > \angle B$. [Hint : Join AC and BD].

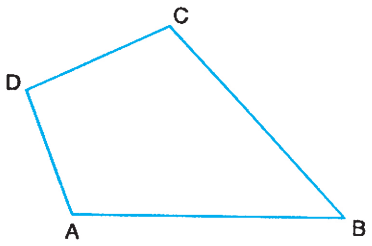


Fig. 11.53

8. ABC is an isosceles triangle in which AB = AC and AD is the altitude from A to the base BC. Prove that BD = DC.

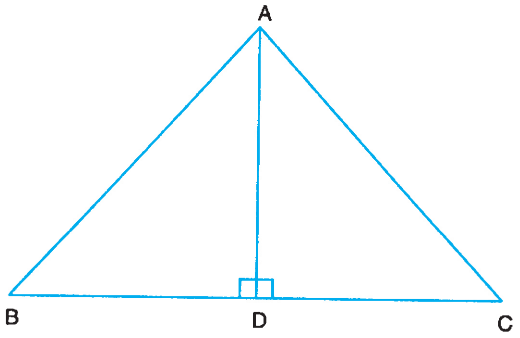


Fig. 11.54

9. Prove that the medians bisecting the equal sides of an isosceles triangle are also equal. [Hint : Show that $\Delta DBC \cong \Delta ECB$]

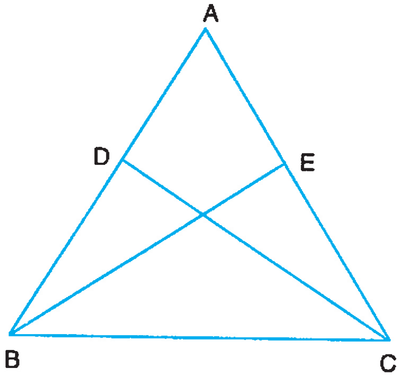


Fig. 11.55

ANSWERS TO CHECK YOUR PROGRESS

11.1

1. (a) 2. (b)

3. (b) 4. (c)

8. $\angle P = \angle C$ $\angle Q = \angle A$ and $\angle R = \angle B$.

11.2

3. $\angle B = \angle C = 65^o$, $\angle A = 50^o$

11.3

2. Greatest angle is A and smallest angle is B.

12

CONCURRENT LINES

You have already learnt about concurrent lines, in the lesson on lines and angles. You have also studied about triangles and some special lines, i.e., medians, right bisectors of sides, angle bisectors and altitudes, which can be drawn in a triangle. In this lesson, we shall study the concurrency property of these lines, which are quite useful.

OBJECTIVES

After studying this lesson, you will be able to

- *define the terms concurrent lines, median, altitude, angle bisector and perpendicular bisector of a side of a triangle.*
- *Verify the concurrnence of medians, altitudes, perpendicular bisectors of sides and angle bisectors of a triangle.*

EXPECTED BACKGROUND KNOWLEDGE

Properties of intersecting lines, such as:

- Two lines in a plane can either be parallel [See Fig 12.1 (a)] or intersecting (See Fig. 12.1 (b) and (c)].

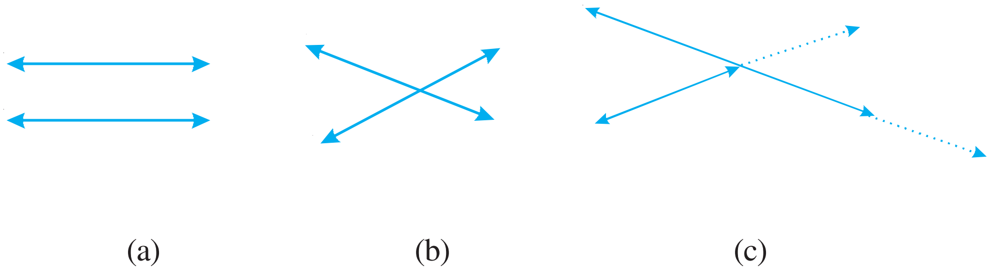
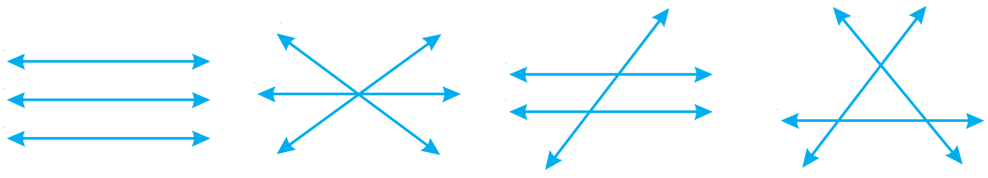

(a) (b) (c)

Fig. 12.1

- Three lines in a plane may
  - (i) be paralled to each other, i.e., intersect in no point [See Fig. 12.2 (a)] or
  - (ii) intersect each other in exactly one point [Fig. 12.2(b)], or
  - (iii) intersect each other in two points [Fig. 12.2(c)], or
  - (iv) intersect each other at the most in three points [Fig. 12.2(d)]

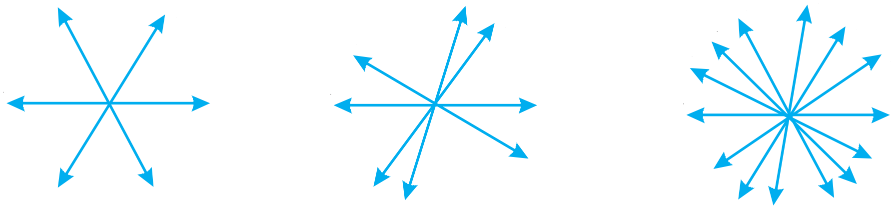


(a) (b) (c) (d)

Fig. 12.2

12.1 CONCURRENT LINES

Three lines in a plane which intersect each other in exactly one point or which pass through the same point are called **concurrent lines** and the common **point** is called the **point of concurrency** (See Fig. 12.3).



(a) (b) (c)

Fig. 12.3

12.1.1 Angle Bisectors of a Triangle

In triangle ABC, the line AD bisects ∠A of the triangle. (See Fig. 12.4)

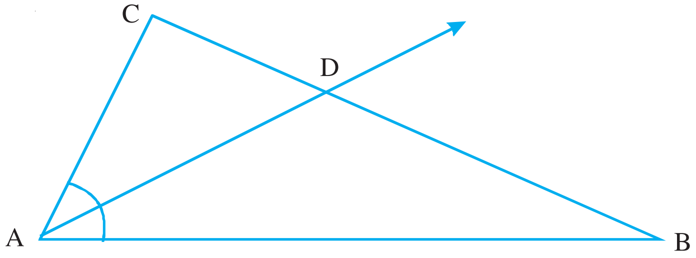


Fig. 12.4

A line which bisects an angle of a triangle is called an **angle bisector** of the triangle.

How many angle bisectors can a triangle have? Since a triangle has three angles, we can draw three angle bisectors in it. AD is one of the three angle bisectors of ΔABC. Let us draw second angle bisector BE of ∠B (See Fig. 12.5)

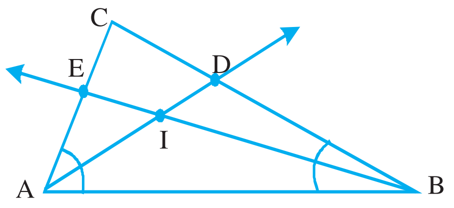


Fig. 12.5

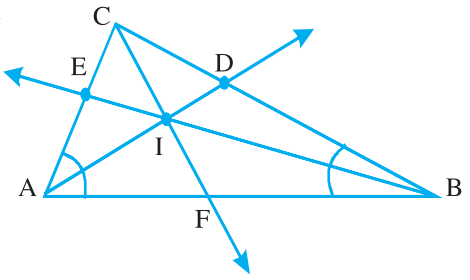


Fig. 12.6

The two angle bisectors of the ΔABC intersect each other at I. Let us draw the third angle bisector CF of ∠C (See Fig. 12.6). We observe that this angle bisector of the triangle also passes through I. In other words they are concurrent and the point of concurrency is I.

We may take any type of triangle—acute, right or obtuse triangle, and draw its angle bisectors, we will always find that the three angle bisectors of a triangle are concurrent (See Fig. 12.7)

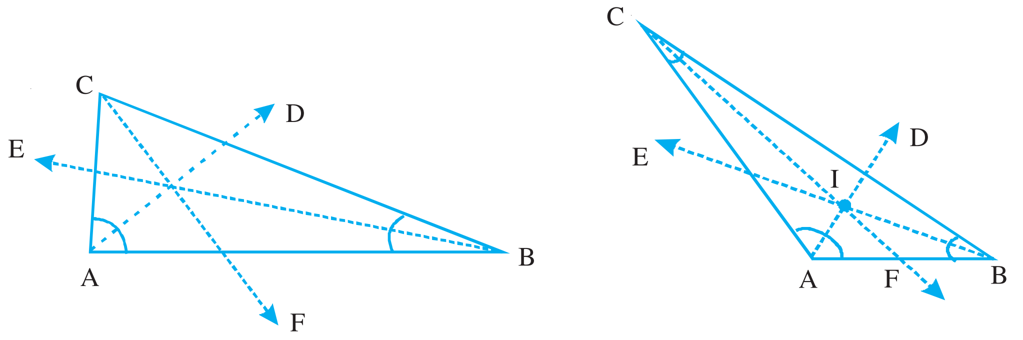


Fig. 12.7

Thus we conclude the following:

Angle bisectors of a triangle pass through the same point, that is they are concurrent

The point of concurrency I is called the 'Incentre' of the triangle.

Can you reason out, why the name incentre for this point?

Recall that the locus of a point equidistant from two intersecting lines is the pair of angle bisectors of the angles formed by the lines. Since I is a point on the bisector of ∠BAC, it must be equidistant from AB and AC. Also I is a point on angle bisector of ∠ABC, (See

Fig. 12.8), it must also be equidistant from AB and BC. Thus point of concurrency I is at the same distance from all the three sides of the triangle.

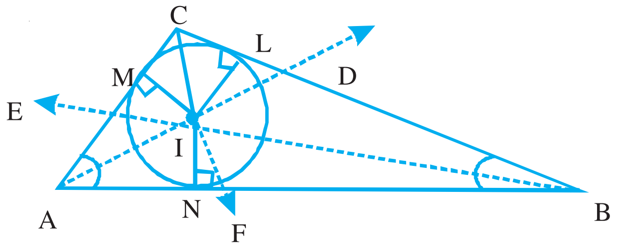


Fig. 12.8

Thus, we have IL = IM = IN (Fig. 12.8). Taking I as the centre and IL as the radius, we can draw a circle touching all the three sides of the triangle called **'Incircle'** of the triangle. I being the centre of the incircle is called the **Incentre** and IL, the radius of the incircle is called the inradius of the triangle.

Note: The incentre always lies in the interior of the triangle.

12.1.2: Perpendicular Bisectors of the Sides of a Triangle

ABC is a triangle, line DP bisects side BC at right angle. A line which bisects a side of a triangle at right angle is called the perpendicular bisector of the side. Since a triangle has three sides, so we can draw three perpendicular bisectors in a triangle. DP is one of the three perpendicular bisectors of ΔABC (Fig. 12.9). We draw the second perpendicular bisector EQ, intersecting DP at O (Fig. 12.10). Now if we also draw the third perpendicular bisector FR, then we observe that it also passes through the point O (Fig. 12.11). In other words, we can say that the three perpendicular bisectors of the sides are concurrent at O.

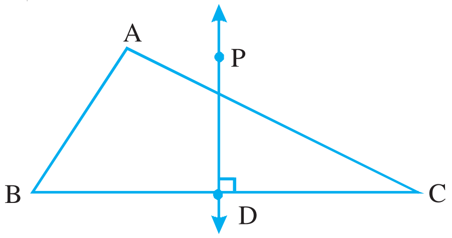


Fig. 12.9

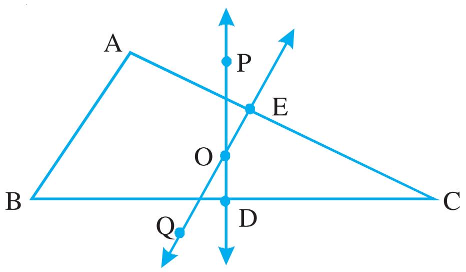


Fig. 12.10

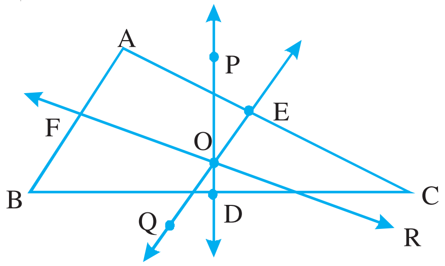
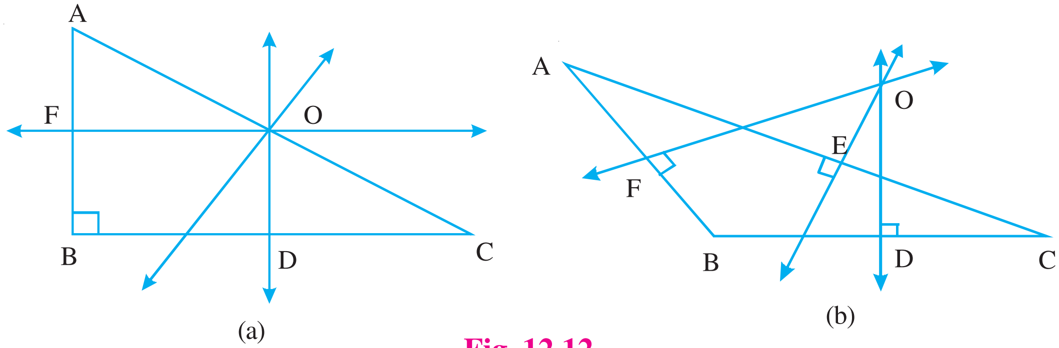


Fig. 12.11

We may repeat this experiment with any type of triangle, but we will always find that the three perpendicular bisectors of the sides of a triangle pass through the same point.



(a) (b)

Fig. 12.12

Thus we conclude that:

The three perpendicular bisectors of the sides of a triangle pass through the same point, that is, they are concurrent.

The point of concurrency O is called the 'circumcentre' of the triangle

Can you reason out: why the name circumcentre for this point?

Recall that the locus of a point equidistant from two given points is the perpendicular bisector of the line joining the two points. Since O lies on the perpendicular bisector of BC, so it must be equidistant from both the point B and C i.e., BO = CO (Fig. 12.13).

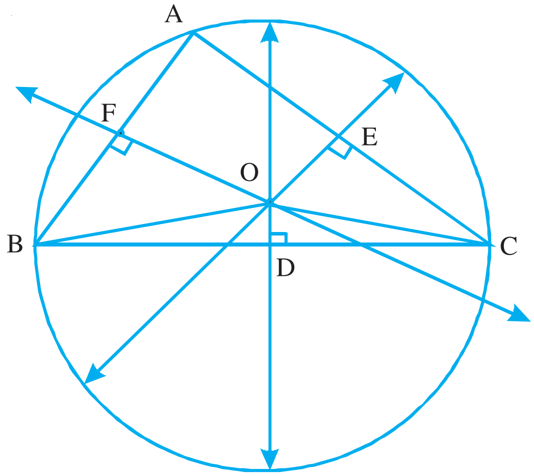


Fig. 12.13

The point O also lies on the perpendicular bisector of AC, so it must be equidistant from both A and C, that is, AO = CO. Thus, we have AO = BO = CO.

If we take O as the centre and AO as the radius, we can draw a circle passing through the three vertices, A, B and C of the triangle, called '**Circumcircle**' of the triangle. O being the centre of this circle is called the **circumcentre** and AO the radius of the circumcircle is called **circumradius** of the triangle.

Note that the circumcentre will be

1. in the interior of the triangle for an acute triangle (Fig. 12.11)
2. on the hypotenuse for a right triangle [Fig. 12.12(a)]
3. in the exterior of the triangle for an obtuse triangle [Fig. 12.12(b)].

12.1.3 Altitudes of a Triangle

In ΔABC, the line AL is the perpendicular drawn from vertex A to the opposite side BC. (Fig. 12.14).

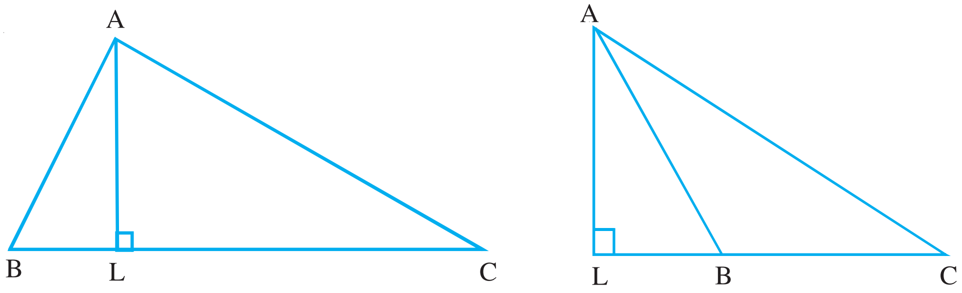


Fig. 12.14

Perpendicular drawn from a vertex of a triangle to the oposite side is called its altitude. How many altitudes can be drawn in a triangle? There are three vertices in a triangle, so we can draw three of its altitudes. AL is one of these altitudes. Now we draw the second altitude BM, which intersects the first altitude at a point H (see Fig. 12.15). We also draw the third altitude CN and observe that it also passes through the point H (Fig. 12.16). This shows that the three altitudes of the triangle pass through the same point.

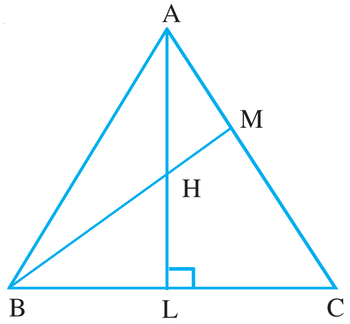


Fig. 12.15

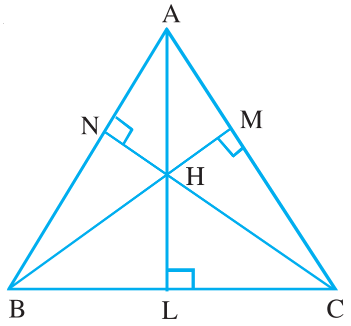


Fig. 12.16

We may take any type of triangle and draw its three altitudes. We always find that the three altitudes of a triangle are concurrent.

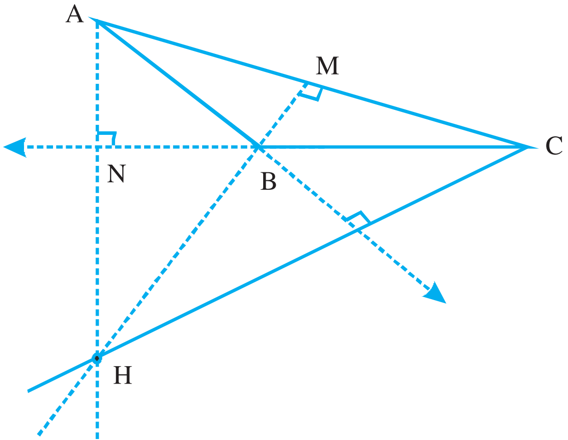


Fig. 12.17

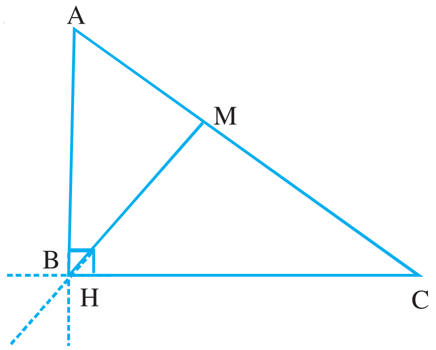


Fig. 12.18

Thus we conclude that:

In a triangle, the three altitudes pass through the same point, that is, they are concurrent.

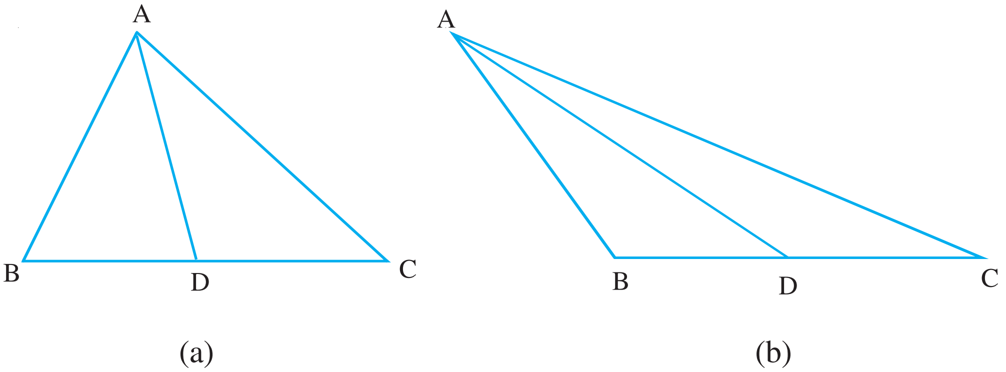
The point of concurrency is called the 'Orthocentre'of the triangle.

Again observe that the orthocentre will be

1. in the interior of the triangle for an acute triangle (Fig. 12.16)
2. in the exterior of the triangle for an obtuse triangle (Fig. 12.17)
3. at the vertex containing the right angle for a right triangle (Fig. 12.18)

12.1.4 Medians of a Triangle

In ΔABC, AD joins the vertex A to the mid point D of the opposite side BC (Fig. 12.19)



(a)

(b)

Fig. 12.19

A line joining a vertex to the mid point of the opposite side of a triangle is called its median. Clearly, three medians can be drawn in a triangle. AD is one of the medians. If we draw all the three medians in any triangle, we always find that the three medians pass through the same point [Fig. 12.20 (a), (b), (c)]

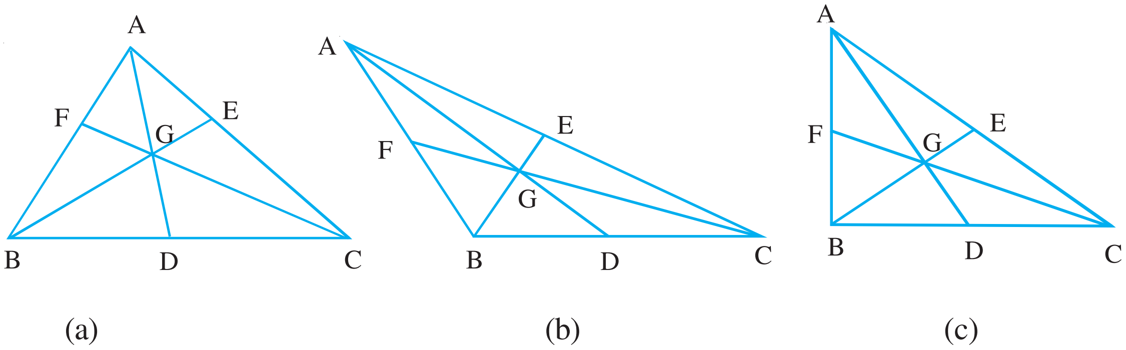


Fig. 12.20

Here in each of the triangles ABC given above (Fig. 12.20) the three medians AD, BE and CF are concurrent at G.. In each triangle we measure the parts into which G divides each median. On measurement, we observe that

AG = 2GD, BG = 2GE

and CG = 2 GF

that is, the point of concurrency G divides each of the medians in the ratio 2 : 1.

Thus we conclude that:

Medians of a triangle pass through the same point, which divides each of the medians in the ratio 2 : 1.

The point of concurrency G is called the 'centroid'of the triangle.

ACTIVITY FOR YOU

Cut out a triangle from a piece of cardboard. Draw its three medians and mark the centroid G of the triangle. Try to balance the triangle by placing the tip of a pointed stick or a needle of compasses below the point G or at G. If the position of G is correctly marked then the weight of the triangle will balance at G (Fig. 12.21).

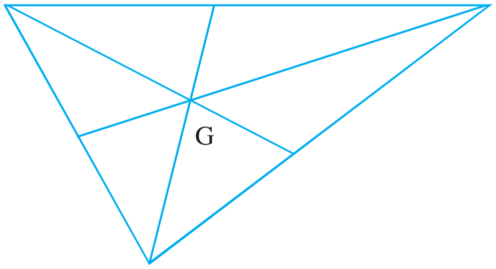


Fig. 12.21

Can you reason out, why the point of concurrency of the medians of a triangle is called its centroid. It is the point where the weight of the triangle is centered or it is the point through which the weight of the triangle acts.

We consider some examples using these concepts.

Example 12.1: In an isosceles triangle, show that the bisector of the angle formed by the equal sides is also a perpendicular bisector, an altitude and a median of the triangle.

Solution: In ΔABD and ΔACD,

$AB = AC$ (Given)

$\angle BAD = \angle CAD$ [$\because$ AD is bisector of $\angle A$]

and $AD = AD$

$\therefore$ $\Delta ABD \cong \Delta ACD$

$\therefore$ $BD = CD$

$\Rightarrow$ AD is also a median

$\Rightarrow$ Also $\angle ADB = \angle ADC = 90^o$

$\Rightarrow$ AD is an altitude

Since, BD = DC,

AD is perpendicular bisector of side BC.

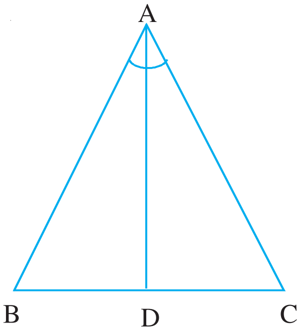


Fig. 12.22

Example 12.2: In an equilateral triangle, show that the three angle bisectors are also the three perpendicular bisectors of sides, three altitudes and the three medians of the triangle.

Solution: Since AB = AC

Therefore, AD, the bisector of $\angle A$ is also a perpendicular bisector of BC, an altitude and a median of the ΔABD.

(Refer Example 12.1 above)

Similarly, since AB = BC and BC = AC

$\therefore$ BE and CF, angle bisectors of $\angle B$ and $\angle C$ respectively, are also perpendicular bisectors, altitudes and medians of the ΔABC.

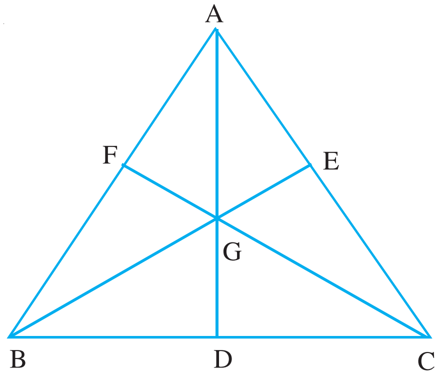


Fig. 12.23

Example 12.3: Find the circumradius of circumcircle and inradius of incircle of an equilateral triangle of side *a*.

Solution: We draw perpendicular from the vertex A to the side BC.

AD is also the angle bisector of $\angle A$, perpendicular bisector of side BC and a median joining vertex to the midpoint of BC.

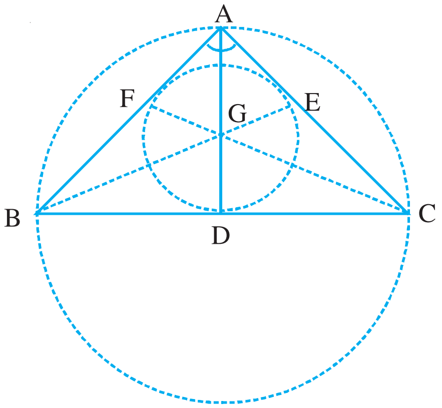


Fig. 12.24

$$\therefore \quad AD = \frac{\sqrt{3}}{2}a, \text{ as } BC = a$$

$$\Rightarrow \quad AG = \text{circumradius in this case} = \frac{2}{3}\times\frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{3}a$$

and $\quad GD = \text{inradius in this case} = \frac{1}{3}\times\frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{6}a$.

CHECK YOUR PROGRESS 12.1

1. In the given figure BF = FC, $\angle BAE = \angle CAE$ and $\angle ADE = \angle GFC = 90^o$, then name a median, an angle bisector, an altitude and a perpendicular bisector of the triangle.

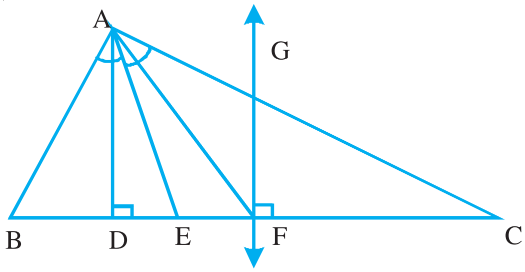


Fig. 12.25

2. In an equilateral triangle show that the incentre, the circumcentre, the orthocentre and the centroid are the same point.

3. In an equilateral ΔABC (Fig. 12.26). G is the centroid of the triangle. If AG is 4.8 cm, find AD and BE.

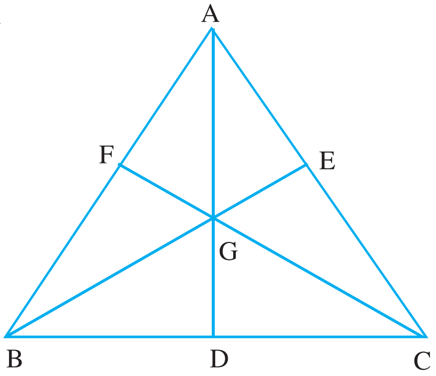


Fig. 12.26

4. If H is the orthocentre of ΔABC, then show that A is the orthocentre of the ΔHBC.
5. Choose the correct answers out of the given alternatives in the following questions:

   (i) In a plane, the point equidistant from vertices of a triangle is called its

   (a) centroid (b) incentre

   (c) circumcentre (d) orthocentre

   (ii) In the plane of a triangle, the point equidistant from the sides of the triangle is called its

   (a) centroid (b) incentre

   (c) circumcentre (d) orthocentre

LET US SUM UP

- Three or more lines in a plane which intersect each other in exactly one point are called concurrent lines.
- A line which bisects an angle of a triangle is called an angle bisector of the triangle.
- A line which bisects a side of a triangle at right angle is called the perpendicular bisector of the side of the triangle.
- A line drawn perpendicular from a vertex of a triangle to its opposite side is called an altitude of the triangle.
- A line which joins a vertex of a triangle to the mid-point of the opposite side is called a median of the triangle.
- In a triangle

  (i) angle bisectors are concurrent and the point of concurrency is called **incentre.**

(ii) perpendicular bisectors of the sides are concurrent and the point of concurrency is called **circumcentre.**

(iii) altitudes are concurrent and the point of concurrency is called **orthocentre.**

(iv) medians are concurrent and the point of concurrency is called **centroid,** which divides each of the medians in the ratio 2 : 1.

TERMINAL EXERCISE

1. In the given Fig. 12.27, D, E and F are the mid points of the sides of ΔABC. Show that $BE+CF>\frac{3}{2}BC$.

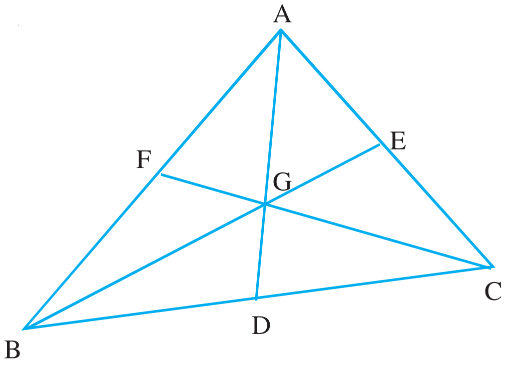


Fig. 12.27

2. ABC is an isoceles triangle such that AB = AC and D is the midpoint of BC. Show that the centroid, the incentre, the circumcentre and the orthocentre, all lie on AD.

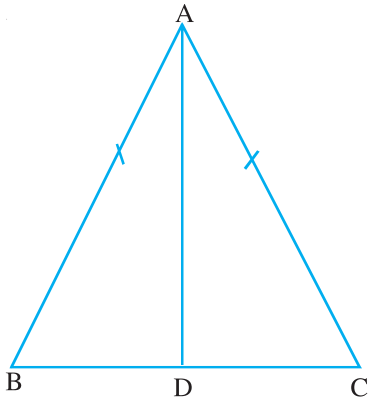


Fig. 12.28

3. ABC is an isoceles triangle such that AB = AC = 17 cm and base BC = 16 cm. If G is the centroid of ΔABC, find AG.

4. ABC is an equilateral triangle of side 12 cm. If G be its centroid, find AG.

ACTIVITIES FOR YOU

1. Draw a triangle ABC and find its circumcentre. Also draw the circumcircle of the triangle.
2. Draw an equilateral triangle. Find its incentre and circumcentre. Draw its incircle and circumcircle.
3. Draw the circumcircle and the incircle for an equilateral triangle of side 5 cm.

ANSWERS TO CHECK YOUR PROGRESS

12.1

1. Median - AF, Angle bisector AE

   Altitude - AD and perpendicular bisector - GF

3. AD = 7.2 cm, BE = 7.2 cm

5. (i) (c) (ii) (b)

ANSWERS TO TERMINAL EXERCISE

3. AG = 10 cm

4. AG = $4\sqrt{3}$ cm

13

QUADRILATERALS

If you look around, you will find many objects bounded by four line-segments. Any surface of a book, window door, some parts of window-grill, slice of bread, the floor of your room are all examples of a closed figure bounded by four line-segments. Such a figure is called a quadrilateral.

The word quadrilateral has its origin from the two words "quadric" meaining four and "lateral" meaning sides. Thus, a quadrilateral is that geometrical figure which has four sides, enclosing a part of the plane.

In this lesson, we shall study about terms and concepts related to quadrilateral with their properties.

OBJECTIVES

After studying this lesson, you will be able to

- *describe various types of quadrilaterals viz. trapeziums, parallelograms, rectangles, rhombuses and squares;*
- *verify properties of different types of quadrilaterals;*
- *verify that in a triangle the line segment joining the mid-points of any two sides is parallel to the third side and is half of it;*
- *verify that the line drawn through the mid-point of a side of a triangle parallel to another side bisects the third side;*
- *verify that if there are three or more parallel lines and the intercepts made by them on a transversal are equal, the corresponding intercepts on any other transversal are also equal;*
- *verify that a diagonal of a parallelogram divides it into two triangles of equal area;*
- *solve problem based on starred results and direct numerical problems based on unstarred results given in the curriculum;*

- *prove that parallelograms on the same or equal bases and between the same parallels are equal in area;*
- *verify that triangles on the same or equal bases and between the same parallels are equal in area and its converse.*

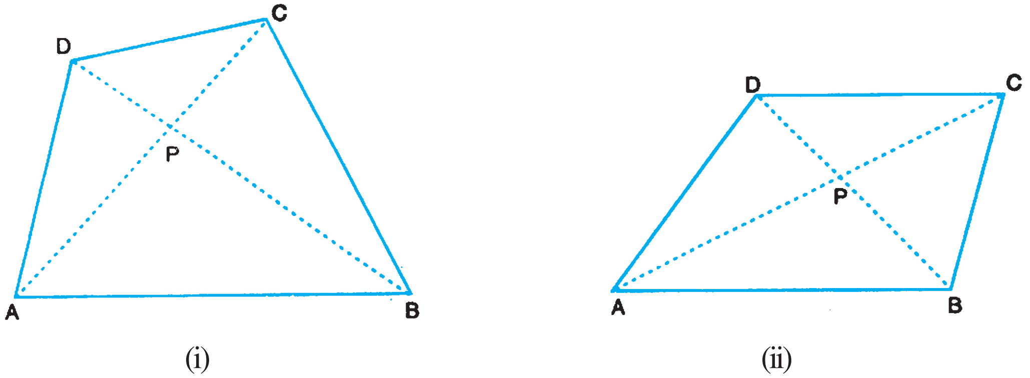
EXPECTED BACKGROUND KNOWLEDGE

- Drawing line-segments and angles of given measure.
- Drawing circles/arcs of given radius.
- Drawing parallel and perpendicular lines.
- Four fundamental operations on numbers.

13.1 QUADRILATERAL

Recall that if A, B, C and D are four points in a plane such that no three of them are collinear and the line segments AB, BC, CD and DA do not intersect except at their end points, then the closed figure made up of these four line segments is called a quadrilateral with vertices A, B, C and D. A quadrilateral with vertices A, B, C and D is generally denoted by quad. ABCD. In Fig. 13.1 (i) and (ii), both the quadrilaterals can be named as quad. ABCD or simply ABCD.

In quadrilateral ABCD,



(i) (ii)

Fig. 13.1

(i) AB and DC ; BC and AD are two pairs of opposite sides.

(ii) $\angle A$ and $\angle C$; $\angle B$ and $\angle D$ are two pairs of opposite angles.

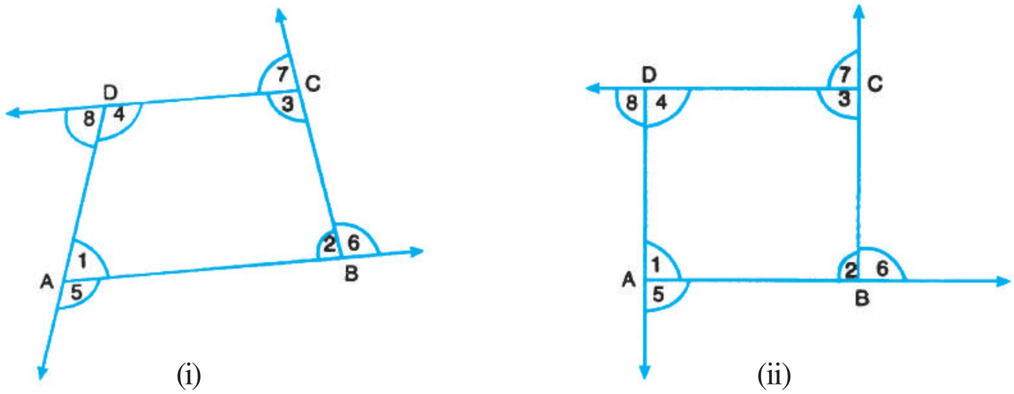
(iii) AB and BC ; BC and CD are two pairs of consecutive or adjacent sides. Can you name the other pairs of consecutive sides?

(iv) $\angle A$ and $\angle B$; $\angle B$ and $\angle C$ are two pairs of consecutive or adjacent angles. Can you name the other pairs of consecutive angles?

(v) AC and BD are the two diagonals.

In Fig. 13.2, angles denoted by l, 2, 3 and 4 are the interior angles or the angles of the quad. ABCD. Angles denoted by 5, 6, 7 and 8 are the exterior angles of the quad. ABCD.

Measure $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$.



(i)

(ii)

Fig. 13.2

What *is* the sum of these angles You will find that $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360°$.

i.e. sum of interior angles of a quadrilateral equals 360°.

Also what is the sum of exterior angles of the quadrilateral ABCD?

You will again find that $\angle 5 + \angle 6 + \angle 7 + \angle 8 = 360°$

i.e., sum of exterior angles of a quadrilateral is also 360°.

13.2 TYPES OF QUADRILATERALS

You are familiar with quadrilaterals and their different shapes. You also know how to name them. However, we will now study different types of quadrilaterals in a systematic way. A family tree of quadrilaterals is given in Fig. 13.3 below:

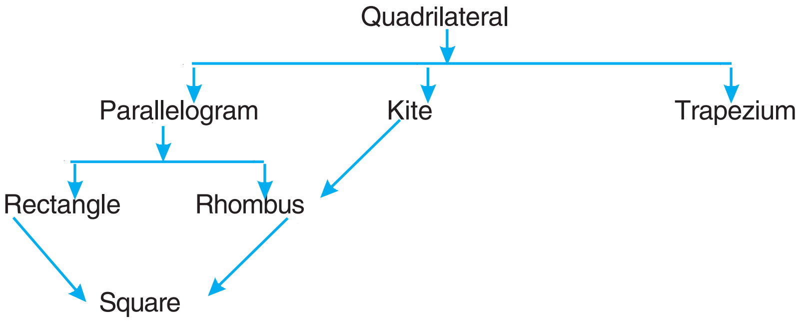


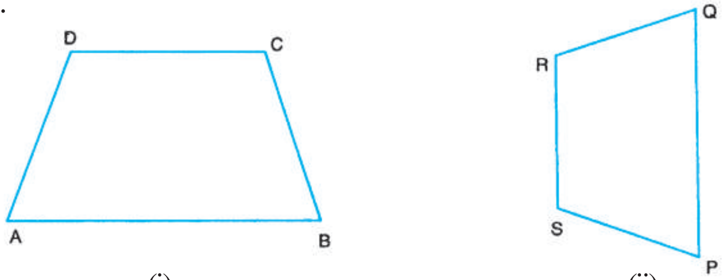
Fig. 13.3

Let us describe them one by one.

1. Trapezium

A quadrilateral which has only one pair of opposite sides parallel is called a trapezium. In

Fig. 13.4 [(i) and (ii)] ABCD and PQRS are trapeziums with AB || DC and PQ || SR respectively.



(i) (ii)

Fig. 13.4

2. Kite

A quadrilateral, which has two pairs of equal sides next to each other, is called a kite. Fig. 13.5 [(i) and (ii)] ABCD and PQRS are kites with adjacent sides AB and AD, BC and CD in (i) PQ and PS, QR and RS in (ii) being equal.

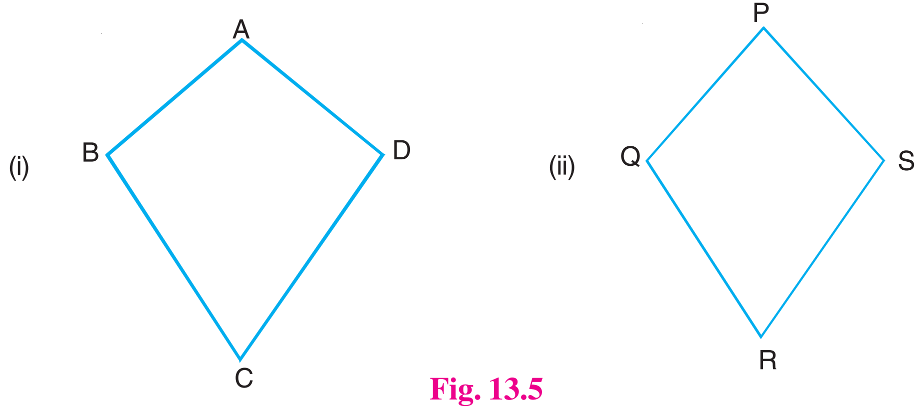
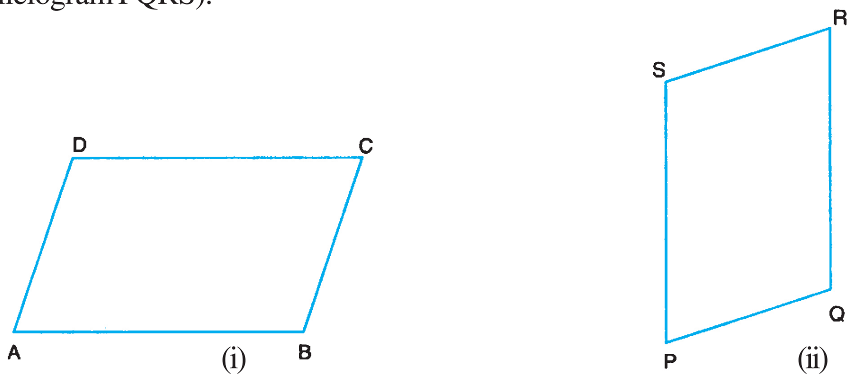


Fig. 13.5

3. Parallelogram

A quadrilateral which has both pairs of opposite sides parallel, is called a parallelogram. In Fig. 13.6 [(i) and (ii)] ABCD and PQRS are parallelograms with AB||DC, AD||BC and PQ||SR, SP||RQ. These are denoted by $||^{gm}$ ABCD (Parallelogram ABCD) and $||^{gm}$ PQRS (Parallelogram PQRS).



(i) (ii)

Fig. 13.6

4. Rhombus

A rhombus is a parallelogram in which any pair of adjacent sides is equal.

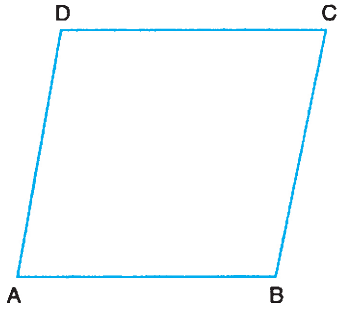


Fig. 13.7

In Fig. 13.7 ABCD is a rhombus.

You may note that ABCD is a parallelogram with AB = BC = CD = DA i.e., each pair of adjacent sides being equal.

5. Rectangle

A parallelogram one of whose angles is a right angle is called a rectangle.

In Fig. 13.8, ABCD is a rectangle in which AB||DC, AD||BC and $\angle A = \angle B = \angle C = \angle D = 90^o$.

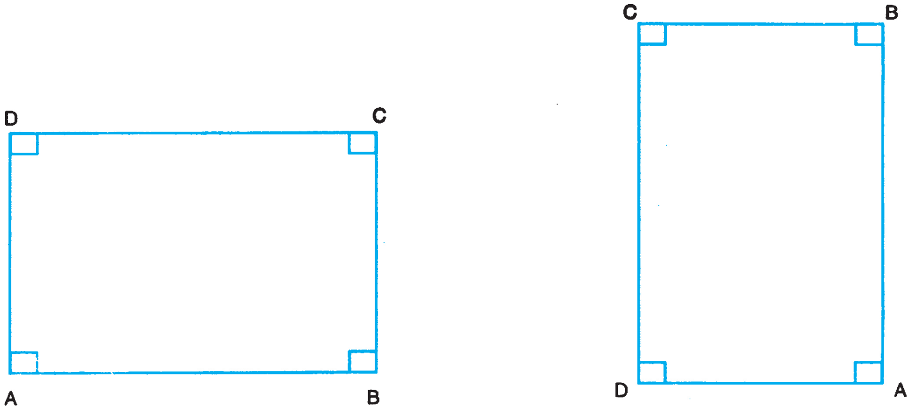


Fig. 13.8

6. Square

A square is a rectangle, with a pair of adjacent sides equal.

In other words, a parallelogram having all sides equal and each angle a right angle is called a square.

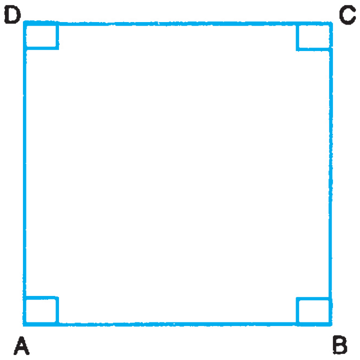


Fig. 13.9

In Fig. 13.9, ABCD is a square in which AB||DC, AD||BC, and AB = BC = CD = DA and $\angle A = \angle B = \angle C = \angle D = 90^\circ$.

Let us take some examples to illustrate different types of quadrilaterals.

Example 13.1: In Fig 13.10, PQR is a triangle. S and T are two points on the sides PQ and PR respectively such that ST||QR. Name the type of quadrilateral STRQ so formed.

Solution: Quadrilateral STRQ is a trapezium, because ST||QR.

Example 13.2: The three angles of a quadrilateral are 100°, 50° and 70°. Find the measure of the fourth angle.

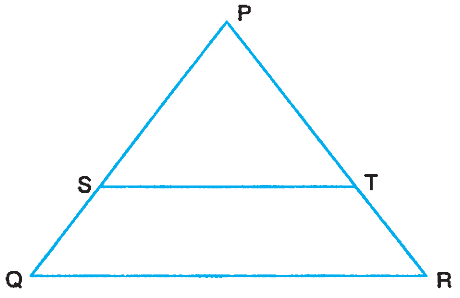


Fig. 13.10

Solution: We know that the sum of the angles of a quadrilateral is 360°.

Then $$100^\circ + 50^\circ + 70^\circ + x^\circ = 360^\circ$$

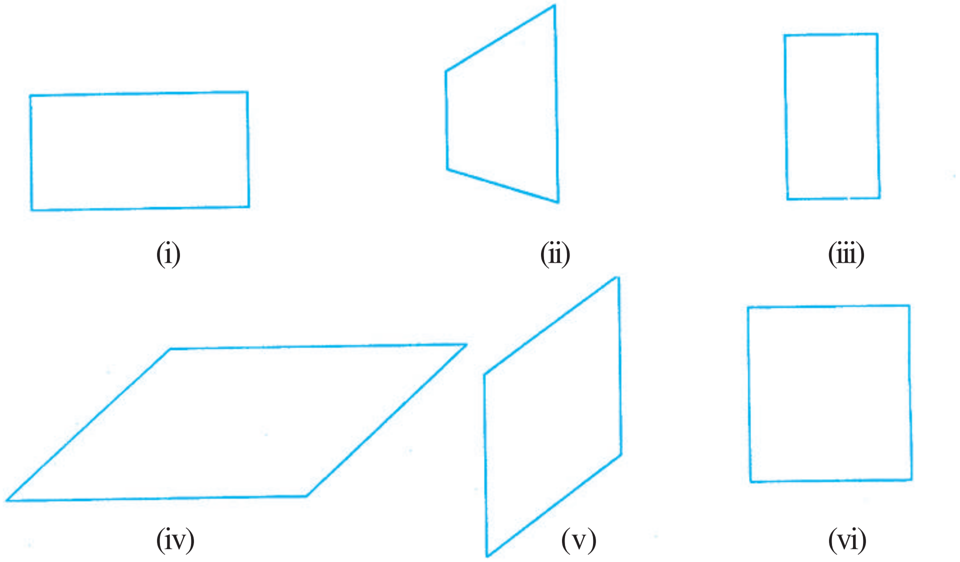
$$220^\circ + x^\circ = 360^\circ$$

$$x = 140$$

Hence, the measure of fourth angle is 140°.

CHECK YOUR PROGRESS 13.1

1. Name each of the following quadrilaterals.



(i) (ii) (iii)

(iv) (v) (vi)

Fig. 13.10

2. State which of the following statements are correct ?

   (i) Sum of interior angles of a quadrilateral is 360°.

   (ii) All rectangles are squares,

   (iii) A rectangle is a parallelogram.

   (iv) A square is a rhombus.

   (v) A rhombus is a parallelogram.

   (vi) A square is a parallelogram.

   (vii) A parallelogram is a rhombus.

   (viii) A trapezium is a parallelogram.

   (ix) A trapezium is a rectangle.

   (x) A parallelogram is a trapezium.

3. In a quadrilateral, all its angles are equal. Find the measure of each angle.

4. The angles of a quadrilateral are in the ratio 5:7:7: 11. Find the measure of each angle.

5. If a pair of opposite angles of a quadrilateral are supplementary, what can you say about the other pair of angles?

13.3 PROPERTIES OF DIFFERENT TYPES OF QUADRILATERALS

1. Properties of a Parallelogram

We have learnt that a parallelogram is a quadrilateral with both pairs of opposite sides parallel. Now let us establish some relationship between sides, angles and diagonals of a parallelogram.

Draw a pair of parallel lines l and m as shown in Fig. 13.12. Draw another pair of parallel lines p and q such that they intersect l and m. You observe that a parallelogram ABCD is formed. Join AC and BD. They intersect each other at O.

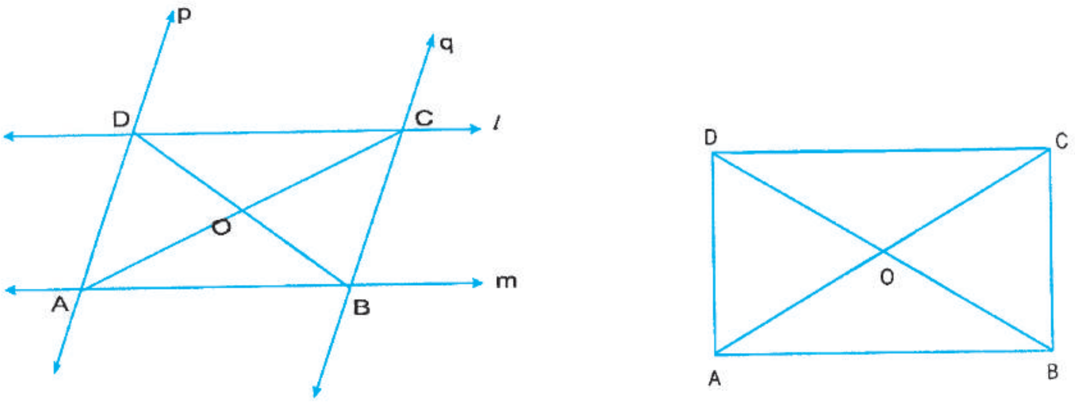


Fig. 13.12

Now measure the sides AB, BC, CD and DA. What do you find?

You will find that AB = DC and BC = AD.

Also measure ∠ABC, ∠BCD, ∠CDA and ∠DAB.

What do you find?

You will find that ∠DAB = ∠BCD and ∠ABC = ∠CDA

Again, Measure OA, OC, OB and OD.

What do you find?

You will find that OA = OC and OB = OD

Draw another parallelogram and repeat the activity. You will find that

The opposite sides of a parallelogram are equal.

The opposite angles of a parallelogram are equal.

The diagonals of a parallelogram bisect each other.

The above mentioned properties of a parallelogram can also be verified by Cardboard model which is as follows:

Let us take a cardboard. Draw any parallelogram ABCD on it. Draw its diagonal AC as shown in Fig 13.13 Cut the parallelogram ABCD from the cardboard. Now cut this parallelogram along the diagonal AC. Thus, the parallelogram has been divided into two parts and each part is a triangle.

In other words, you get two triangles, ΔABC and ΔADC. Now place ΔADC on ΔABC in such a way that the vertex D falls on the vertex B and the side CD falls along the side AB.

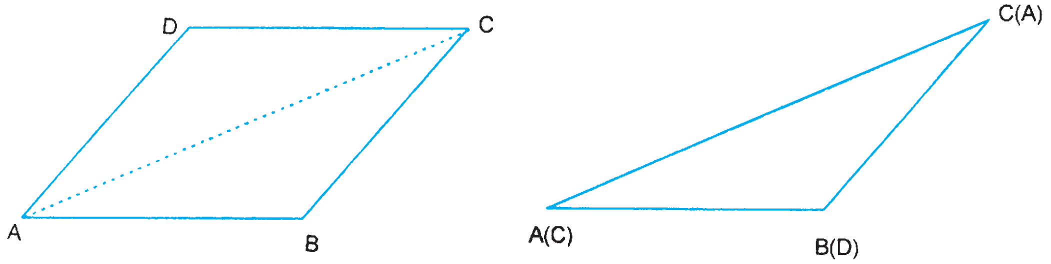


Fig. 13.13

Where does the point C fall?

Where does the point A fall?

You will observe that ΔADC will coincide with ΔABC. In other words ΔABC ≅ ΔADC. Also AB = CD and BC = AD and ∠B = ∠D.

You may repeat this activity by taking some other parallelograms, you will always get the same results as verified earlier, thus, proving the above two properties of the parallelogram.

Now you can prove the third property of the parallelogram, i.e., the diagonals of a parallelogram bisect each other.

Again take a thin cardboard. Draw any parallelogram PQRS on it. Draw its diagonals

PR and QS which intersect each other at O as shown in Fig. 13.14. Now cut the parallelogram PQRS.

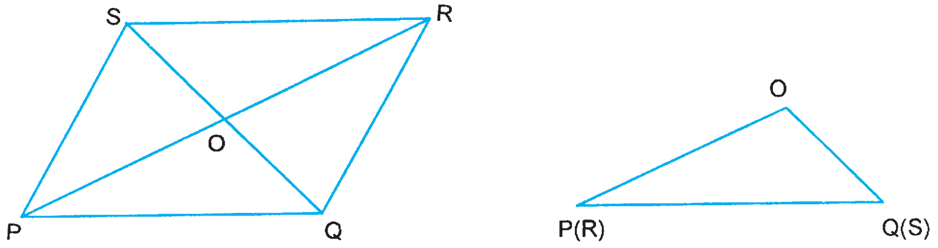


Fig. 13.14

Also cut ΔPOQ and ΔROS.

Now place ΔROS and ΔPOQ in such a way that the vertex R coincides with the vertex P and RO coincides with the side PO.

Where does the point S fall?

Where does the side OS fall?

Is ΔROS ≅ ΔPOQ? Yes, it is.

So, what do you observe?

We find that RO = PO and OS = OQ

You may also verify this property by taking another pair of triangles i.e. ΔPOS and ΔROQ You will again arrive at the same result.

You may also verify the following properties which are the converse of the properties of a parallelogram verified earlier.

A quadrilateral is a parallelogram if its opposite sides are equal.

A quadrilateral is a parallelogram if its opposite angles are equal.

A quadrilateral is a parallelogram if its diagonals bisect each other.

2. Properties of a Rhombus

In the previous section we have defined a rhombus. We know that a rhombus is a parallelogram in which a pair of adjacent sides is equal. In Fig. 13.15, ABCD is a rhombus.

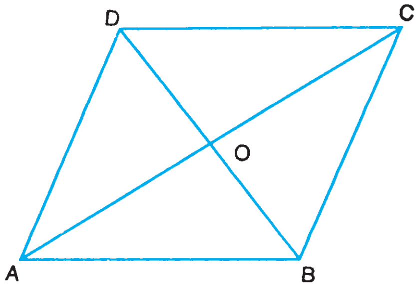


Fig. 13.15

Thus, ABCD is a parallelogram with AB = BC. Since every rhombus is a parallelogram, therefore all the properties of a parallelogram are also true for rhombus, i.e.

(i) Opposite sides are equal,

i.e., AB = DC and AD = BC

(ii) Opposite angles are equal,

i.e., $\angle A = \angle C$ and $\angle B = \angle D$

(iii) Diagonals bisect each other

i.e., AO = OC and DO = OB

Since adjacent sides of a rhombus are equal and by the property of a parallelogram opposite sides are equal. Therefore,

AB = BC = CD = DA

Thus, all the sides of a rhombus are equal. Measure ∠AOD and ∠BOC.

What is the measures of these angles?

You will find that each of them equals 90°

Also $\angle AOB = \angle COD$ (Each pair is a vertically opposite angles)

and $\angle BOC = \angle DOA$

$\therefore \angle AOB = \angle COD = \angle BOC = \angle DOA = 90°$

Thus, the diagonals of a rhombus bisect each other at right angles.

You may repeat this experiment by taking different rhombuses, you will find in each case, the diagonals of a rhombus bisect each other.

Thus, we have the following properties of a rhombus.

All sides of a rhombus are equal

Opposite angles of a rhombus are equal

The diagonals of a rhombus bisect each other at right angles.

3. Properties of a Rectangle

We know that a rectangle is a parallelogram one of whose angles is a right angle. Can you say whether a rectangle possesses all the properties of a parallelogram or not?

Yes it possesses. Let us study some more properties of a rectangle.

Draw a parallelogram ABCD in which $\angle B = 90°$.

Join AC and BD as shown in the Fig. 13.16

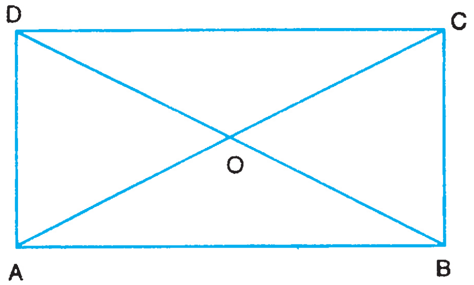


Fig. 13.16

Measure ∠BAD, ∠BCD and ∠ADC, what do you find?

What are the measures of these angles?

The measure of each angle is 90°. Thus, we can conclude that

$$\angle A = \angle B = \angle C = \angle D = 90^o$$

i.e., each angle of a rectangle measures 90°. Now measure the diagonals AC and BD. Do you find that AC = BD.

Now, measure AO, OC, BO and OD.

You will find that AO = OC and BO = OD.

Draw some more rectangles of different dimensions. Label them again by ABCD. Join AC and BD in each case. Let them intersect each other at O. Also measure AO, OC and BO, OD for each rectangle. In each case you will find that

The diagonals of a rectangle are equal and they bisect each other. Thus, we have the following properties of a rectangle;

The opposite sides of a rectangle are equal

Each angle of a rectangle is a right-angle.

The diagonals of a rectangle are equal.

The diagonals of a rectangle bisect each other.

4. Properties of a Square

You know that a square is a rectangle, with a pair of adjacent sides equal. Now, can you conclude from definition of a square that a square is a rectangle and possesses all the properties of a rectangle? Yes it is. Let us now study some more properties of a square.

Draw a square ABCD as shown in Fig. 13.17.

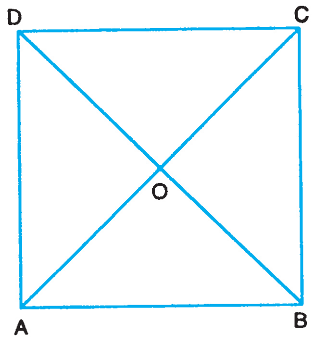


Fig 13.17

Since ABCD is a rectangle, therefore we have

(i) AB = DC, AD = BC

(ii) $\angle A = \angle B = \angle C = \angle D = 90^o$

(iii) AC = BD and AO = OC, BO = OD

But in a square we have AB = AD

∴ By property (i) we have

AB = AD = CD = BC.

Since a square is also a rhombus. Therefore, we conclude that the diagonals AC and BD of a square bisect each other at right angles.

Thus, we have the following properties of a square.

All the sides of a square are equal

Each of the angles measures 90°.

The diagonals of a square are equal.

The diagonals of a square bisect each other at right angles.

Let us study some examples to illtustrate the above properties:

Example 13.3: In Fig. 13.17, ABCD is a parallelogram. If $\angle A = 80^o$, find the measures of the remaining angles

Solution: As ABCD is a parallelogram.

$\angle A = \angle C$ and $\angle B = \angle D$

It is given that

$\angle A = 80^o$

∴ $\angle C = 80^o$

∴ AB || DC

∴ $\angle A + \angle D = 180^o$

∴ $\angle D = (180 - 80)^o = 100^o$

∴ $\angle B = \angle D = 100^o$

Hence $\angle C = 80^o$, $\angle B = 100^o$ and $\angle D \ \ 100^o$

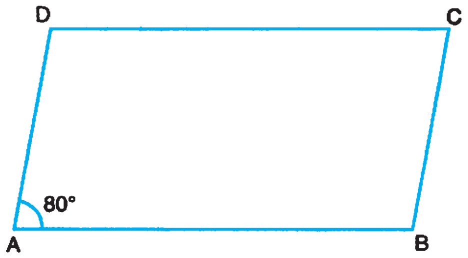


Fig 13.18

Example 13.4: Two adjacent angles of a rhombus are in the ratio 4 : 5. Find the measure of all its angles.

Solution: Since opposite sides of a rhombus are parallel, the sum of two adjacent angles of a rhombus is 180°.

Let the measures of two angles be $4x^o$ and $5x^o$,

Therefore, $4x + 5x = 180$

i.e. $9x = 180$

$$x = 20$$

$\therefore$ The two measures of angles are 80° and 100°.

i.e. $\angle A = 80^\circ$ and $\angle B = 100^\circ$

Since $\angle A = \angle C \Rightarrow \angle C = 100^\circ$

Also, $\angle B = \angle D \Rightarrow \angle D = 100^\circ$

Hence, the measures of angles of the rhombus are 80°, 100°, 80° and 100°.

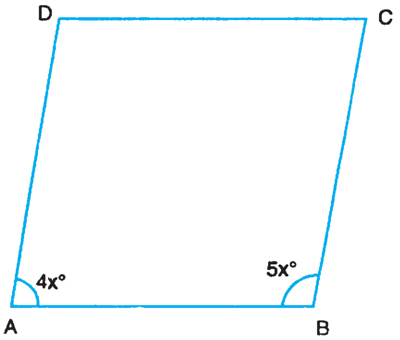


Fig 13.19

Example 13.5: One of the diagonals of a rhombus is equal to one of its sides. Find the angles of the rhombus.

Solution: Let in rhombus, ABCD,

$$AB = AD = BD$$

$\therefore$ ΔABD is an equilateral triangle.

$\therefore \quad \angle DAB = \angle 1 = \angle 2 = 60^\circ \quad$ ....(1)

Similarly $\angle BCD = \angle 3 = \angle 4 = 60^\circ \quad$ ....(2)

Also from (1) and (2)

$$\angle ABC = \angle B = \angle 1 + \angle 3 = 60^\circ + 60^\circ = 120^\circ$$

$$\angle ADC = \angle D = \angle 2 + \angle 4 = 60^\circ + 60^\circ = 120^\circ$$

Hence, $\angle A = 60^\circ, \angle B = 120^\circ, \angle C = 60^\circ$ and $\angle D = 120^\circ$

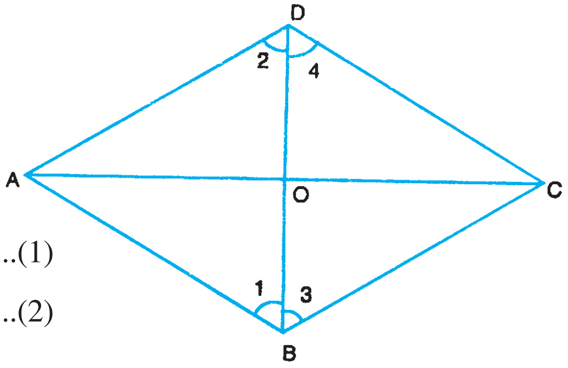


Fig 13.20

Example 13.6: The diagonals of a rhombus ABCD intersect at O. If $\angle ADC = 120^\circ$ and $OD = 6$ cm, find

(a) $\angle OAD$

(b) side AB

(c) perimeter of the rhombus ABCD

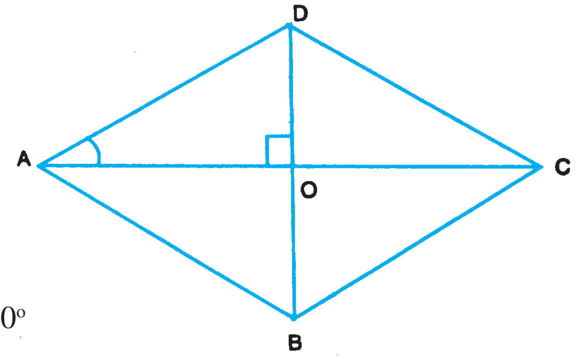


Fig 13.21

Solution: (a) Given that

$$\angle ADC = 120^\circ$$

i.e. $\quad \angle ADO + \angle ODC = 120^\circ$

But $\quad \angle ADO = \angle ODC \quad (\Delta AOD \cong \Delta COD)$

$\therefore \quad 2\angle ADO = 120^\circ$

i.e. $\quad \angle ADO = 60^\circ \quad$ ...(i)

Also, we know that the diagonals of a rhombus bisect each that at 90°.

$$\therefore \quad \angle DOA = 90^o \quad ...(ii)$$

Now, in ΔDOA

$$\angle ADO + \angle DOA + \angle OAD = 180^o$$

From (i) and (ii), we have

$$60^o + 90^o + \angle OAD = 180^o$$

$$\Rightarrow \quad \angle OAD = 30^o$$

(b) Now, $\angle DAB = 60^o$ [since $\angle OAD = 30^o$, similarly $\angle OAB = 30^o$]

$\therefore \Delta DAB$ is an equilateral triangle.

$$OD = 6 \text{ cm} \quad \text{[given]}$$

$$\Rightarrow \quad OD + OB = BD$$

$$6 \text{ cm} + 6 \text{ cm} = BD$$

$$\Rightarrow \quad BD = 12 \text{ cm}$$

so, $AB = BD = AD = 12$ cm

$$AB = 12 \text{ cm}$$

(c) Now Perimeter $= 4 \times$ side

$$= (4 \times 12) \text{ cm}$$

$$= 48 \text{ cm}$$

Hence, the perimeter of the rhombus = 48 cm.

CHECK YOUR PROGRESS 13.2

1. In a parallelogram ABCD, $\angle A = 62^o$. Fing the measures of the other angles.
2. The sum of the two opposite angles of a parallelogram is 150^o. Find all the angles of the parallelogram.
3. In a parallelogram ABCD, $\angle A = (2x + 10)^o$ and $\angle C = (3x - 20)^o$. Find the value of x.
4. ABCD is a parallelogram in which $\angle DAB = 70^o$ and $\angle CBD = 55^o$. Find $\angle CDB$ and $\angle ADB$.
5. ABCD is a rhombus in which $\angle ABC = 58^o$. Find the measure of $\angle ACD$.

6. In Fig. 13.22, the diagonals of a rectangle PQRS intersect each other at O. If ∠ROQ = 40°, find the measure of ∠OPS.

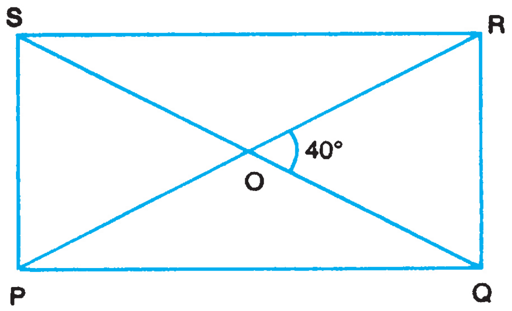


Fig 13.22

7. AC is one diagonal of a square ABCD. Find the measure of ∠CAB.

13.4 MID POINT THEOREM

Draw any triangle ABC. Find the mid points of side AB and AC. Mark them as D and E respectively. Join DE, as shown in Fig. 13.23.

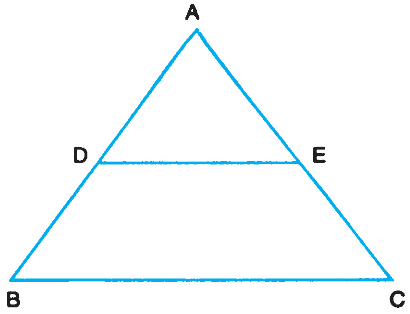


Fig 13.23

Measure BC and DE.

What relation do you find between the length of BC and DE?

Of course, it is $DE = \frac{1}{2} BC$

Again, measure ∠ADE and ∠ABC.

Are these angles equal?

Yes, they are equal. You know that these angles make a pair of corresponding angles. You know that when a pair of corresponding angles are equal, the lines are parallel

$$\therefore \quad DE \parallel BC$$

You may repeat this expreiment with another two or three triangles and naming each of them as triangle ABC and the mid point as D and E of sides AB and AC respectively.

You will always find that $DE = \frac{1}{2} BC$ and $DE \parallel BC$.

Thus, we conclude that

In a triangle the line-segment joining the mid points of any two sides is parallel to the third side and is half of it.

We can also verify the converse of the above stated result.

Draw any ΔPQR. Find the mid point of side RQ, and mark it as L. From L, draw a line LX || PQ, which intersects, PR at M.

Measure PM and MR. Are they equal? Yes, they are equal.

You may repeat with different triangles and by naming each of them as PQR and taking each time L as the mid-point of RQ and drawing a line LM || PQ, you will find in each case that RM = MP. Thus, we conclude that

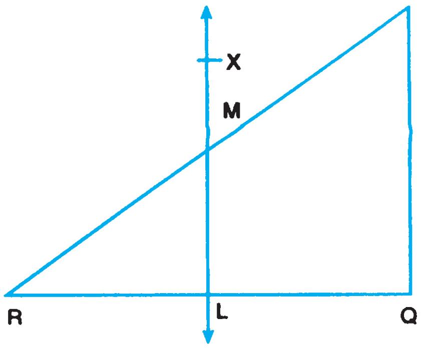


Fig 13.24

"The line drawn through the mid point of one side of a triangle parallel to the another side bisects the third side."

Example 13.7: In Fig. 13.25, D is the mid-point of the side AB of ΔABC and DE || BC. If AC = 8 cm, find AE.

Solution: In ΔABC, DE || BC and D is the mid point of AB

∴ E is also the mid point of AC

$$\text{i.e. } AE = \frac{1}{2} AC$$

$$= \left(\frac{1}{2} \times 8\right) \text{cm} \qquad [\because AC = 8 \text{ cm}]$$

$$= 4 \text{ cm}$$

Hence, AE = 4 cm

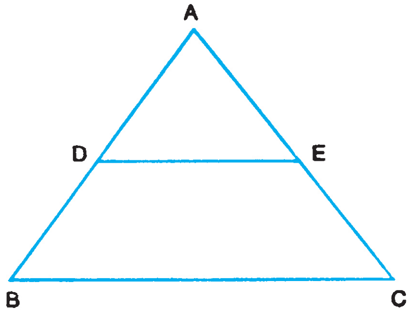


Fig 13.25

Example 13.8: In Fig. 13.26, ABCD is a trapezium in which AD and BC are its non-parallel sides and E is the mid-point of AD. EF || AB. Show that F is the mid-point of BC.

Solution: Since EG || AB and E is the mid-point of AD (considering ΔABD)

∴ G is the mid point of DB

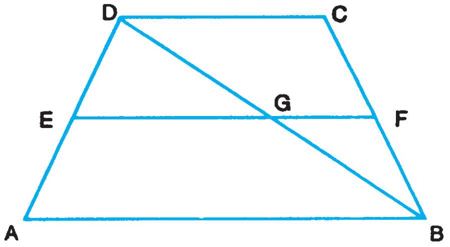


Fig 13.26

In ΔDBC, GF || DC and G is the mid-point of DB,

∴ F is the mid-point of BC.

Example 13.9: ABC is a triangle, in which P, Q and R are mid-points of the sides AB, BC and CA respectively. If AB = 8 cm, BC = 7 cm and CA = 6 cm, find the sides of the triangle PQR.

Solution: P is the mid-point of AB and R the mid-point of AC.

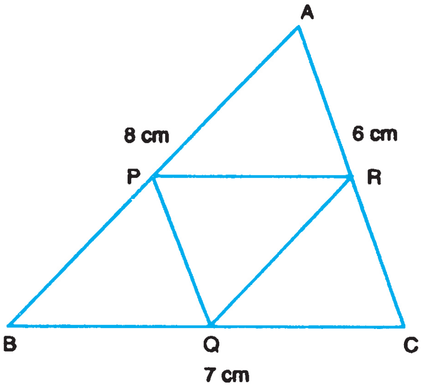


Fig 13.27

$$\therefore \quad PR \parallel BC \text{ and } PR = \frac{1}{2}BC$$

$$= \frac{1}{2} \times 7 \text{ cm} \qquad [\because BC = c$$

$$= 3.5 \text{ cm}$$

$$\text{Similarly,} \qquad PQ = \frac{1}{2}AC$$

$$= \frac{1}{2} \times 6 \text{ cm} \qquad [\because AC = 6 \text{ cm}]$$

$$= 3 \text{ cm}$$

$$\text{and} \qquad QR = \frac{1}{2}AB$$

$$= \frac{1}{2} \times 8 \text{ cm} \qquad [\because AB = 8 \text{ cm}]$$

$$= 4 \text{ cm}$$

Hence, the sides of ΔPQR are PQ = 3 cm, QR = 4 cm and PR = 3.5 cm.

CHECK YOUR PROGRESS 13.3

1. In Fig. 13.28, ABC is an equilateral triangle. D, E and F are the mid-points of the sides AB, BC and CA respectively. Prove that DEF is also an equilateral triangle.

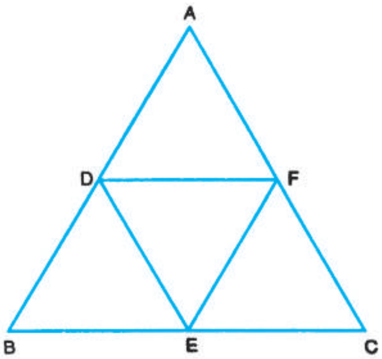


Fig. 13.28

2. In Fig. 13.29, D and E are the mid-points of the sides AB and AC respectively of a ΔABC. If BC = 10 cm; find DE.

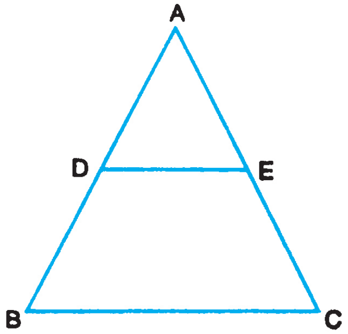


Fig. 13.29

3. In Fig. 13.30, AD is a median of the ΔABC and E is the mid-point of AD, BE is produced to meet AC at F. DG || EF, meets AC at G. If AC = 9 cm, find AF.

   [Hint: First consider ΔADG and next consider ΔCBF]

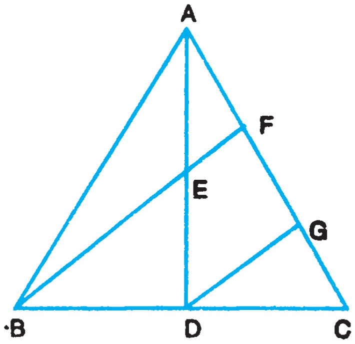


Fig. 13.30

4. In Fig. 13.31, A and C divide the side PQ of ΔPQR into three equal parts, AB||CD||QR. Prove that B and D also divide PR into three equal parts.

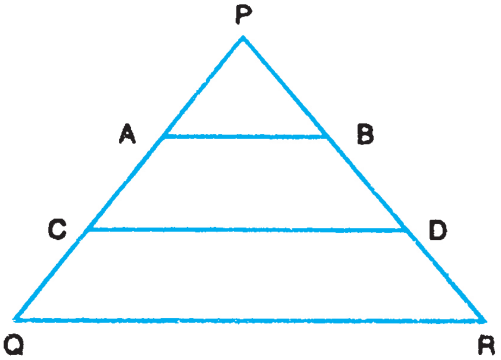


Fig. 13.31

5. In Fig. 13.32, ABC is an isosceles triangle in which AB = AC. M is the mid-point of AB and MN||BC. Show that ΔAMN is also an isosceles triangle.

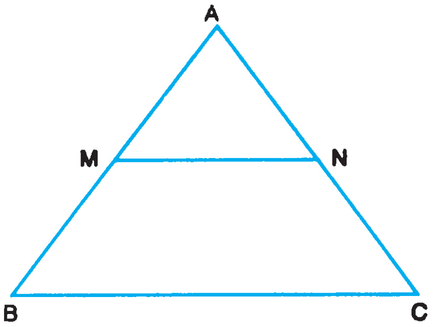


Fig. 13.32

13.5 EQUAL INTERCEPT THEORM

Recall that a line which intersects two or more lines is called a transversal. The line-segment cut off from the transversal by a pair of lines is called an intercept. Thus, in Fig. 13.33, XY is an intercept made by line *l* and m on transversal n.

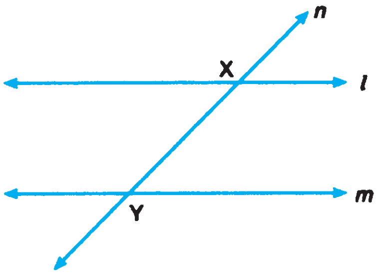
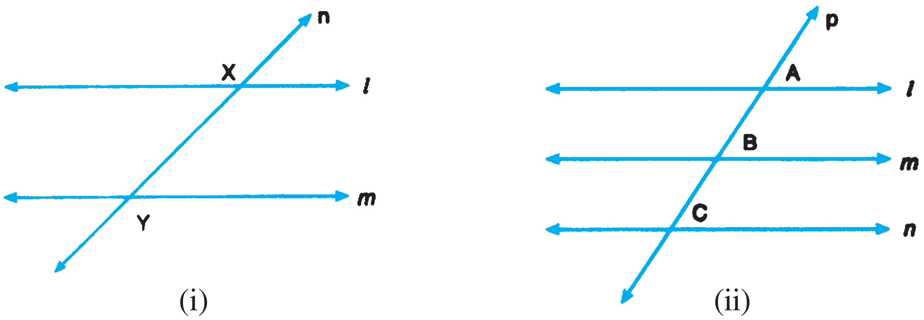


Fig. 13.33

The intercepts made by parallel lines on a transversal have some special properties which we shall study here.

Let *l* and m be two parallel lines and XY be an intercept made on the transversal "n". If there are three parallel lines and they are intersected by a transversal, there will be two intercepts AB and BC as shown in Fig. 13.34 (ii).



(i) (ii)

Fig. 13.34

Now let us learn an important property of intercepts made on the transversals by the parallel lines.

On a page of your note-book, draw any two transversals *l* and m intersecting the equidistant parallel lines p, q, r and s as shown in Fig. 13.35. These transversals make different intercepts. Measure the intercept AB, BC and CD. Are they equal? Yes, they are equal.

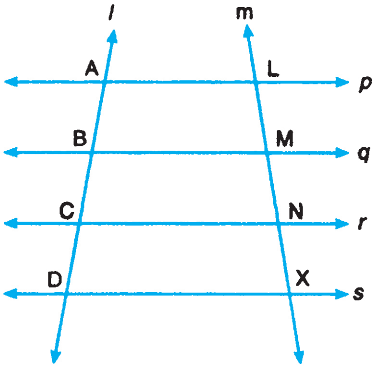


Fig. 13.35

Also, measure LM, MN and NX. Do you find that they are also equal? Yes, they are.

Repeat this experiment by taking another set of two or more equidistant parallel lines and measure their intercepts as done earlier. You will find in each case that the intercepts made are equal.

Thus, we conclude the following:

If there are three or more parallel lines and the intercepts made by them on a transversal are equal, the corresponding intercepts made on any other transversal are also equal.

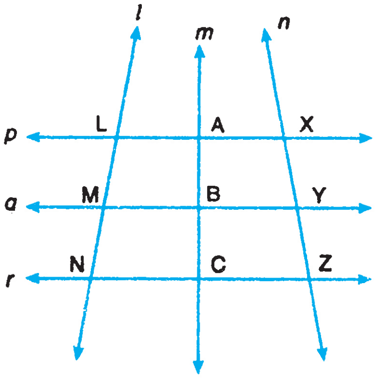


Fig. 13.36

Let us illustrate it by some examples: This result is known as Equal Intercept Theorm.

Example 13.10: In Fig. 13.36, p || q ||r. The transversal *l*, m and n cut them at L, M, N; A, B, C and X, Y, Z respectively such that XY = YZ. Show that AB = BC and LM = MN.

Solution: Given that XY = YZ

∴ AB = BC (Equal Intercept theorem)

and LM = MN

Thus, the other pairs of equal intercepts are

AB = BC and LM = MN.

Example 13.11: In Fig. 13.37, *l* || m || n and PQ = QR. If XZ = 20 cm, find YZ.

Solution: We have PQ = QR

∴ By intercept theorem,

$XY = YZ$

Also $XZ = XY + YZ$

$= YZ + YZ$

∴ $20 = 2YZ \Rightarrow YZ = 10$ cm

Hence, YZ = 10 cm

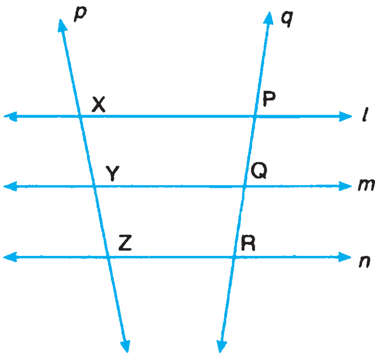


Fig. 13.37

CHECK YOUR PROGRESS 13.4

1. In Fig. 13.38, *l*, m and n are three equidistant parallel lines. AD, PQ and GH are three transversal, If BC = 2 cm and LM = 2.5 cm and AD || PQ, find MS and MN.

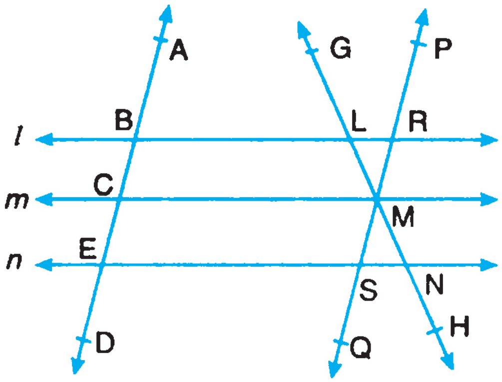


Fig. 13.38

2. From Fig. 13.39, when can you say that AB = BC and XY = YZ?

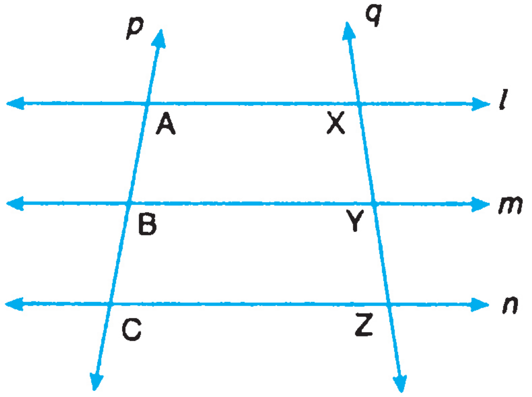


Fig. 13.39

3. In Fig. 13.40, LM = MZ = 3 cm, find XY, XP and BZ. Given that $l \parallel m \parallel n$ and PQ = 3.2 cm, AB = 3.5 cm and YZ = 3.4 cm.

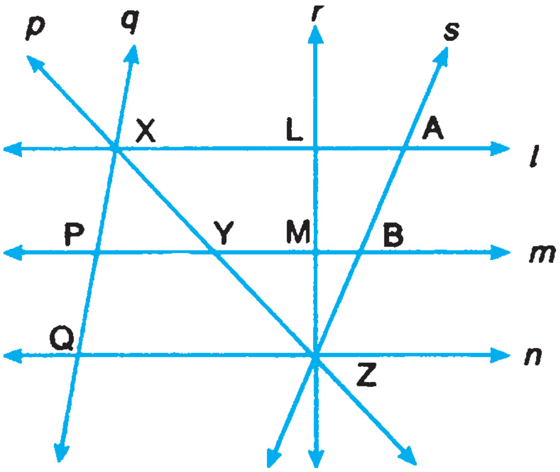


Fig. 13.40

13.6 THE DIAGONAL OF A PARALLELOGRAM AND RELATION TO THE AREA

Draw a parallelogram ABCD. Join its diagonal AC. DP ⊥ DC and QC ⊥ DC.

Consider the two triangles ADC and ACB in which the parallelogram ABCD has been divided by the diagonal AC. Because AB || DC, therefore PD = QC.

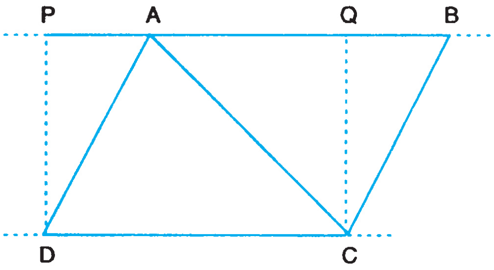


Fig. 13.41

Now, Area of $\Delta ADC = \frac{1}{2}$ DC × PD ....(i)

Area of $\Delta ACB = \frac{1}{2}$ AB × QC ....(ii)

As AB = DC and PD = QC

∴ Area (ΔADC) = Area (ΔACB)

Thus, we conclude the following:

A diagonal of a parallelogram divides it into two triangles of equal area.

13.7 PARALLELOGRAMS AND TRIANGLES BETWEEN THE SAME PARALLELS

Two parallelograms or triangles, having same or equal bases and having their other vertices on a line parallel to their bases, are said to be on the same or equal bases and between the same parallels.

We will prove an important theorem on parallelogram and their area.

Theorm: Parallelogrm on the same base (or equal bases) and between the same parallels are equal in area.

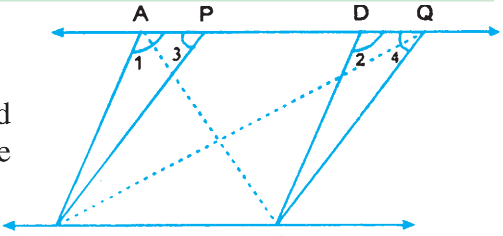


Fig. 13.42

Let us prove it logically.

Given: Parallelograms ABCD and PBCQ stand on the same base BC and between the same parallels BC and AQ.

To prove: Area (ABCD) = Area (BCQP)

we have $AB = DC$ (Opposite sides of a parallelogram)

and $BP = CQ$ (Opposite sides of a parallelogram)

$\angle 1 = \angle 2$

$\therefore \quad \Delta ABP \cong \Delta DCQ$

$\therefore$ Area (ΔABP) = Area (ΔDCQ) ...(i)

Now, Area ($||^{gm}$ ABCD) = Area (ΔABP) + Area Trapezium, BCDP) ...(ii)

Area ($||^{gm}$ BCQP) = Area (ΔDCQ) + Area Trapezium, BCDP) ...(iii)

From (i), (ii) and (iii), we get

Area ($||^{gm}$ ABCD) = Area ($||^{gm}$ BCQP)

Parallelogram on the same base (or equal bases) and between the same parallels are equal in area.

Note: $||^{gm}$ stands for parallelogram.

Result: Triangles, on the same base and between the same parallels, are equal in area.

Consider Fig. 13.42. Join the diagonals BQ and AC of the two parallelograms BCQP and ABCD respectively. We know that a diagonals of a $||^{gm}$ divides it in two triangles of equal area.

∴ Area (ΔBCQ) = Area (ΔPBQ) [Each half of $||^{gm}$ BCQP]

and Area (ΔABC) = Area (ΔCAD) [Each half of $||^{gm}$ ABCD]

∴ Area (ΔABC) = Area (ΔBCQ) [Since area of $||^{gm}$ ABCD = Area of $||^{gm}$ BCQP]

Thus we conclude the following:

Triangles on the same base (or equal bases) and between the same parallels are equal in area.

13.8 TRIANGLES ON THE SAME OR EQUAL BASES HAVING EQUAL AREAS HAVE THEIR CORRESPONDING ALTITUDES EQUAL

Recall that the area of triangle = $\frac{1}{2}$ (Base) × Altitude

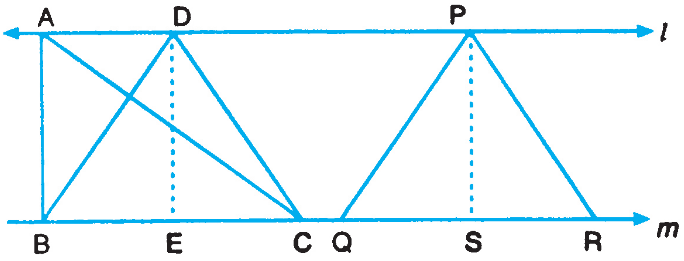


Fig. 13.43

Here BC = QR

and Area (ΔABC) = Area (ΔDBC) = Area (ΔPQR) [Given] ..(i)

Draw perpendiculars DE and PS from D and P to the line m meeting it in E and S respectively.

Now Area (ΔABC) = $\frac{1}{2}$ BC × DE

Area (ΔDBC) = $\frac{1}{2}$ BC × DE ...(ii)

and Area (ΔPQR) = $\frac{1}{2}$ QR × PS

Also, BC = QR (given) ...(iii)

From (i), (ii) and (iii), we get

$$\frac{1}{2}BC \times DE = \frac{1}{2}QR \times PS$$

or $$\frac{1}{2}BC \times DE = \frac{1}{2}BC \times PS$$

$$\therefore \quad DE = PS$$

i.e., Altitudes of ΔABC, ΔDBC and ΔPQR are equal in length.

Thus, we conclude the following:

Triangles on the same or equal bases, having equal areas have their corresponding altitudes equal.

Let us consider some examples:

Example 13.12: In Fig. 13.44, the area of parallelogram ABCD is 40 sq cm. If BC = 8 cm, find the altitude of parallelogram BCEF.

Solution: Area of $||^{gm}$ BCEF = Area of $||^{gm}$ ABCD = 40 sq cm

we know that Area ($||^{gm}$ BCEF) = EF × Altitude

or $40 = BC \times$ Altitude of $||^{gm}$ BCEF

or $40 = BC \times$ Altitude of $||^{gm}$ BCEF

∴ Altitude of $||^{gm}$ BCEF = $\frac{40}{8}$ cm or 5 cm.

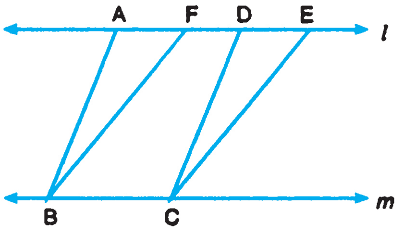


Fig. 13.44

Example 13.13: In Fig. 13.45, the area of ΔABC is given to be 18 cm^2. If the altitude DL equals 4.5 cm, find the base of the ΔBCD.

Solution: Area (ΔBCD) = Area (ΔABC) = 18 cm^2

Let the base of ΔBCD be x cm

$$\therefore \quad \text{Area of } \Delta BCD = \frac{1}{2} x \times DL$$

$$= \left(\frac{1}{2} x \times 4.5\right) cm^2$$

or $$18 = \left(\frac{9}{4} x\right)$$

$$\therefore \quad x = \left(18 \times \frac{4}{9}\right) cm = 8 \text{ cm.}$$

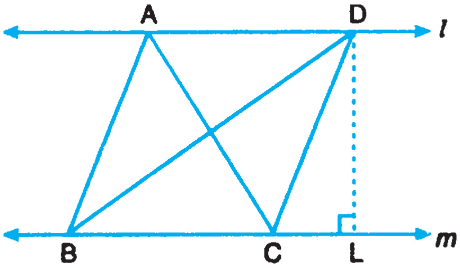


Fig. 13.45

Example 13.14: In Fig. 13.46, ABCD and ACED are two parallelograms. If area of ΔABC equals 12 cm^2, and the length of CE and BC are equal, find the area of the trapezium ABED.

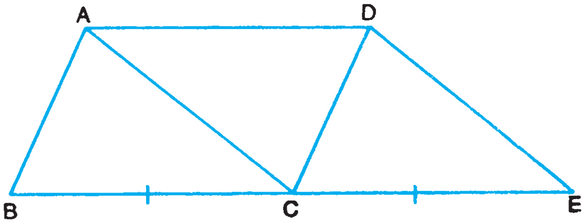


Fig. 13.46

Solution: Area (||^{gm} ABCD) = Area (||^{gm} ACED)

The diagonal AC divides the ||^{gm} ABCD into two triangles of equal area.

$$\therefore \quad \text{Area } (\Delta BCD) = \frac{1}{2} \text{ Area } (||^{gm} ABCD)$$

$$\therefore \quad \text{Area } (||^{gm} ABCD) = \text{Area } (||^{gm} ACED) = 2 \times 12 \text{ cm}^2$$

$$= 24 \text{ cm}^2$$

∴ Area of Trapezium ABED

$$= \text{Area } (\Delta ABC) + \text{Area } (||^{gm} ACED)$$

$$= (12 + 24) \text{ cm}^2$$

$$= 36 \text{ cm}^2$$

CHECK YOUR PROGRESS 13.5

1. When do two parallelograms on the same base (or equal bases) have equal areas?
2. The area of the triangle ABC formed by joining the diagonal AC of a ||^{gm} ABCD is 16 cm^2. Find the area of the ||^{gm} ABCD.
3. The area of ΔACD in Fig. 13.47 is 8 cm^2. If EF = 4 cm, find the altitude of ||^{gm} BCFE.

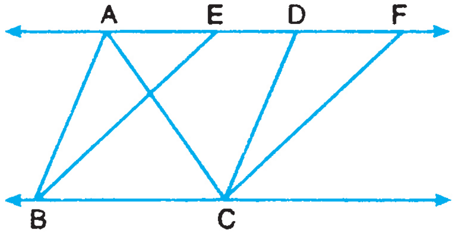


Fig. 13.47

LET US SUM UP

- A quadrilateral is a four sided closed figure, enclosing some region of the plane.
- The sum of the interior or exterior angles of a quadrilateral is equal to 360° each.
- A quadrilateral is a trapezium if its only one pair of opposite sides is parallel.
- A quadrilateral is a parallelogrm if both pairs of sides are parallel.
- In a parallelogram:
  (i) opposite sides and angles are equal.
  (ii) diagonals bisect each other.
- A parallelogram is a rhombus if its adjacent sides are equal.
- The diagonals of a rhombus bisect each other at right angle.
- A parallelogram is a rectangle if its one angle is 90°.
- The diagonals of a rectangle are equal.
- A rectangle is a square if its adjacent sides are equal.
- The diagonals of a square intersect at right angles.
- The diagonal of a parallelogram divides it into two triangles of equal area.
- Parallelogram on the same base (or equal bases) and between the same parallels are equal in area.
- The triangles on the same base (or equal bases) and between the same parallels are equal in area.
- Triangles on same base (or equal bases) having equal areas have their corrsponding altitudes equal.

TERMINAL EXERCISE

1. Which of the following are trapeziums?

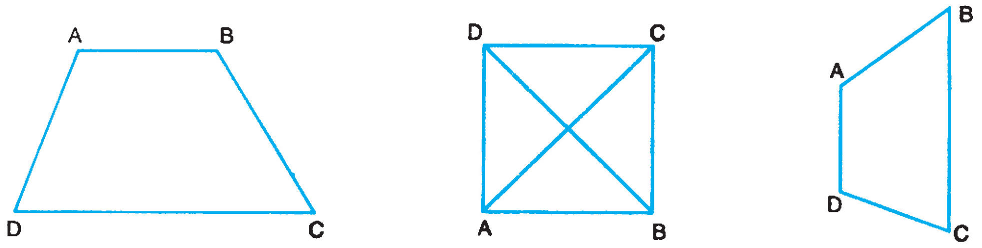


Fig. 13.48

2. In Fig. 13.49, PQ || FG || DE || BC. Name all the trapeziums in the figure.

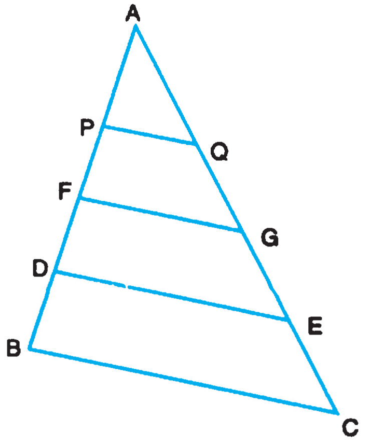


Fig. 13.49

3. In Fig. 13.50, ABCD is a parallelogram with an area of 48 cm^2. Find the area of (i) shaded region (ii) unshaded region.

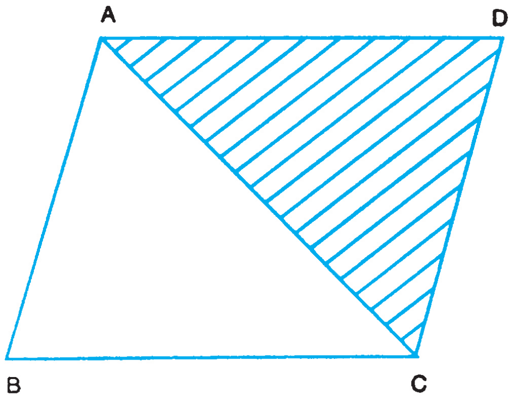


Fig. 13.49

4. Fill in the blanks in each of the following to make them true statements:
   (i) A quadrilateral is a trapezium if ....
   (ii) A quadrilateral is a parallelogram if ....
   (iii) A rectangle is a square if ...
   (iv) the diagonals of a quadrilateral bisect each other at right angle. If none of the angles of the quadrilateral is a right angle, it is a ...
   (v) The sum of the exterior angles of a quadrilateral is ...
5. If the angles of a quadrilateral are $(x-20)^o$, $(x+20)^o$, $(x-15)^o$ and $(x+15)^o$, find x and the angles of the quadrilateral.
6. The sum of the opposite angles of a parallelograms is 180^o. What type of a parallelogram is it?

7. The area of a ΔABD in Fig. 13.51 is 24 cm^2. If DE = 6 cm, and AB || CD, BD || CE, AE || BC, find

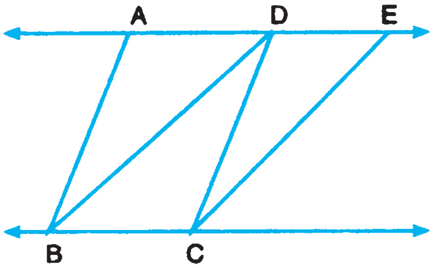


Fig. 13.51

(i) Altitude of the parallelogram BCED.

(ii) Area of the parallelogram BCED.

8. In Fig. 13.52, the area of parallelogram ABCD is 40 cm^2. If EF = 8 cm, find the altitude of ΔDCE.

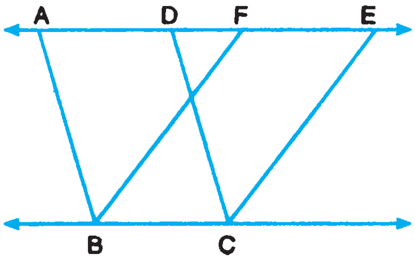


Fig. 13.52

ANSWERS TO CHECK YOUR PROGRESS

13.1

1. (i) Rectangle (ii) Trapezium (iii) Rectangle (iv) Parallelogram
(v) Rhombus (vi) Square

2. (i) True (ii) False (iii) True (iv) True
(v) True (vi) True (vii) False (viii) False
(ix) False (x) False

3. 90^o

4. 60^o, 84^o, 84^o and 132^o

5. Other pair of opposite angles will also be supplementary.

13.2

1. $\angle B = 118^o$, $\angle C = 62^o$ and $\angle D = 118^o$

2. $\angle A = 105^o$, $\angle B = 75^o$, $\angle C = 105^o$ and $\angle D = 75^o$

3. 30
4. $\angle CDB = 55^\circ$ and $\angle ADB = 55^\circ$
5. $\angle ACD = 61^\circ$
6. $\angle OPS = 70^\circ$ 7. $\angle CAB = 45^\circ$

13.3

2. 5 cm
3. 3 cm

13.4

1. MS = 2 cm and MN = 2.5 cm
2. 1, m and n are three equidistant parallel lines
3. XY = 3.4 cm, XP = 3.2 cm and BZ = 3.5 cm

13.5

1. When they are lying between the same parallel lines
2. 32 cm^2
3. 4 cm

ANSWERS TO TERMINAL EXERCISE

1. (i) and (iii)
2. PFGQ, FDEG, DBCE, PDEQ, FBCG and PBCQ
3. (i) 24 cm^2 (ii) 24 cm^2
4. (i) any one pair of opposite sides are parallel.
   (ii) both pairs of opposite sids are parallel
   (iii) pair of adjacent sides are equal
   (iv) rhombus
   (v) 360°
5. $x = 90^\circ$, angles are 70°, 110°, 75° and 105° respectively.
6. Rectangle.
7. (i) 8 cm (ii) 48 cm^2
8. 5 cm

14

SIMILARITY OF TRIANGLES

Looking around you will see many objects which are of the same shape but of same or different sizes. For examples, leaves of a tree have almost the same shape but same or different sizes. Similarly, photographs of different sizes developed from the same negative are of same shape but different sizes, the miniature model of a building and the building itself are of same shape but different sizes. **All those objects which have the same shape but not necessarily the same size are called similar objects.**

Let us examine the similarity of plane figures (Fig. 14.1):

(i) Two line-segments of the same length are congruent as well as similar and of different lengths are similar but not congruent.

Fig. 14.1 (i)

(ii) Two circles of the same radius are congruent as well as similar and circles of different radii are similar but not congruent.

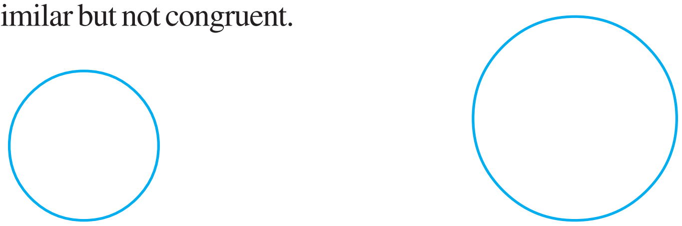


Fig. 14.1 (ii)

(iii) Two equilateral triangles of different sides are similar but not congruent.

Fig. 14.1 (iii)

(iv) Two squares of different sides are similar but not congruent.

Fig. 14.1 (iv)

In this lesson, we shall study about the concept of similarity, particularly similarity of triangles and the conditions thereof. We shall also study about various results related to them.

OBJECTIVES

After studying this lesson, you will be able to

- *identify similar figures;*
- *distinguish between congruent and similar plane figures;*
- *prove that if a line is drawn parallel to one side of a triangle then the other two sides are divided in the same ratio;*
- *state and use the criteria for similarity of triangles viz. AAA, SSS and SAS;*
- *verify and use unstarred results given in the curriculum based on similarity experimentally;*
- *prove the Baudhayan/Pythagoras Theorem;*
- *apply these results in verifying experimentally (or proving logically) problems based on similar triangles.*

EXPECTED BACKGROUND KNOWLEDGE

- knowledge of plane figures like triangles, quadrilaterals, circles, rectangles, squares, etc.
- criteria of congruency of triangles.
- finding squares and square-roots of numbers.
- ratio and proportion.
- Interior and exterior angles of a triangle.

14.1 SIMILAR PLANE FIGURES

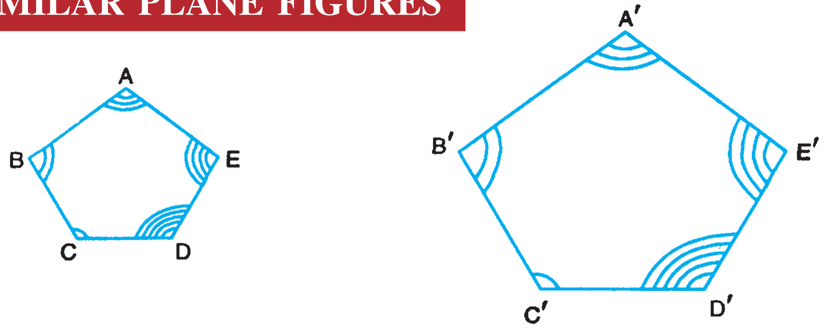


Fig. 14.2

In Fig. 14.2, the two pentagons seem to be of the same shape.

We can see that if ∠A = ∠A′, ∠B = ∠B′, ∠C = ∠C′, ∠D = ∠D′ and ∠E = E′ and $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}$. then the two pentagons are similar. Thus we say that

Any two polygons, with corresponding angles equal and corresponding sides proportional, are similar.

Thus, two polygons are similar, if they satisfiy the following two conditions:

(i) Corresponding angles are equal.

(ii) The corresponding sides are proportional.

Even if one of the conditions does not hold, the polygons are not similar as in the case of a rectangle and square given in Fig. 14.3. Here all the corresponding angles are equal but the corresponding sides are not proportional.

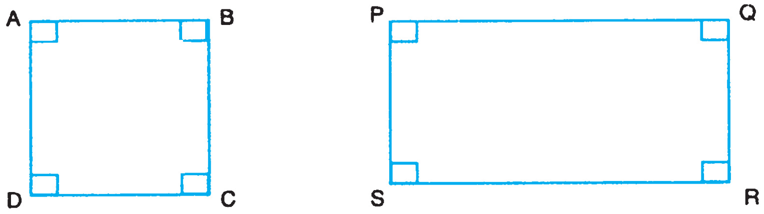


Fig. 14.3

14.2 BASIC PROPORTIONALITY THEORM

We state below the Basic Proportionality Theorm:

If a line is drawn parallel to one side of a triangle intersecting the other two sides, the other two sides of the triangle are divided proportionally.

Thus, in Fig. 14.4, DE || BC, According to the above result

$$\frac{AD}{DB} = \frac{AE}{EC}$$

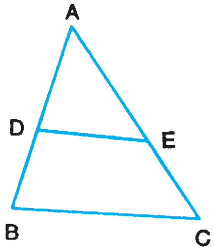


Fig. 14.4

We can easily verify this by measuring AD, DB, AE and EC. You will find that

$$\frac{AD}{DB} = \frac{AE}{EC}$$

We state the converse of the above result as follows:

If a line divides any two sides of a triangle in the same ratio, the line is parallel to third side of the triangle.

Thus, in Fig 14.4, if DE divides side AB and AC of ΔABC such that $\frac{AD}{DB} = \frac{AE}{EC}$, then DE || BC.

We can verify this by measuring ∠ADE and ∠ABC and finding that

$$\angle ADE = \angle ABC$$

These being corresponding angles, the line DE and BC are parallel.

We can verify the above two results by taking different triangles.

Let us solve some examples based on these.

Example 14.1: In Fig. 14.5, DE || BC. If AD = 3 cm, DB = 5 cm and AE = 6 cm, find AC.

Solution: DE || BC (Given). Let EC = x

$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC}$$

$$\therefore \quad \frac{3}{5} = \frac{6}{x}$$

$$\Rightarrow \quad 3\,x = 30$$

$$\Rightarrow \quad x = 10$$

$$\therefore \quad EC = 10 \text{ cm}$$

$$\therefore \quad AC = AE + EC = 16 \text{ cm}$$

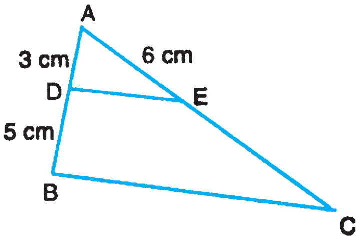


Fig. 14.5

Example 14.2: In Fig. 14.6, AD = 4 cm, DB = 5 cm, AE = 4.5 cm and EC = $5\frac{5}{8}$ cm.

Is DE || BC? Given reasons for your answer.

Solution: We are given that AD = 4 cm and DB = 5 cm

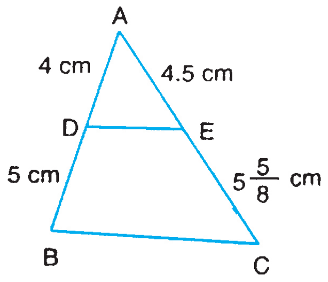


Fig. 14.6

$$\therefore \qquad \frac{AD}{DB} = \frac{4}{5}$$

Similarly, $$\frac{AE}{EC} = \frac{4.5}{\frac{45}{8}} = \frac{9}{2} \times \frac{8}{45} = \frac{4}{5}$$

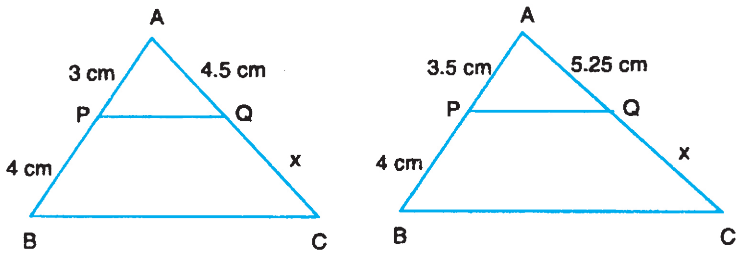
$$\therefore \qquad \frac{AD}{DB} = \frac{AE}{EC}$$

∴ According to converse of Basic Proportionality Theorem

DE || BC

CHECK YOUR PROGRESS 14.1

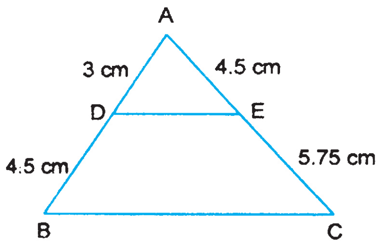
1. In Fig. 14.7 (i) and (ii), PQ || BC. Find the value of x in each case.



(i) (ii)

Fig. 14.7

2. In Fig. 14.8 [(i)], find whether DE || BC is parallel to BC or not? Give reasons for your answer.



(i)

Fig. 14.8

14.3 BISECTOR OF AN ANGLE OF A TRIANGLE

We now state an important result as given below:

> **The bisector of an interior angle of a triangle divides the opposite side in the ratio of sides containing the angle.**

According to the above result, if AD is the internal bisector of ∠A of ΔABC, then

$$\frac{BD}{DC} = \frac{AB}{AC} \text{ (Fig. 14.9)}$$

We can easily verify this by measuring BD, DC, AB and AC and finding the ratios. We will find that

$$\frac{BD}{DC} = \frac{AB}{AC}$$

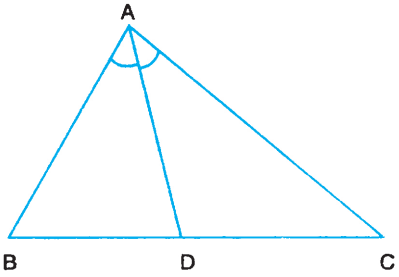


Fig. 14.9

Repeating the same activity with other triangles, we may verify the result.

Let us solve some examples to illustrate this.

Example 14.3: The sides AB and AC of a triangle are of length 6 cm and 8 cm respectively. The bisector AD of ∠A intersects the opposite side BC in D such that BD = 4.5 cm (Fig. 14.10). Find the length of segment CD.

Solution: According to the above result, we have

$$\frac{BD}{DC} = \frac{AB}{AC}$$

(∵ AD is internal bisector of ∠A of ΔABC)

or $$\frac{4.5}{x} = \frac{6}{8}$$

$$\Rightarrow \quad 6x = 4.5 \times 8$$

$$x = 6$$

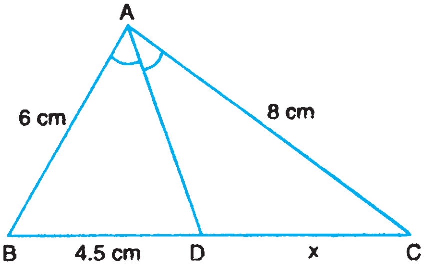


Fig. 14.10

i.e., the length of line-segment CD = 6 cm.

Example 14.4: The sides of a triangle are 28 cm, 36 cm and 48 cm. Find the lengths of the line-segments into which the smallest side is divided by the bisector of the angle opposite to it.

Solution: The smallest side is of length 28 cm and the sides forming ∠A opposite to it are 36 cm and 48 cm. Let the angle bisector AD meet BC in D (Fig. 14.11).

$$\therefore \quad \frac{BD}{DC} = \frac{36}{48} = \frac{3}{4}$$

$$\Rightarrow \quad 4BD = 3DC \quad \text{or} \quad BD = \frac{3}{4}DC$$

$$BC = BD + DC = 28 \text{ cm}$$

$$\therefore \quad DC + \frac{3}{4}DC = 28$$

$$\therefore \quad DC = \left(28 \times \frac{4}{7}\right)\text{cm} = 16 \text{ cm}$$

$$\therefore \quad BD = 12 \text{ cm and } DC = 16 \text{ cm}$$

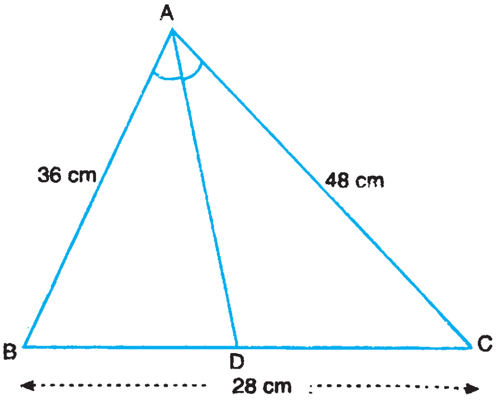


Fig. 14.11

CHECK YOUR PROGRESS 14.2

1. In Fig. 14.12, AD is the bisector of $\angle A$, meeting BC in D. If AB = 4.5 cm, BD = 3 cm, DC = 5 cm, find x.

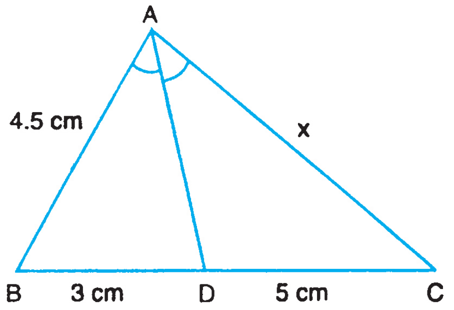


Fig. 14.12

2. In Fig. 14.13, PS is the bisector of $\angle P$ of ΔPQR. The dimensions of some of the sides are given in Fig. 14.13. Find x.

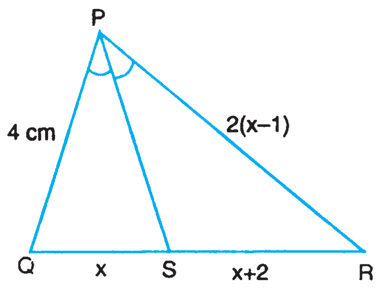


Fig. 14.13

3. In Fig. 14.14, RS is the bisector of ∠R of ΔPQR. For the given dimensions, express p, the length of QS in terms of x, y and z.

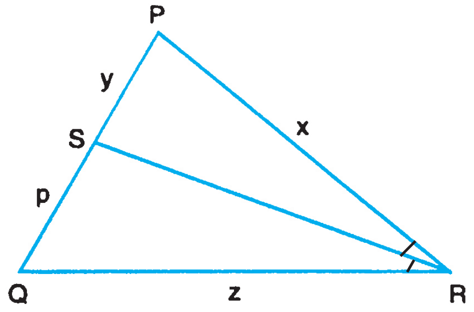


Fig. 14.14

14.4 SIMILARITY OF TRIANGLES

Triangles are special type of polygons and therefore the conditions of similarity of polygons also hold for triangles. Thus,

Two triangles are similar if

(i) their corresponding angles are equal, and

(ii) their corresponding sides are proportional.

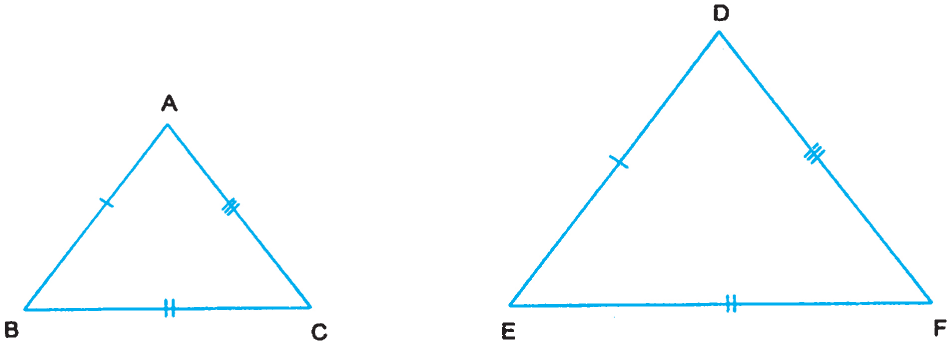


Fig. 14.15

We say that ΔABC is similar to ΔDEF and denote it by writing

ΔABC ~ ΔDEF (Fig. 14.15)

The symbol '~' stands for the phrase "is similar to"

If ΔABC ~ ΔDEF, then by definition

$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ and $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

14.4.1 AAA Criterion for Similarity

We shall show that in the case of triangles if either of the above two conditions is satisfied then the other automatically holds.

Let us perform the following experiment.

Construct two Δ's ABC and PQR in which $\angle P = \angle A$, $\angle Q = \angle B$ and $\angle R = \angle C$ as shown in Fig. 14.16.

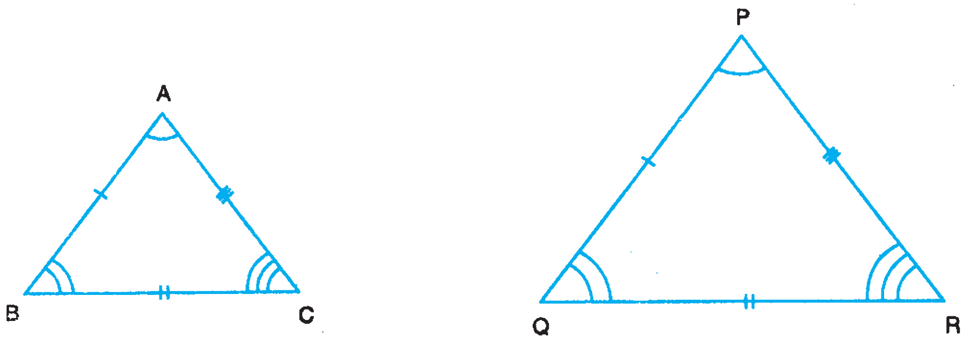


Fig. 14.16

Measure the sides AB, BC and CA of the ΔABC and also measure the sides PQ, QR and RP of ΔPQR.

Now find the ratio $\frac{AB}{PQ}, \frac{BC}{QR}$ and $\frac{CA}{RP}$.

What do you find? You will find that all the three ratios are equal and therefore the triangles are similar.

Try this with different triangles with equal corresponding angles. You will find the same result.

Thus, we can say that:

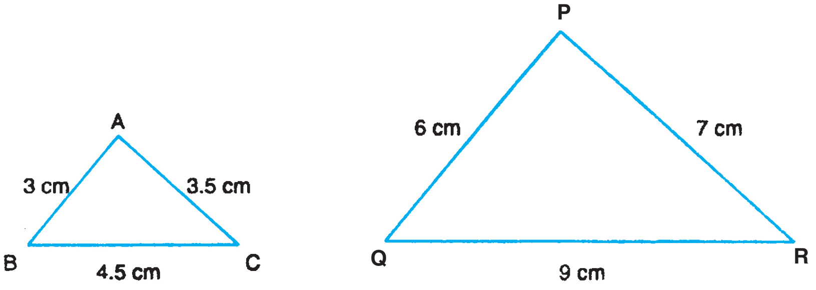
If in two triangles, the corresponding angles are equal the triangles are similar

This is called AAA similarity criterion.

14.4.2 SSS Criterion for Similarity

Let us now perform the following experiment:

Draw a triangle ABC with AB = 3 cm, BC = 4.5 cm and CA = 3.5 cm [Fig. 14.17 (i)].



(i) (ii)

Fig. 14.17

Draw another ΔPQR as shown in Fig. 14.17(ii), with PQ = 6 cm, QR = 9 cm and PR = 7 cm.

We can see that $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$

i.e., the sides of the two triangles are proportional.

Now measure ∠A, ∠B and ∠C of ΔABC and ∠P, ∠Q and ∠R of ΔPQR.

You will find that ∠A = ∠P, ∠B = ∠Q and ∠C = ∠R.

Repeat the experiment with another two triangles having corresponding sides proportional, you will find that the corresponding angles are equal and so the triangles are similar.

Thus, we can say that

> **If the corresponding sides of two triangles are proportional the triangles are similar.**

14.4.3 SAS Criterian for Similarity

Let us conduct the following experiment.

Take a line AB = 3 cm and at A construct an angle of 60°. Cut off AC = 4.5 cm. Join BC.

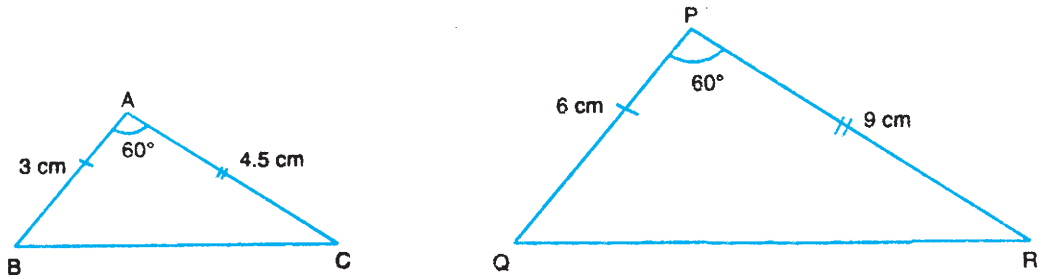


Fig. 14.18

Now take PQ = 6 cm. At P, draw an angle of 60° and cut off PR = 9 cm (Fig. 14.18) and join QR.

Measure ∠B, ∠C, ∠Q and ∠R. We shall find that ∠B = ∠Q and ∠C = ∠R

Thus, ΔABC ~ ΔPQR

Thus, we conclude that

If one angle of a triangle is equal to one angle of the other triangle and the sides containing these angles are proportional, the triangles are similar.

Thus, we have three important criteria for the similarity of triangles. They are given below:

(i) **If in two triangles, the corresponding angles are equal, the triangles are similar.**

(ii) **If the corresponding sides of two triangles are proportional, the triangles are similar.**

(iii) **If one angle of a triangle is equal to one angle of the other triangle and the sides containing these angles are proportional, the triangles are similar.**

Example 14.5: In Fig. 14.19 two triangles ABC and PQR are given in which ∠A = ∠P and ∠B = ∠Q. Is ΔABC ~ ΔPQR?.

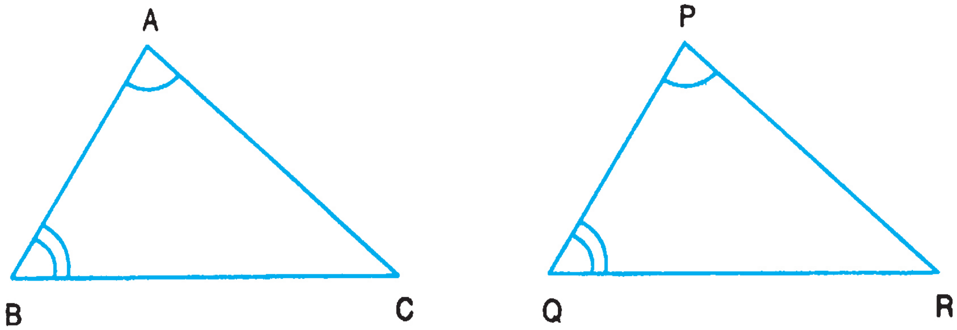


Fig. 14.19

Solution: We are given that

∠A = ∠P and ∠B = ∠Q

We also know that

∠A + ∠B + ∠C = ∠P + ∠Q + ∠R = 180°

Therefore ∠C = ∠R

Thus, according to first criterion of similarity (AAA)

ΔABC ~ ΔPQR

Example 14.6: In Fig. 14.20, ΔABC ~ ΔPQR. If AC = 4.8 cm, AB = 4 cm and PQ = 9 cm, find PR.

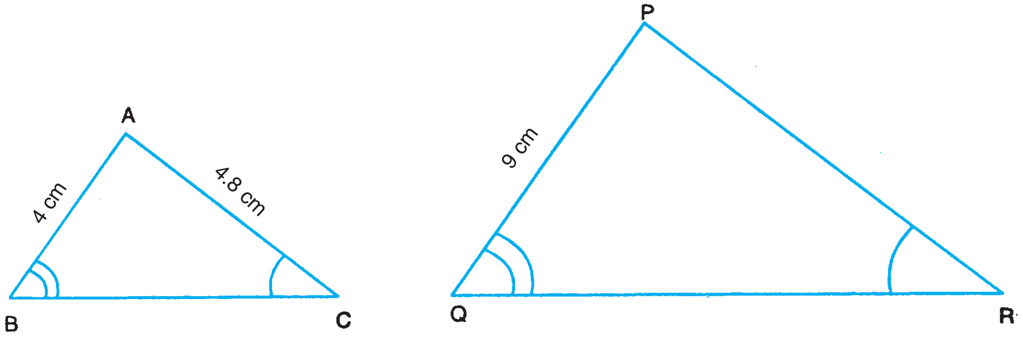


Fig. 14.20

Solution: It is given that ΔABC ~ ΔPQR

$$\therefore \qquad \frac{AB}{PQ}=\frac{AC}{PR}$$

Let PR = x cm

$$\therefore \qquad \frac{4}{9}=\frac{4.8}{x}$$

$$\Rightarrow \qquad 4\,x = 9 \times 4.8$$

$$\Rightarrow \qquad x = 10.8$$

i.e., PR = 10.8 cm.

CHECK YOUR PROGRESS 14.3

Find values of x and y of ΔABC ~ ΔPQR in the following figures:

(i)

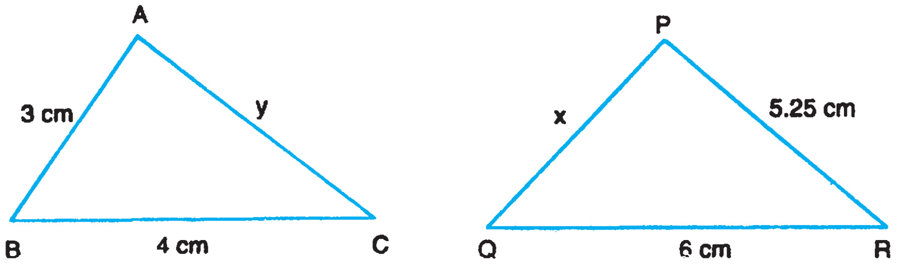


Fig. 14.21

(ii)

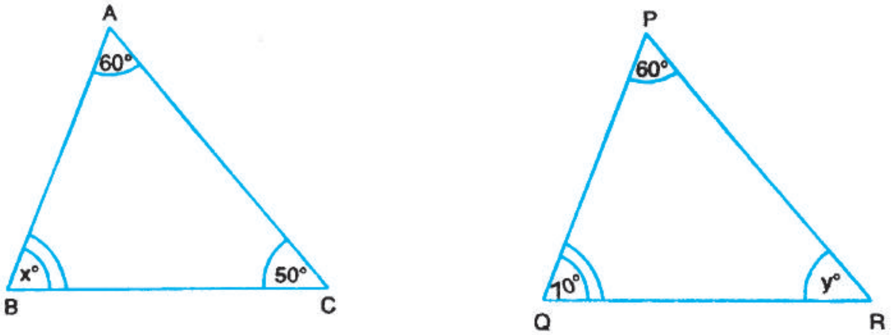


Fig. 14.22

(iii)

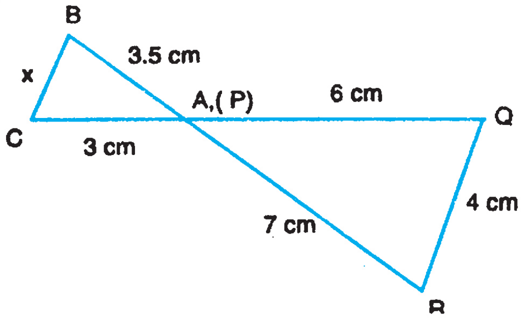


Fig. 14.23

14.5 SOME MORE IMPORTANT RESULTS

Let us study another important result on similarity in connection with a right triangle and the perpendicular from the vertex of right angle to the opposite side. We state the result below and try to verify the same.

> **If a perpendicualr is drawn from the vertex of the right angle of a right triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to each other and to the original triangle.**

Let us try to verify this by an activity.

Draw a ΔABC, right angled at A. Draw $AD \perp$ to the hypoenuse BC, meeting it in D.

Let $\angle DBA = \alpha$,

As $\angle ADB = 90^o$, $\angle BAD = 90^o - \alpha$

As $\angle BAC = 90^o$ and $\angle BAD = 90^o - \alpha$

Therefore $\angle DAC = \alpha$

Similarly $\angle DCA = 90^o - \alpha$

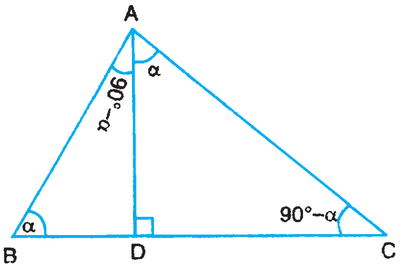


Fig. 14.24

$\therefore$ ΔADB and ΔCDA are similar, as it has all the corresponding angles equal.

Also, the angles B, A and C of ΔBAC are α, 90^{o} and $90^{o} - \alpha$ respectively.

$$\therefore \quad \Delta ADB \sim \Delta CDA \sim \Delta CAB$$

Another important result is about relation between corresponding sides and areas of similar triangles.

It states that

The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.

Let us verify this result by the following activity. Draw two right triangles ABC and PQR which are similar i.e., their sides are proportional (Fig. 14.25).

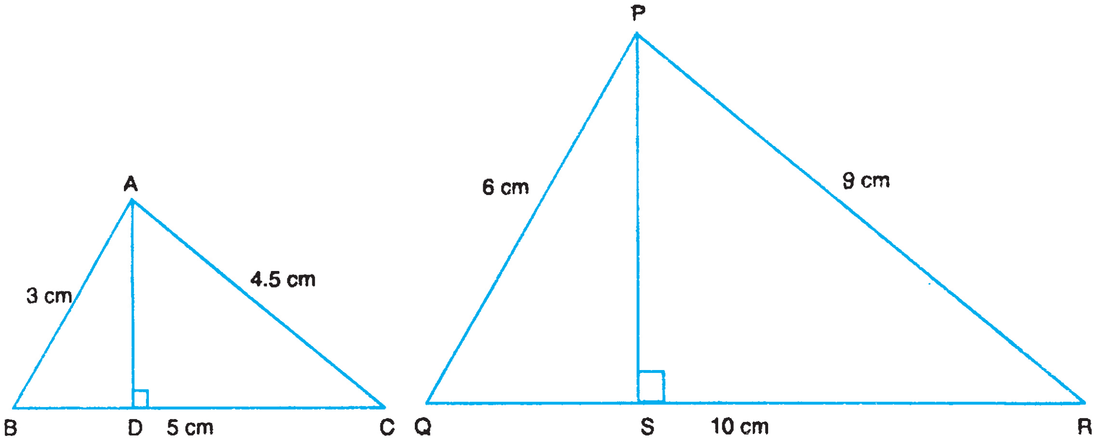


Fig. 14.25

Draw AD ⊥ BC and PS ⊥ QR.

Measure the lengths of AD and PS.

Find the product $AD \times BC$ and $PS \times QR$

You will find that $AD \times BC = BC^2$ and $PS \times QR = QR^2$

$$\text{Now} \quad AD \times BC = 2 \,.\, \text{Area of } \Delta ABC$$

$$PS \times QR = 2.\ \text{Area of } \Delta PQR$$

$$\therefore \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{AD \times BC}{PS \times QR} = \frac{BC^2}{QR^2} \qquad \text{...(i)}$$

$$\text{As} \quad \frac{BC}{QR} = \frac{AB}{PQ} = \frac{AC}{PR}$$

$$\therefore \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{BC^2}{QR^2} = \frac{AB^2}{PQ^2} = \frac{AC^2}{PR^2}$$

The activity may be repeated by taking different pairs of similar triangles.

Let us illustrate these results with the help of examples.

Example 14.7: Find the ratio of the area of two similar triangles if one pair of their corresponding sides are 2.5 cm and 5.0 cm.

Solution: Let the two triangles be ABC and PQR

Let BC = 2.5 cm and QR = 5.0 cm

$$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta PQR)}=\frac{BC^2}{QR^2}=\frac{(2.5)^2}{(5.0)^2}=\frac{1}{4}$$

Example 14.8: In a ΔABC, PQ || BC and intersects AB and AC at P and Q respectively. If $\frac{AP}{BP}=\frac{2}{3}$ find the ratio of areas ΔAPQ and ΔABC.

Solution: In Fig 14.26

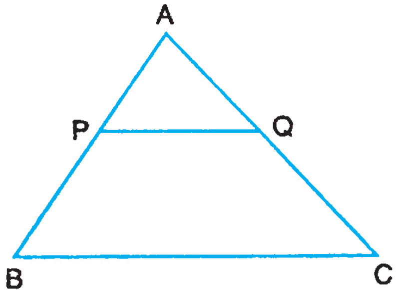


Fig. 14.26

PQ || BC

$$\therefore \quad \frac{AP}{BP}=\frac{AQ}{QC}=\frac{2}{3}$$

$$\therefore \quad \frac{BP}{AP}=\frac{QC}{AQ}=\frac{3}{2}$$

$$\therefore 1+\frac{BP}{AP}=1+\frac{QC}{AQ}=1+\frac{3}{2}=\frac{5}{2}$$

$$\Rightarrow \frac{AB}{AP}=\frac{AC}{AQ}=\frac{5}{2} \quad \Rightarrow \frac{AP}{AB}=\frac{AQ}{AC}=\frac{2}{5}$$

$$\therefore \quad \Delta APQ \sim \Delta ABC$$

$$\therefore \frac{\text{Area}(\Delta APQ)}{\text{Area}(\Delta ABC)}=\frac{AP^2}{AB^2}=\left(\frac{AP}{AB}\right)^2=\left(\frac{2}{5}\right)^2=\frac{4}{25}(\because \Delta APQ \sim \Delta ABC)$$

CHECK YOUR PROGRESS 14.4

1. In Fig. 14.27, ABC is a right triangle with A = 90° and C = 30°. Show that ΔDAB ~ ΔDCA ~ ΔACB.

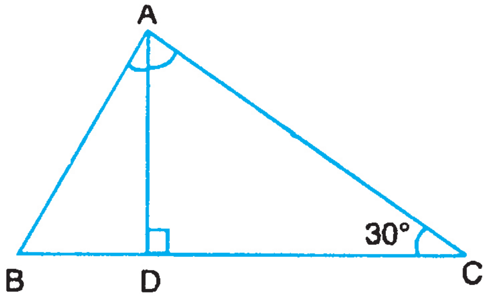


Fig. 14.27

2. Find the ratio of the areas of two similar triangles if two of their corresponding sides are of length 3 cm and 5 cm.

3. In Fig. 14.28, ABC is a triangle in which DE || BC. If AB = 6 cm and AD = 2 cm, find the ratio of the areas of ΔADC and trapezium DBCE.

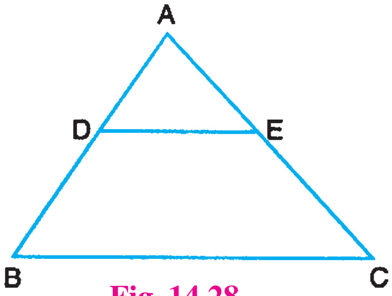


Fig. 14.28

4. P, Q and R are respectively the mid-points of the sides AB, BC and CA of the ΔABC. Show that the area of ΔPQR is one-fourth the area of ΔABC.

5. In two similar triangles ABC and PQR, if the corresponding altitudes AD and PS are in the ratio of 4 : 9, find the ratio of the areas of ΔABC and ΔPQR.

$$\left[\text{Hint : Use } \frac{AB}{PQ} = \frac{AD}{PS} = \frac{BC}{QR} = \frac{CA}{PR}\right]$$

6. If the ratio of the areas of two similar triangles is 16 : 25, find the ratio of their corresponding sides.

14.6 BAUDHYAN/PYTHAGORAS THEOREM

We now prove an important theorem, called Baudhayan/Phythagoras Theorem using the concept of similarity.

Theorem: In a right triangle, the square on the hypotenuse is equal to sum of the squares on the other two sides.

Given: A right triangle ABC, in which $\angle B = 90^o$.

To Prove: $AC^2 = AB^2 + BC^2$

Construction: From B, draw $BD \perp AC$ (See Fig. 14.29)

Proof: $BD \perp AC$

$\therefore \quad \Delta ADB \sim \Delta ABC \quad$ ...(i)

and $\quad \Delta BDC \sim \Delta ABC \quad$ ...(ii)

From (i), we get $\frac{AB}{AC} = \frac{AD}{AB}$

$\Rightarrow \quad AB^2 = AC \,.\, AD \quad$ ...(X)

From (ii), we get $\frac{BC}{AC} = \frac{DC}{BC}$

$\Rightarrow \quad BC^2 = AC \,.\, DC \quad$ ...(Y)

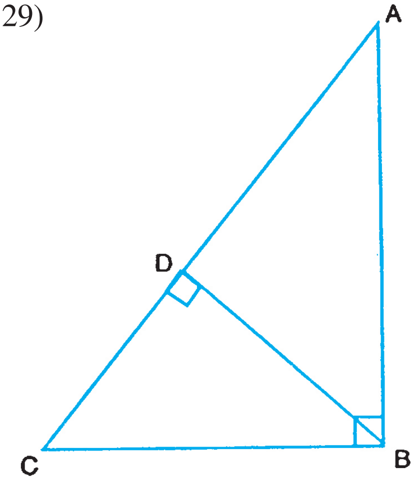


Fig. 14.29

Adding (X) and (Y), we get

$$AB^2 + BC^2 = AC\ (AD + DC)$$

$$= AC \,.\, AC = AC^2$$

The theorem is known after the name of famous Greek Mathematician Pythagoras. This was originally stated by the Indian mathematician Baudhayan about 200 years before Pythagoras in about 800 BC.

14.6.1 Converse of Pythagoras Theorem

The conserve of the above theorem states:

In a triangle, if the square on one side is equal to sum of the squares on the other two sides, the angle opposite to first side is a right angle.

This result can be verified by the following activity.

Draw a triangle ABC with side 3 cm, 4 cm and 5 cm.

i.e., $\quad AB = 3$ cm, $BC = 4$ cm

and $AC = 5$ cm (Fig. 14.30)

You can see that $AB^2 + BC^2 = (3)^2 + (4)^2$

$$= 9 + 16 = 25$$

$AC^2 = (5)^2 = 25$

$\therefore AB^2 + BC^2 = AC^2$

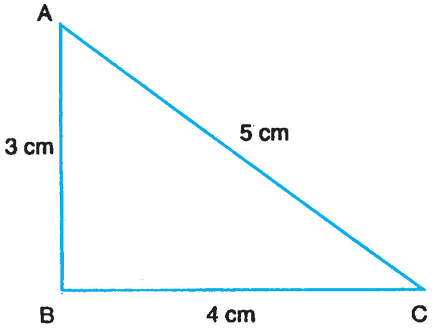


Fig. 14.30

The triangle in Fig. 14.30 satisfies the condition of the above result.

Measure $\angle ABC$, you will find that $\angle ABC = 90^\circ$. Construct triangles of sides 5 cm, 12 cm and 13 cm, and of sides 7 cm, 24 cm, 25 cm. You will again find that the angles opposite to side of length 13 cm and 25 cm are 90° in each case.

Example 14.9: In a right triangle, the sides containing the right angle are of length 5 cm and 12 cm. Find the length of the hypotenuse.

Solution: Let ABC be the right triangle, right angled at B.

$\therefore$ AB = 5 cm, BC = 12 cm

Also, $AC^2 = BC^2 + AB^2$

$= (12)^2 + (5)^2$

$= 144 + 125$

$= 169$

$\therefore$ $AC = 13$

i.e., the length of the hypotenuse is 13 cm.

Example 14.10: Find the length of diagonal of a rectangle the lengths of whose sides are 3 cm and 4 cm.

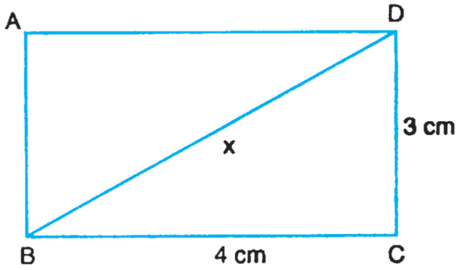


Fig. 14.31

Solution: In Fig. 14.31, is a rectangle ABCD. Join the diagonal BD. Now DCB is a right triangle.

$\therefore$ $BD^2 = BC^2 + CD^2$

$= 4^2 + 3^2$

$= 16 + 9 = 25$

$BD = 5$

i.e., the length of diagonal of rectangle ABCD is 5 cm.

Example 14.11: In an equilateral triangle, verify that three times the square on one side is equal to four times the square on its altitude.

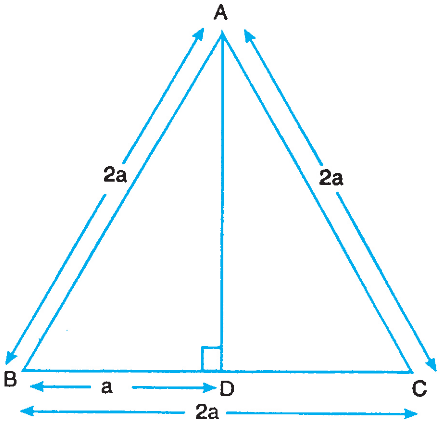


Fig. 14.32

Solution: The altitude $AD \perp BC$

and BD = CD (Fig. 14.32)

Let AB = BC = CA = 2a

and BD = CD = a

Let AD = x

$\therefore$ $x^2 = (2a)^2 - (a)^2 = 3a^2$

$3.\ (\text{Side})^2 = 3.\ (2a)^2 = 12\ a^2$

$4.\ (\text{Altitude})^2 = 4.\ 3a^2 = 12a^2$

Hence the result.

Example 14.12: ABC is a right triangle, right angled at C. If CD, the length of perpendicular from C on AB is p, BC = a, AC = b and AB = c (Fig. 14.33), show that:

(i) $pc = ab$

(ii) $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$

Solution: (i) $CD \perp AB$

$\therefore \Delta ABC \sim \Delta ACD$

$\therefore \frac{c}{b} = \frac{a}{p}$

$\Rightarrow pc = ab$

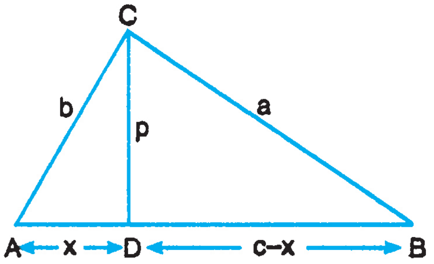


Fig. 14.33

(ii) $AB^2 = AC^2 + BC^2$

or $c^2 = b^2 + a^2$

$$\left(\frac{ab}{p}\right)^2 = b^2 + a^2$$

or $$\frac{1}{p^2} = \frac{a^2 + b^2}{a^2 b^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

CHECK YOUR PROGRESS 14.5

1. The sides of certain triangles are given below. Determine which of them are right triangles: [AB = c, BC = a, CA = b]

   (i) a = 4 cm, b = 5 cm, c = 3 cm

   (ii) a = 1.6 cm, b = 3.8 cm, c = 4 cm

   (iii) a = 9 cm, b = 16 cm, c = 18 cm

   (iv) a = 7 cm, b = 24 cm, c = 25 cm

2. Two poles of height 6 m and 11 m, stand on a plane ground. If the distance between their feet is 12 m, find the distance between their tops.

3. Find the length of the diagonal of a square of side 10 cm.

4. In Fig. 14.34, ∠C is acute and AD ⊥ BC. Show that $AB^2 = AC^2 + BC^2 - 2\,BC.\,DC$.

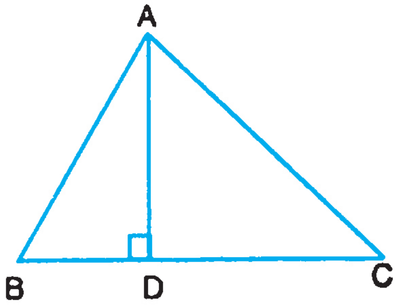


Fig. 14.34

5. L and M are the mid-points of the sides AB and AC of ΔABC, right angled at B. Show that $4LC^2 = AB^2 + 4\,BC^2$

6. P and Q are points on the sides CA and CB respectively of ΔABC, right angled at C Prove that $AQ^2 + BP^2 = AB^2 + PQ^2$

7. PQR is an isosceles right triangle with $\angle Q = 90^o$. Prove that $PR^2 = 2PQ^2$.

8. A ladder is placed against a wall such that its top reaches upto a height of 4 m of the wall. If the foot of the ladder is 3 m away from the wall, find the length of the ladder.

LET US SUM UP

- Objects which have the same shape but different or same sizes are called similar objects.
- Any two polygons, with corresponding angles equal and corresponding sides proportional are similar.
- If a line is drawn parallel to one-side of a triangle, it divides the other two sides in the same ratio and its converse.
- The bisector of an interior angle of a triangle divides the opposite side in the ratio of sides containing the angle.
- Two triangles are said to be similar, if

  (a) their corresponding angles are equal **and**

  (b) their corresponding sides are proportional
- Criteria of similarity

  - AAA criterion

  - SSS criterion

  - SAS criterion

- If a perpendicular is drawn from the vertex of the right angle of a right angled triangle to the hypotenuse, the triangles so formed are similar to each other and to the given triangle.
- The ratio of the areas of two similar triangles is equal to the ratio of squares of their corresponding sides.
- In a right triangle, the square on the hypotenuse is equal to sum of the squares on the remaining two sides – (Baudhayan Pythagoras Theorem).
- In a triangle, if the square on one side is equal to the sum of the squares on the remaining two sides, then the angle opposite to the first side is a right angle – converse of (Baudhayan) Pythagoras Theorem.

TERMINAL EXERCISE

1. Write the criteria for the similarity of two polygons.
2. Enumerate different criteria for the similarity of the two triangles.
3. In which of the following cases, Δ's ABC and PQR are similar.

   (i) $\angle A = 40^o, \angle B = 60^o, \angle C = 80^o, \angle P = 40^o, \angle Q = 60^o$ and $\angle R = 80^o$

   (ii) $\angle A = 50^o, \angle B = 70^o, \angle C = 60^o, \angle P = 50^o, \angle Q = 60^o$ and $\angle R = 70^o$

   (iii) AB = 2.5 cm, BC = 4.5 cm, CA = 3.5 cm

   PQ = 5.0 cm, QR = 9.0 cm, RP = 7.0 cm

   (iv) AB = 3 cm, QR = 7.5 cm, RP = 5.0 cm

   PQ = 4.5 cm, QR = 7.5 cm, RP = 6.0 cm.
4. In Fig. 14.35, AD = 3 cm, AE = 4.5 cm, DB = 4.0 cm, find CE, give that DE || BC.

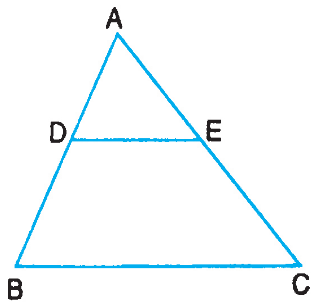


Fig. 14.35

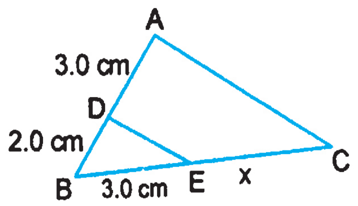


Fig. 14.36

5. In Fig. 14.36, DE || AC. From the dimensions given in the figure, find the value of x.

6. In Fig. 14.37 is shown a ΔABC in which AD = 5 cm, DB = 3 cm, AE = 2.50 cm and EC = 1.5 cm. Is DE || BC? Give reasons for your answer.

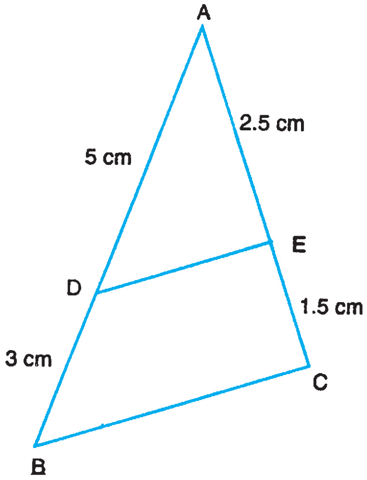


Fig. 14.37

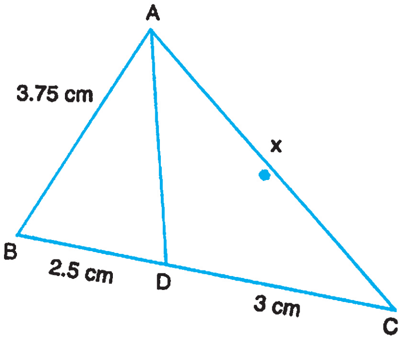


Fig. 14.38

7. In Fig. 14.38, AD is the internal bisector of ∠A of ΔABC. From the given dimensions, find x.

8. The perimeter of two similar triangles ABC and DEF are 12 cm and 18 cm. Find the ratio of the area of ΔABC to that of ΔDEF.

9. The altitudes AD and PS of two similar triangles ABC and PQR are of length 2.5 cm and 3.5 cm. Find the ratio of area of ΔABC to that of ΔPQR.

10. Which of the following are right triangles?

(i) AB = 5 cm, BC = 12 cm, CA = 13 cm

(ii) AB = 8 cm, BC = 6 cm, CA = 10 cm

(iii) AB = 10 cm, BC = 5 cm, CA = 6 cm

(iv) AB = 25 cm, BC = 24 cm, 7 = 13 cm

(v) AB = $a^2 + b^2$, BC = 2ab, CA = $a^2 - b^2$

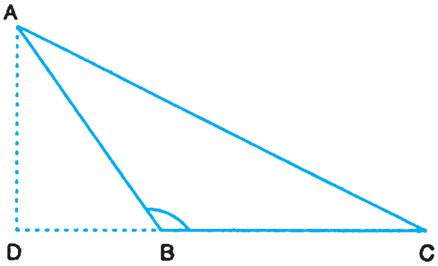


Fig. 14.39

11. Find the area of an equilateral triangle of side 2a.

12. Two poles of heights 12 m and 17 m, stand on a plane ground and the distance between their feet is 12 m. Find the distance between their tops.

13. In Fig. 13.39, show that:

$AB^2 = AC^2 + BC^2 + 2\ BC.\ CD$

14. A ladder is placed against a wall and its top reaches a point at a height of 8 m from the ground. If the distance between the wall and foot of the ladder is 6 m, find the length of the ladder.

15. In an equilateral triangle, show that three times the square of a side equals four times the square of medians.

ANSWERS TO CHECK YOUR PROGRESS

14.1

1. (i) 6 (ii) 6 (iii) 10 cm
2. (i) No (ii) Yes (iii) Yes

14.2

1. 7.5 cm 2. 4 cm

3. $\frac{yz}{x}$ (x = –1 is not possible)

14.3

1. (i) x = 4.5, y = 3.5 (ii) x = 70, y = 50 (iii) x = 2 cm, y = 7 cm

14.4

2. 9 : 25 3. 1 : 8 5. 16 : 81 6. 4 : 5

14.5

1. (i) Yes (ii) No (iii) No (iv) Yes
2. 13 m 3. $10\sqrt{2}$ cm 8. 5 m

ANSWERS TO TERMINAL EXERCISE

3. (i) and (iii) 4. 6 cm 5. 4.5 cm 6. Yes : $\frac{AD}{DB} = \frac{AE}{EC}$

7. 4.5 cm 8. 4 : 9 9. 25 : 49 10. (i), (ii), (iv) and (v)

11. $\sqrt{3}\,a^2$ 12. 13 m 14. 10 m

15

CIRCLES

You are already familiar with geometrical figures such as a line segment, an angle, a triangle, a quadrilateral and a circle. Common examples of a circle are a wheel, a bangle, alphabet O, etc. In this lesson we shall study in some detail about the circle and related concepts.

OBJECTIVES

After studying this lesson, you will be able to

- *define a circle*
- *give examples of various terms related to a circle*
- *illustrate congruent circles and concentric circles*
- *identify and illustrate terms connected with circles like chord, arc, sector, segment, etc.*
- *verify experimentally results based on arcs and chords of a circle*
- *use the results in solving problems*

EXPECTED BACKGROUND KNOWLEDGE

- Line segment and its length
- Angle and its measure
- Parallel and perpendicular lines
- Closed figures such as triangles, quadrilaterals, polygons, etc.
- Perimeter of a closed figure
- Region bounded by a closed figure
- Congruence of closed figures

15.1 CIRCLE AND RELATED TERMS

15.1.1 Circle

A circle is a collection of all points in a plane which are at a constant distance from a fixed point in the same plane.

Radius : A line segment joining the centre of the circle to a point on the circle is called its radius.

In Fig. 15.1, there is a circle with centre O and one of its radius is OA. OB is another radius of the same circle.

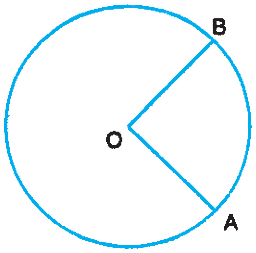


Fig. 15.1

Activity for you : Measure the length OA and OB and observe that they are equal. Thus

All radii (plural of radius) of a circle are equal

The length of the radius of a circle is generally denoted by the letter 'r'. It is customry to write radius instead of the length of the radius.

A closed geometric figure in the plane divides the plane into three parts namely, the inner part of the figure, the figure and the outer part. In Fig. 15.2, the shaded portion is the inner part of the circle, the boundary is the circle and the unshaded portion is the outer part of the circle.

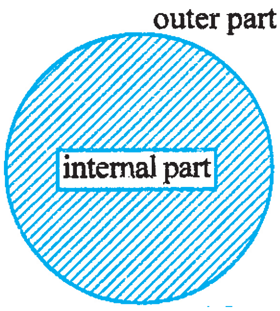


Fig. 15.2

Activity for you

(a) Take a point Q in the inner part of the circle (See Fig. 15.3). Measure OQ and find that OQ < r. The inner part of the circle is called **the interior of the circle**.

(b) Now take a point P in the outer part of the circle (Fig. 15.3). Measure OP and find that OP > r. The outer part of the circle is called **the exterior of the circle.**

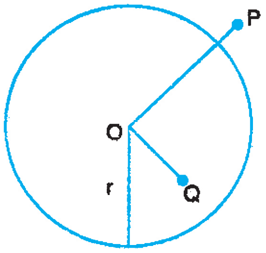


Fig. 15.3

15.1.2 Chord

A line segment joining any two points of a circle is called a chord. In Fig. 15.4, AB, PQ and CD are three chords of a circle with centre O and radius r. The chord PQ passes through the centre O of the circle. Such a chord is called a diameter of the circle. Diameter is usually denoted by 'd'.

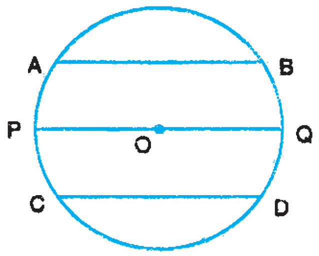


Fig. 15.4

A chord passing though the centre of a circle is called its diameter.

Activity for you :

Measure the length d of PQ, the radius r and find that d is the same as 2r. Thus we have

$d = 2r$

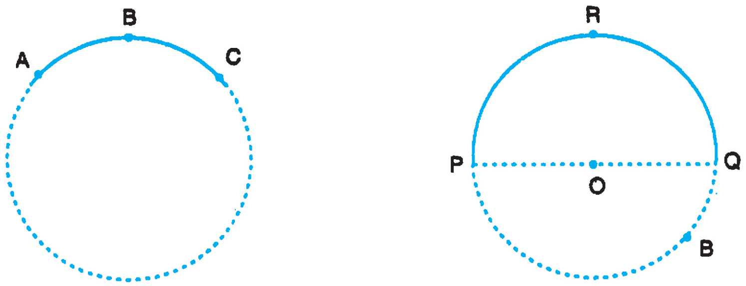
i.e. the diameter of a circle = twice the radius of the circle.

Measure the length PQ, AB and CD and find that PQ > AB and PQ > CD, we may conclude

Diameter is the longest chord of a circle.

15.1.3 Arc

A part of a circle is called an arc. In Fig. 15.5(a) ABC is an arc and is denoted by arc ABC



(a) (b)

Fig. 15.5

15.1.4 Semicircle

A diameter of a circle divides a circle into two equal arcs, each of which is known as a semicircle.

In Fig. 15.5(b), PQ is a diameter and $\overset{\frown}{PRQ}$ is semicircle and so is $\overset{\frown}{PBQ}$.

15.1.5 Sector

The region bounded by an arc of a circle and two radii at its end points is called a sector.

In Fig. 15.6, the shaded portion is a sector formed by the arc PRQ and the unshaded portion is a sector formed by the arc PTQ.

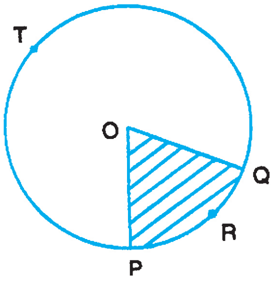


Fig. 15.6

15.1.6 Segment

A chord divides the interior of a circle into two parts,

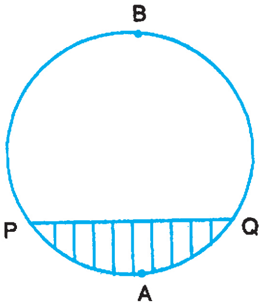


Fig. 15.7

each of which is called a segment. In Fig. 15.7, the shaded region PAQP and the unshaded region PBQP are both segments of the circle. PAQP is called a minor segment and PBQP is called a major segment.

15.1.7 Circumference

Choose a point P on a circle. If this point moves along the circle once and comes back to its original position then the distance covered by P is called the circumference of the circle

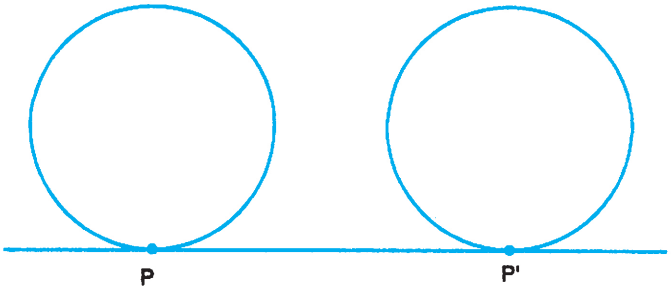


Fig. 15.8

Activity for you :

Take a wheel and mark a point P on the wheel where it touches the ground. Rotate the wheel along a line till the point P comes back on the ground. Measure the distance between the Ist and last position of P along the line. This distance is equal to the circumference of the circle. Thus,

The length of the boundary of a circle is the circumference of the circle.

Activity for you

Consider different circles and measures their circumference(s) and diameters. Observe that in each case the ratio of the circumference to the diameter turns out to be the same.

The ratio of the circumference of a circle to its diameter is always a constant. This constant is universally denoted by Greek letter π.

Therefore, $\frac{c}{d} = \frac{c}{2r} = \pi$, where c is the circumference of the circle, d its diameter and r is its radius.

An approximate value of π is $\frac{22}{7}$. Aryabhata -I (476 A.D.), a famous Indian Mathematician gave a more accurate value of π which is 3.1416. In fact this number is an irrational number.

15.2 MEASUREMENT OF AN ARC OF A CIRCLE

Consider an arc PAQ of a circle (Fig. 15.9). To measure its length we put a thread along PAQ and then measure the length of the thread with the help of a scale.

Similarly, you may measure the length of the arc PBQ.

15.2.1 Minor arc

An arc of circle whose length is less than that of a semi-circle of the same circle is called a minor arc. PAQ is a minor arc (See Fig. 15.9)

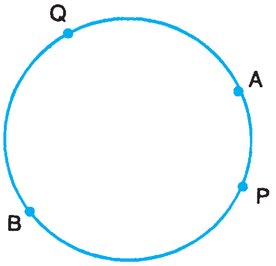


Fig. 15.9

15.2.2 Major arc

An arc of a circle whose length is greater than that of a semicircle of the same circle is called a major arc. In Fig. 15.9, arc PBQ is a major arc.

15.3 CONCENTRIC CIRCLES

Circles having the same centre but different radii are called concentric circles (See Fig. 15.10).

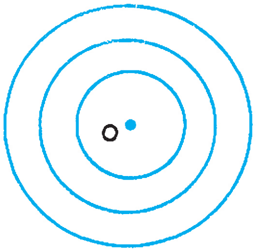


Fig. 15.10

15.4 CONGRUENT CIRCLES OR ARCS

Two cirlces (or arcs) are said to be congruent if we can superimpose (place) one over the other such that they cover each other completely.

15.5 SOME IMPORTANT RULES

Activity for you :

(i) Draw two circles with cenre O_1 and O_2 and radius r and s respectively (See Fig. 15.11)

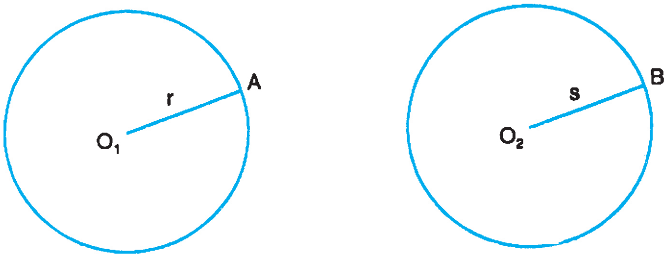


Fig. 15.11

(ii) Superimpose the circle (i) on the circle (ii) so that O_1 coincides with O_2.

(iii) We observe that circle (i) will cover circle (ii) if and only if r = s

Two circles are congruent if and only if they have equal radii.

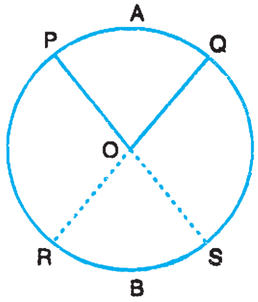


Fig. 15.12

In Fig. 15.12 if arc PAQ = arc RBS then ∠ POQ = ∠ ROS and conversely if ∠ POQ = ∠ ROS then arc PAQ = arc RBS.

Two arcs of a circle are congruent if and only if the angles subtended by them at the centre are equal.

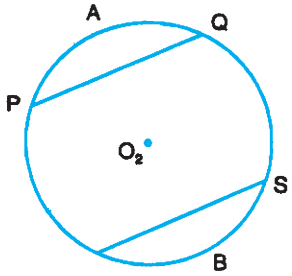


Fig. 15.13

In Fig. 15.13, if arc PAQ = arc RBS

then PQ = RS

and conversely if PQ = RS then

arc PAQ = arc RBS.

Two arcs of a circle are congruent if and only if their corresponding chords are equal.

Activity for you :

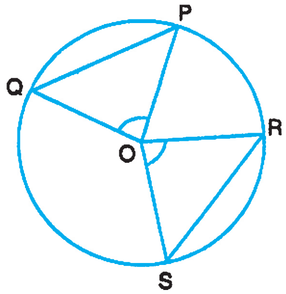


Fig. 15.14

(i) Draw a circle with centre O

(ii) Draw equal chords PQ and RS (See Fig. 15.14)

(iii) Join OP, OQ, OR and OS

(iv) Measure ∠ POQ and ∠ ROS

We observe that ∠ POQ = ∠ ROS

Conversely if ∠ POQ = ∠ ROS

then PQ = RS

Equal chords of a circle subtend equal angles at the centre and conversely if the angles subtended by the chords at the centre of a circle are equal, then the chords are equal.

Note : *The above results also hold good in case of congruent circles.*

We take some examples using the above properties :

Example 15.1 : In Fig. 15.15, chord PQ = chord RS. Show that chord PR = chord QS.

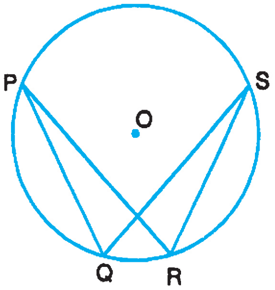


Fig. 15.15

Solution : The arcs corresponding to equal chords PQ and RS are equal.

Add to each arc, the arc QR,

yielding arc PQR = arc QRS

$\therefore$ chord PR = chord QS

Example 15.2 : In Fig. 15.16, arc AB = arc BC, $\angle$ AOB = 30° and $\angle$ AOD = 70°. Find $\angle$ COD.

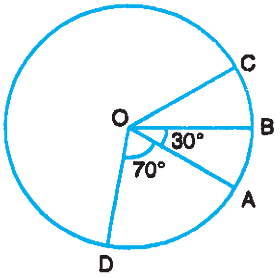


Fig. 15.16

Solution : Since arc AB = arc BC

$\therefore$ $\angle AOB = \angle BOC$

(Equals arcs subtend equal angles at the centre)

$\therefore$ $\angle BOC = 30^\circ$

Now $\angle COD = \angle COB + \angle BOA + \angle AOD$

$= 30^\circ + 30^\circ + 70^\circ$

$= 130^\circ$.

Activity for you :

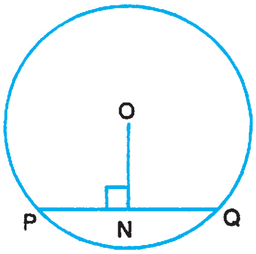


Fig. 15.17

(i) Draw a circle with centre O (See Fig. 15.17).

(ii) Draw a chord PQ.

(iii) From O draw ON $\perp$ PQ

(iv) Measure PN and NQ

You will observe that

PN = NQ.

The perpendicular drawn from the centre of a circle to a chord bisects the chord.

Activity for you :

(i) Draw a circle with centre O (See Fig. 15.18).

(ii) Draw a chord PQ.

(iii) Find the mid point M of PQ.

(iv) Join O and M.

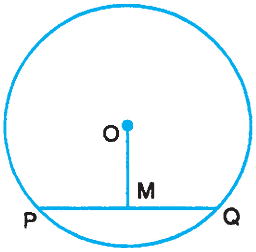


Fig. 15.18

(v) Measure $\angle OMP$ or $\angle OMQ$ with set square or protractor.

We observe that $\angle OMP = \angle OMQ = 90^0$.

The line joining the centre of a circle to the mid point of a chord is perpendicular to the chord.

Activity for you :

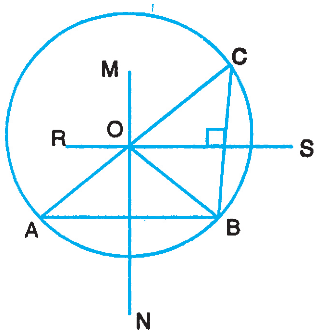


Fig. 15.20

Take three non collinear points A, B and C. Join AB and BC. Draw perpendicular bisectors MN and RS of AB and BC respectively.

Since A, B, C are not collinear, MN is not parallel to RS. They will intersect only at one point O. Join OA, OB and OC and measure them.

We observe that OA = OB = OC

Now taking O as the centre and OA as radius draw a circle which passes through A, B and C.

Repeat the above procedure with another three non-collinear points and observe that there is only one circle passing through three given non-collinear points.

There is one and only one circle passing through three non-collinear points.

Note. It is important to note that a circle can not be drawn to pass through three collinear points.

Activity for you :

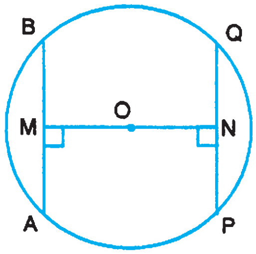


Fig. 15.20a

(i) Draw a circle with centre O [Fig. 15.20a]

(ii) Draw two equal chords AB and PQ of the circle.

(iii) Draw $OM \perp PQ$ and $ON \perp PQ$

(iv) Measure OM and ON and observe that they are equal.

Equal chords of a circle are equidistant from the centre.

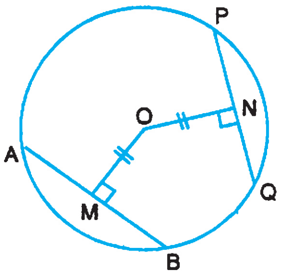


Fig. 15.20b

In Fig. 15.20 b, OM = ON

Measure and observe that AB = PQ. Thus,

Chords, that are equidistant from the centre of a circle, are equal.

The above results hold good in case of congruent circles also.

We now take a few examples using these properties of circle.

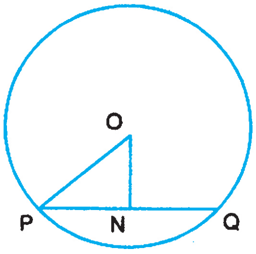


Fig. 15.21

Examples 15.3 : In Fig. 15.21, O is the centre of the circle and $ON \perp PQ$. If PQ = 8 cm and ON = 3 cm, find OP.

Solution: $ON \perp PQ$ (given) and since perpendicular drawn from the centre of a circle to a chord bisects the chord.

$\therefore$ PN = NQ = 4 cm

In a right triangle OPN,

$$\therefore \quad OP^2 = PN^2 + ON^2$$

$$\text{or} \quad OP^2 = 4^2 + 3^2 = 25$$

$$\therefore \quad OP = 5 \text{ cm.}$$

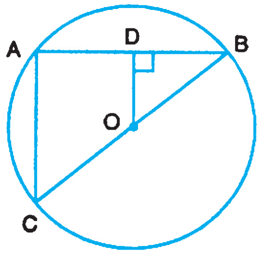


Fig. 15.22

Examples 15.4 : In Fig. 15.22, OD is perpendicular to the chord AB of a circle whose centre is O and BC is a diameter. Prove that CA = 2OD.

Solution : Since $OD \perp AB$ (Given)

$\therefore$ D is the mid point of AB (Perpendicular through the centre bisects the chord)

Also O is the mid point of CB (Since CB is a diameter)

Now in ΔABC, O and D are mid points of the two sides BC and BA of the triangle ABC. Since the line segment joining the mid points of any two sides of a triangle is parallel and half of the third side.

$$\therefore \quad OD = \frac{1}{2} CA$$

$$\text{i.e.} \quad CA = 2OD.$$

Example 15.5 : A regular hexagon is inscribed in a circle. What angle does each side of the hexagon subtend at the centre?

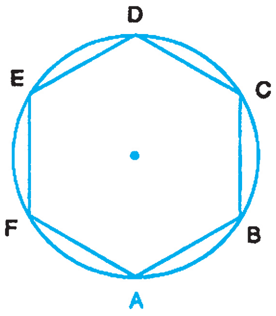


Fig. 15.23

Solution : A regular hexagon has six sides which are equal. Therefore each side subtends the same angle at the centre.

Let us suppose that a side of the hexagon subtends an angle x° at the centre.

Then, we have

$$6x^\circ = 360^\circ \Rightarrow x = 60^\circ$$

Hence, each side of the hexagon subtends an angle of 60° at the centre.

Example 15.6 : In Fig. 15.24, two parallel chords PQ and AB of a circle are of lengths 7 cm and 13 cm respectively. If the distance between PQ and AB is 3 cm, find the radius of the circle.

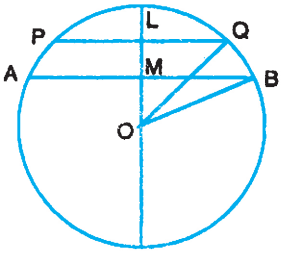


Fig. 15.24

Solution : Let O be the centre of the circle. Draw perpendicular bisector OL of PQ which also bisects AB at M. Join OQ and OB (Fig. 15.24)

Let OM = x cm and radius of the circle be r cm

Then $OB^2 = OM^2 + MB^2$ and $OQ^2 = OL^2 + LQ^2$

$$\therefore \quad r^2 = x^2 + \left(\frac{13}{2}\right)^2 \qquad \text{...(i)}$$

$$\text{and} \quad r^2 = (x+3)^2 + \left(\frac{7}{2}\right)^2 \qquad \text{...(ii)}$$

Therefore from (i) and (ii),

$$x^2 + \left(\frac{13}{2}\right)^2 = (x+3)^2 + \left(\frac{7}{2}\right)^2$$

$$\therefore \quad 6x = \frac{169}{4} - 9 - \frac{49}{4}$$

or 6x = 21

$$\therefore \quad x = \frac{7}{2}$$

$$\therefore \quad r^2 = \left(\frac{7}{2}\right)^2 + \left(\frac{13}{2}\right)^2 = \frac{49}{4} + \frac{169}{4} = \frac{218}{4}$$

$$\therefore \quad r = \frac{\sqrt{218}}{2}$$

Hence the radius of the circle is $r = \frac{\sqrt{218}}{2}$ cm.

CHECK YOUR PROGRESS 15.1

In questions 1 to 5, fill in the blanks to make each of the statements true.

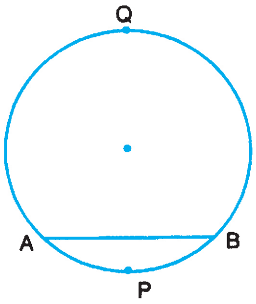


Fig. 15.25

1. In Fig. 15.25,

   (i) AB is a ... of the circle.

   (ii) Minor arc corresponding to AB is... .

2. A ... is the longest chord of a circle.

3. The ratio of the circumference to the diameter of a circle is always ... .

4. The value of π as 3.1416 was given by great Indian Mathematician... .

5. Circles having the same centre are called ... circles.

6. Diameter of a circle is 30 cm. If the length of a chord is 20 cm, find the distance of the chord from the centre.

7. Find the circumference of a circle whose radius is

   (i) 7 cm (ii) 11 cm. $\left(\text{Take } \pi = \dfrac{22}{7}\right)$

8. In the Fig. 15.26, RS is a diameter which bisects the chords PQ and AB at the points M and N respectively. Is PQ || AB ? Given reasons.

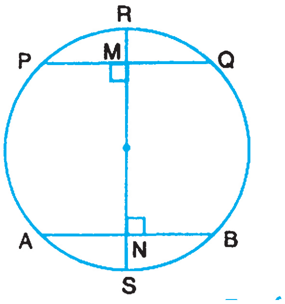


Fig. 15.26

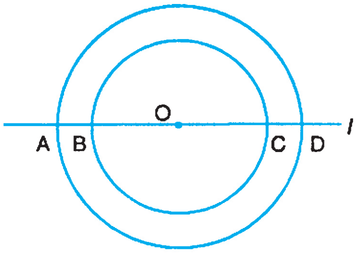


Fig. 15.27

9. In Fig. 15.27, a line *l* intersects the two concentric circles with centre O at points A, B, C and D. Is AB = CD ? Give reasons.

LET US SUM UP

- The circumference of a circle of radius r is equal to 2 π r.

- Two arcs of a circle are congruent if and only if either the angles subtended by them at the centre are equal or their corresponding chords are equal.
- Equal chords of a circle subtend equal angles at the centre and vice versa.
- Perpendicular drawn from the centre of a circle to a chord bisects the chord.
- The line joining the centre of a circle to the mid point of a chord is perpendicular to the chord.
- There is one and only one circle passng through three non-collinear points.
- Equal chords of a circle are equidistant from the centre and the converse.

TERMINAL EXERCISE

1. If the length of a chord of a circle is 16 cm and the distance of the chord from the centre is 6 cm, find the radius of the circle.
2. Two circles with centres O and O′ (See Fig. 15.28) are congruent. Find the length of the arc CD.

Fig. 15.28

3. A regular pentagon is inscribed in a circle. Find the angle which each side of the pentagon subtends at the centre.
4. In Fig. 15.29, AB = 8 cm and CD = 6 cm are two parallel chords of a circle with centre O. Find the distance between the chords.

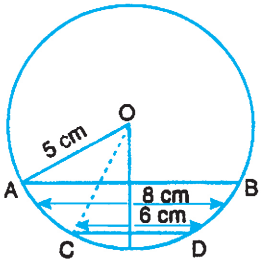


Fig. 15.29

5. In Fig.15.30 arc PQ = arc QR, ∠POQ = 15^0 and ∠SOR = 110°. Find ∠SOP.

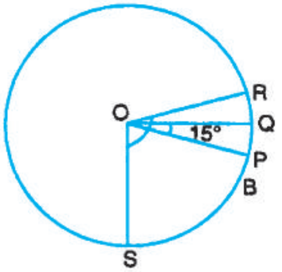


Fig. 15.30

6. In Fig. 15.31, AB and CD are two equal chords of a circle with centre O. Is chord BD = chord CA? Give reasons.

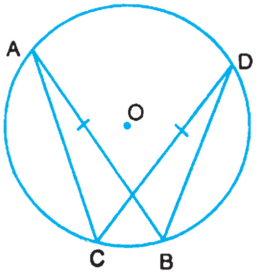


Fig. 15.31

7. If AB and CD are two equal chords of a circle with centre O (Fig. 15.32) and OM ⊥ AB, ON ⊥ CD. Is OM = ON ? Give reasons.

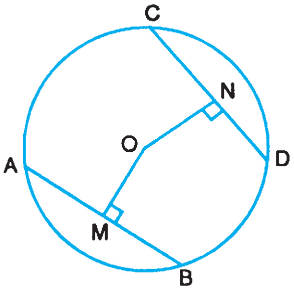


Fig. 15.32

8. In Fig. 15.33, AB = 14 cm and CD = 6 cm are two parallel chords of a circle with centre O. Find the distance between the chords AB and CD.

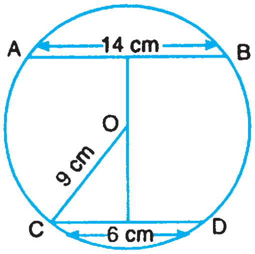


Fig. 15.33

9. In Fig. 15.34, AB and CD are two chords of a circle with centre O, intersecting at a point P inside the circle.

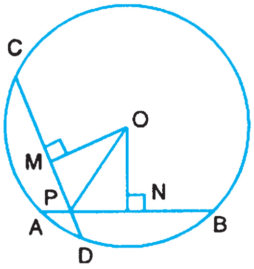


Fig. 15.34

OM $\perp$ CD, ON $\perp$ AB and $\angle$OPM = $\angle$OPN. Now answer:

Is (i) OM = ON, (ii) AB = CD ? Give reasons.

10. C_1 and C_2 are concentric circles with centre O (See Fig. 15.35), *l* is a line intersecting C_1 at points P and Q and C_2 at points A and B respectively, ON $\perp$ *l*, is PA = BQ? Give reasons.

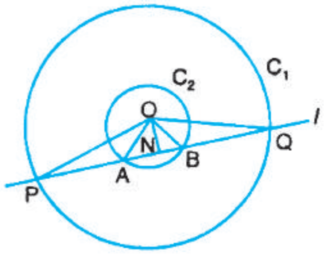


Fig. 15.35

ANSWERS TO CHECK YOUR PROGRESS

15.1

1. (i) Chord (ii) APB 2. Diameter 3. Constant

4. Aryabhata-I 5. Concentric 6. $5\sqrt{5}$ cm.

7. (i) 44 cm (ii) 69.14 cm 8. Yes 9. Yes

ANSWERS TO TERMINAL EXERCISE

1. 10 cm 2. 2a cm 3. 72^o

4. 1 cm 5. 80^o

6. Yes (Equal arcs have corresponding equal chrods of acircle)

7. Yes (equal chords are equidistant from the centre of the circle)

8. $10\sqrt{2}$ cm 9. (i) Yes (ii) Yes ($\Delta OMP \cong \Delta ONP$)

10. Yes (N is the middle point of chords PQ and AB).

16

ANGLES IN A CIRCLE AND CYCLIC QUADRILATERAL

You must have measured the angles between two straight lines. Let us now study the angles made by arcs and chords in a circle and a cyclic quadrilateral.

OBJECTIVES

After studying this lesson, you will be able to

- *verify that the angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle;*
- *prove that angles in the same segment of a circle are equal;*
- *cite examples of concyclic points;*
- *define cyclic quadrilterals;*
- *prove that sum of the opposite angles of a cyclic quadrilateral is* 180^o*;*
- *use properties of cyclic qudrilateral;*
- *solve problems based on Theorems (proved) and solve other numerical problems based on verified properties;*
- *use results of other theorems in solving problems.*

EXPECTED BACKGROUND KNOWLEDGE

- Angles of a triangle
- Arc, chord and circumference of a circle
- Quadrilateral and its types

16.1 ANGLES IN A CIRCLE

Central Angle. The angle made at the centre of a circle by the radii at the end points of an arc (or a chord) is called the central angle or angle subtended by an arc (or chord) at the centre.

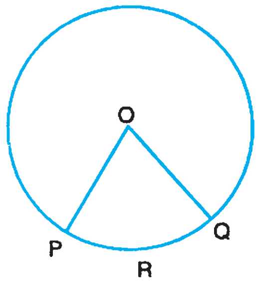


Fig. 16.1

In Fig. 16.1, ∠POQ is the central angle made by arc PRQ.

The length of an arc is closely associated with the central angle subtended by the arc. Let us define the "degree measure" of an arc in terms of the central angle.

The degree measure of a minor arc of a circle is the measure of its corresponding central angle.

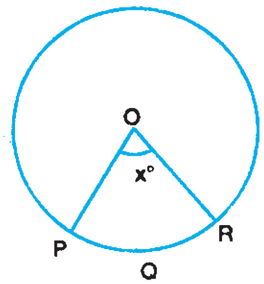


Fig. 16.2

In Fig. 16.2, Degree measure of PQR = x^o

The degree measure of a semicircle is 180^o and that of a major arc is 360^o minus the degree measure of the corresponding minor arc.

Relationship between length of an arc and its degree measure.

$$\text{Length of an arc} = \text{circumference} \times \frac{\text{degree measure of the arc}}{360^o}$$

If the degree measure of an arc is 40^o

then length of the arc PQR = $2\pi r.\frac{40^o}{360^o} = \frac{2}{9}\pi r$

Inscribed angle : The angle subtended by an arc (or chord) on any point on the remaining part of the circle is called an inscribed angle.

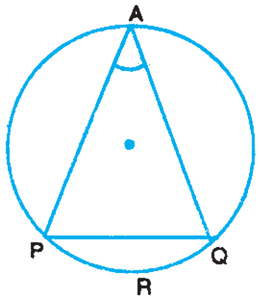


Fig. 16.3

In Fig. 16.3, ∠PAQ is the angle inscribed by arc PRQ at point A of the remaining part of the circle or by the chord PQ at the point A.

16.2 SOME IMPORTANT PROPERTIES

ACTIVITY FOR YOU :

Draw a circle with centre O. Let PAQ be an arc and B any point on the remaining part of the circle.

Measure the central angle POQ and an inscribed angle PBQ by the arc at remaining part of the circle. We observe that

$\angle POQ = 2\angle PBQ$

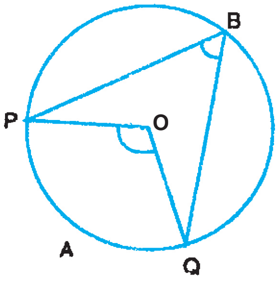


Fig. 16.4

Repeat this activity taking different circles and different arcs. We observe that

> **The angle subtended at the centre of a circle by an arc is double the angle subtended by it on any point on the remaining part of the circle.**

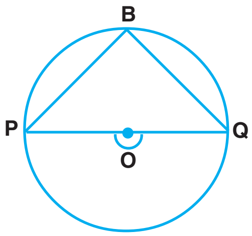


Fig. 16.5

Let O be the centre of a circle. Consider a semicircle PAQ and its inscribed angle PBQ

$$\therefore\ 2\angle PBQ = \angle POQ$$

(Since the angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle)

But $\angle POQ = 180^0$

$$2\angle PBQ = 180^0$$

$\therefore\ \angle PBQ = 90^0$

Thus, we conclude the following:

Angle in a semicircle is a right angle.

Theorem : *Angles in the same segment of a circle are equal*

Given : A circle with centre O and the angles $\angle PRQ$ and $\angle PSQ$ in the same segment formed by the chord PQ (or arc PAQ)

To prove : $\angle PRQ = \angle PSQ$

Construction : Join OP and OQ.

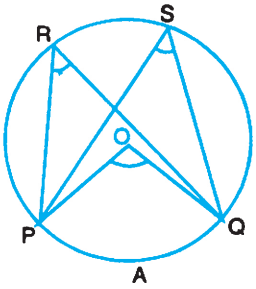


Fig. 16.6

Proof : As the angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle, therefore we have

$$\angle POQ = 2\angle PRQ \qquad ...(i)$$

and $$\angle POQ = 2\angle PSQ \qquad ...(ii)$$

From (i) and (ii), we get

$$2\angle PRQ = 2\angle PSQ$$

$$\therefore \quad \angle PRQ = \angle PSQ$$

We take some examples using the above results

The converse of the result is also true, which we can state as under and verify by the activity.

> **"If a line segment joining two points subtends equal angles at two other points on the same side of the line containing the segment, the four points lie on a circle"**

For verification of the above result, draw a line segment AB (of say 5 cm). Find two points C and D on the same side of AB such that $\angle ACB = \angle ADB$.

Now draw a circle through three non-collinear points A, C, B. What do you observe?

Point D will also lie on the circle passing through A, C and B. i.e. all the four points A, B, C and D are concyclic.

Repeat the above activity by taking another line segment. Every time, you will find that the four points will lie on the same circle.

This verifies the given result.

Example 16.1 : In Fig. 16.7, O is the centre of the circle and $\angle AOC = 120^0$. Find $\angle ABC$.

Solution : It is obvious that $\angle x$ is the central angle subtended by the arc APC and $\angle ABC$ is the inscribed angle.

$$\therefore \quad \angle x = 2\angle ABC$$

$$\text{But} \quad \angle x = 360^\circ - 120^0 = 240^\circ$$

$$\therefore \quad 2\angle ABC = 240^\circ$$

$$\therefore \quad \angle ABC = 120^\circ$$

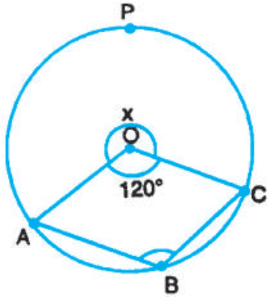


Fig. 16.7

Example 16.2 : In Fig. 16.8, O is the centre of the circle and $\angle PAQ = 35^\circ$. Find $\angle OPQ$.

Solution : $\angle POQ = 2\angle PAQ = 70^\circ$ ...(i)

(Angle at the centre is double the angle on the remaining part of the circle)

Since OP = OQ (Radii of the same circle)

$$\therefore \quad \angle OPQ = \angle OQP \quad \text{...(ii)}$$

(Angles opposite to equal sides are equal)

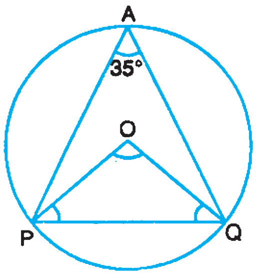


Fig. 16.8

But $\angle OPQ + \angle OQP + \angle POQ = 180^o$

$\therefore \quad 2\angle OPQ = 180^o – 70^o = 110^o$

$\therefore \quad \angle OPQ = 55^o$

Example 16.3 : In Fig. 16.9, O is the centre of the circle and AD bisects $\angle BAC$. Find $\angle BCD$.

Solution : Since BC is a diameter

$\angle BAC = 90^o$

(Angle in the semicircle is a right angle)

As AD bisects $\angle BAC$

$\therefore \quad \angle BAD = 45^o$

But $\angle BCD = \angle BAD$

(Angles in the same segment).

$\therefore \quad \angle BCD = 45^o$

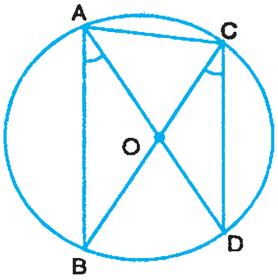


Fig. 16.9

Example 16.4 : In Fig. 16.10, O is the centre of the circle, $\angle POQ = 70^o$ and $PS \perp OQ$. Find $\angle MQS$.

Solution :

$2\angle PSQ = \angle POQ = 70^o$

(Angle subtended at the centre of a circle is twice the angle subtended by it on the remaining part of the circle)

$\therefore \quad \angle PSQ = 35^o$

Since $\angle MSQ + \angle SMQ + \angle MQS = 180^o$

(Sum of the angles of a triangle)

$\therefore \quad 35^o + 90^o + \angle MQS = 180^o$

$\therefore \quad \angle MQS = 180^o – 125^o = 55^o$

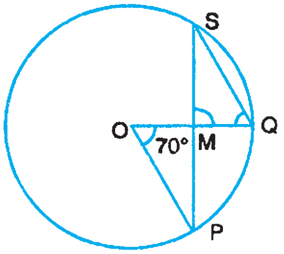


Fig. 16.10

CHECK YOUR PROGRESS 16.1

1. In Fig. 16.11, ADB is an arc of a circle with centre O, if $\angle ACB = 35^o$, find $\angle AOB$.

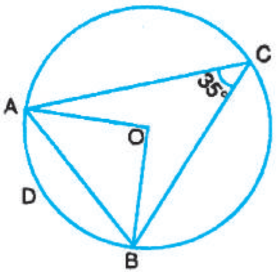


Fig. 16.11

2. In Fig. 16.12, AOB is a diameter of a circle with centre O. Is $\angle APB = \angle AQB = 90^o$. Give reasons.

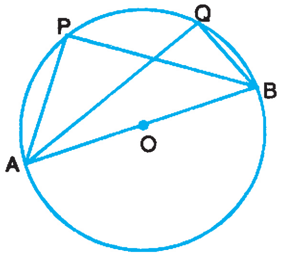


Fig. 16.12

3. In Fig. 16.13, PQR is an arc of a circle with centre O. If $\angle PTR = 35^0$, find $\angle PSR$.

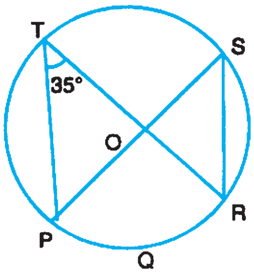


Fig. 16.13

4. In Fig. 16.14, O is the centre of a circle and $\angle AOB = 60^0$. Find $\angle ADB$.

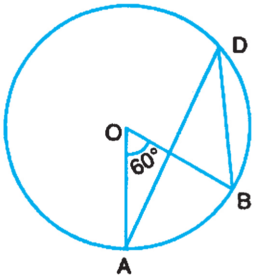


Fig. 16.14

16.3 CONCYLIC POINTS

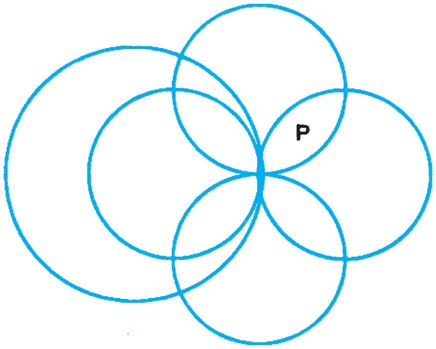


Fig. 16.15

Definition : Points which lie on a circle are called concyclic points.

Let us now find certain conditions under which points are concyclic.

If you take a point P, you can draw not only one but many circles passing through it as in Fig. 16.15.

Now take two points P and Q on a sheet of a paper. You can draw as many circles as you wish, passing through the points. (Fig. 16.16).

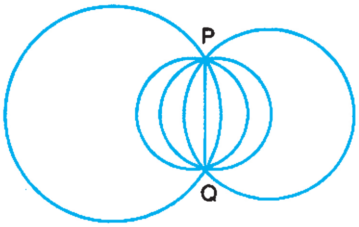


Fig. 16.16

Let us now take three points P, Q and R which do not lie on the same straight line. In this case you can draw only one circle passing through these three non-colinear points (Fig. 16.17).

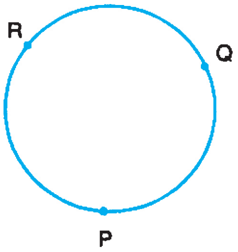
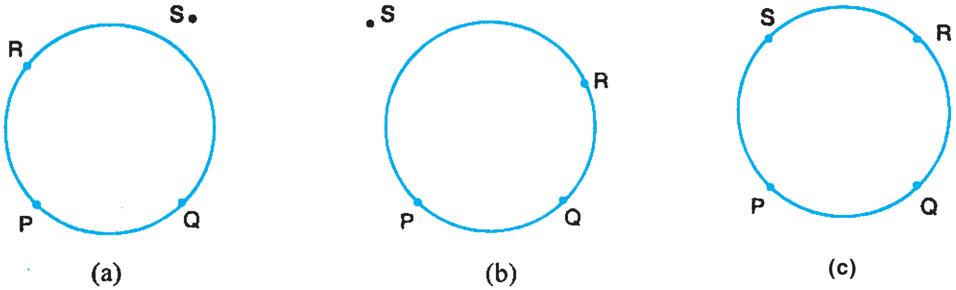


Fig. 16.17

Further let us now take four points P, Q, R, and S which do not lie on the same line. You will see that it is not always possible to draw a circle passing through four non-collinear points.

In Fig. 16.18 (a) and (b) points are noncyclic but concyclic in Fig. 16.18(c)



(a) (b) (c)

Fig. 16.18

Note. If the points, P, Q and R are collinear then it is not possible to draw a circle passing through them.

Thus we conclude

1. **Given one or two points there are infinitely many circles passing through them.**
2. **Three non-collinear points are always concyclic and there is only one circle passing through all of them.**
3. **Three collinear points are not concyclic (or noncyclic).**
4. **Four non-collinear points may or may not be concyclic.**

16.3.1 Cyclic Quadrilateral

A quadrilateral is said to be a cyclic quadrilateral if there is a circle passing through all its four vertices.

For example, Fig. 16.19 shows a cyclic quadrilateral PQRS.

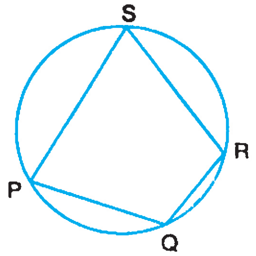


Fig. 16.19

Theorem. *Sum of the opposite angles of a cyclic quadrilateral is* 180^0.

Given : A cyclic quadrilateral ABCD

To prove : $\angle BAD + \angle BCD = \angle ABC + \angle ADC = 180^0$.

Construction : Draw the diagonals AC and DB

Proof : $\angle ACB = \angle ADB$

and $\angle BAC = \angle BDS$

[Angles in the same segment]

$\therefore \quad \angle ACB + \angle BAC = \angle ADB + \angle BDC = \angle ADC$

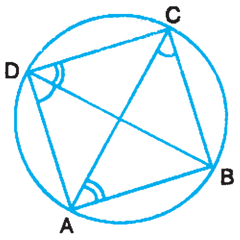


Fig. 16.20

Adding $\angle ABC$ on both the sides, we get

$\angle ACB + \angle BAC + \angle ABC = \angle ADC + \angle ABC$

But $\angle ACB + \angle BAC + \angle ABC = 180^o$ [Sum of the angles of a triangle]

$\therefore \quad \angle ADC + \angle ABC = 180^o$

$\therefore \quad \angle BAD + \angle BCD = \angle ADC + \angle ABC = 180^0$.

Hence proved.

Converse of this theorem is also true.

If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.

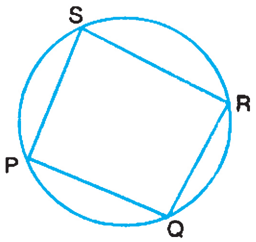


Fig. 16.21

Verification :

Draw a quadrilateral PQRS

Since in quadrilateral PQRS,

$\angle P + \angle R = 180^o$

and $\angle S + \angle Q = 180^o$

Therefore draw a circle passing through the point P, Q and R and observe that it also passes through the point S. So we conclude that quadrilateral PQRS is cyclic quadrilateral.

We solve some examples using the above results.

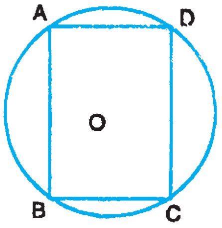


Fig. 16.22

Example 16.5 : ABCD is a cyclic parallelogram. Show that it is a rectangle.

Solution : $\angle A + \angle C = 180^o$

(ABCD is a cyclic quadrilateral)

Since $\angle A = \angle C$

[Opposite angles of a parallelogram]

or $\angle A + \angle A = 180^o$

$\therefore$ $2\angle A = 180^o$

$\therefore$ $\angle A = 90^o$

Thus ABCD is a rectangle.

Example 16.6 : A pair of opposite sides of a cyclic quadrilateral is equal. Prove that its diagonals are also equal (See Fig. 16.23)

Solution : Let ABCD be a cyclic quadrilateral and AB = CD.

$\Rightarrow$ arc AB = arc CD (Corresponding arcs)

Adding arc AD to both the sides;

arc AB + arc AD = arc CD + arc AD

$\therefore$ arc BAD = arc CDA

$\Rightarrow$ Chord BD = Chord CA

$\Rightarrow$ BD = CA

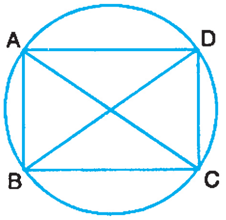


Fig. 16.23

Example 16.7 : In Fig. 16.24, PQRS is a cyclic quadrilateral whose diagonals intersect at A. If $\angle SQR = 80^o$ and $\angle QPR = 30^o$, find $\angle SRQ$.

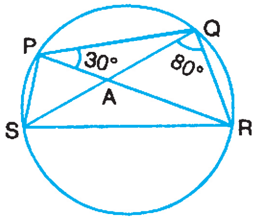


Fig. 16.24

Solution : Given $\angle SQR = 80^o$

Since $\angle SQR = \angle SPR$

[Angles in the same segment]

$\therefore \angle SPR = 80^o$

$\therefore \angle SPQ = \angle SPR + \angle RPQ$

$= 80^o + 30^o$.

or $\angle SPQ = 110^o$.

But $\angle SPQ + \angle SRQ = 180^o$. (Sum of the opposite angles of a cyclic quadrilateral is 180^o)

$\therefore \angle SRQ = 180^o - \angle SPQ$

$= 180^o - 110^o = 70^o$

Example 16.8 : PQRS is a cyclic quadrilateral.

If $\angle Q = \angle R = 65^o$, find $\angle P$ and $\angle S$.

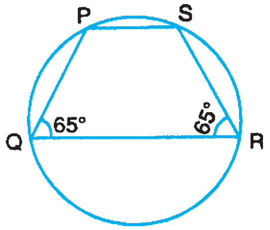


Fig. 16.25

Solution : $\angle P + \angle R = 180^o$

$\therefore \angle P = 180^o - \angle R = 180^o - 65^o$

$\therefore \angle P = 115^o$

Similarly, $\angle Q + \angle S = 180^o$

$\therefore \angle S = 180^o - \angle Q = 180^o - 65^o$

$\therefore \angle S = 115^o$.

CHECK YOUR PROGRESS 16.2

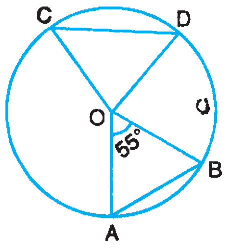


Fig. 16.26

1. In Fig. 16.26, AB and CD are two equal chords of a circle with centre O. If $\angle AOB = 55^0$, find $\angle COD$.

2. In Fig. 16.27, PQRS is a cyclic quadrilateral, and the side PS is extended to the point A. If $\angle PQR = 80^0$, find $\angle ASR$.

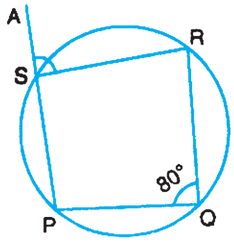


Fig. 16.27

3. In Fig. 16.28, ABCD is a cyclic quadrilateral whose diagonals intersect at O. If ∠ACB = 50° and ∠ABC = 110°, find ∠BDC.

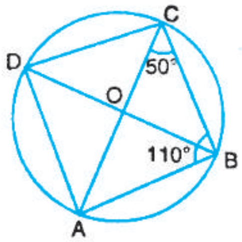


Fig. 16.28

4. In Fig. 16.29, ABCD is a quadrilateral. If ∠A= ∠BCE, is the quadrilateral a cyclic quadrilateral? Give reasons.

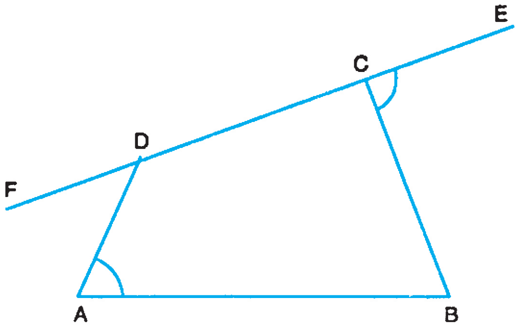


Fig. 16.29

LET US SUM UP

- The angle subtended by an arc (or chord) at the centre of a circle is called central angle and an ngle subtended by it at any point on the remaining part of the circle is called inscribed angle.
- Points lying on the same circle are called concyclic points.
- The angle subtended by an arc at the centre of a circle is double the angle subtended by it at any point on the remaining part of the circle.

- Angle in a semicircle is a right angle.
- Angles in the same segment of a circle are equal.
- Sum of the opposite angles of cyclic quadrilateral is 180^0.
- If a pair of opposite angles of a quadrilateral is supplementary, then the quadrilateral is cyclic.

TERMINAL EXERCISE

1. A square PQRS is inscribed in a circle with centre O. What angle does each side subtend at the centre O?

2. In Fig. 16.30, C_1 and C_2 are two circles with centre O_1 and O_2 and intersect each other at points A and B. If O_1O_2 intersect AB at M then show that

   (i) $\Delta O_1AO_2 \cong \Delta O_1BO_2$

   (ii) M is the mid point of AB

   (iii) $AB \perp O_1O_2$

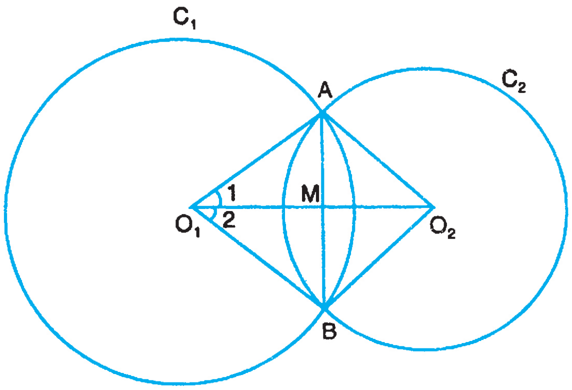


Fig. 16.30

[(Hint. From (i) conclude that $\angle 1 = \angle 2$ and then prove that $\Delta AO_1M \cong \Delta BO_1M$ (by SAS rule)].

3. Two circles intersect in A and B. AC and AD are the diameters of the circles. Prove that C, B and D are collinear.

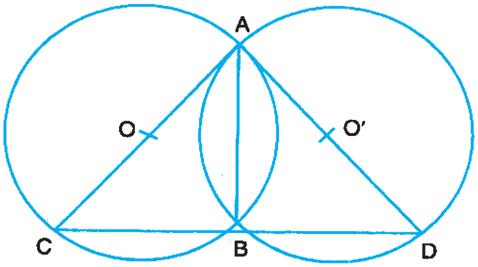


Fig. 16.31

[Hint. Join CB, BD and AB, Since ∠ABC = 90° and ∠ABD = 90°]

4. In Fig. 16.32, AB is a chord of a circle with centre O. If ∠ACB = 40°, find ∠OAB.

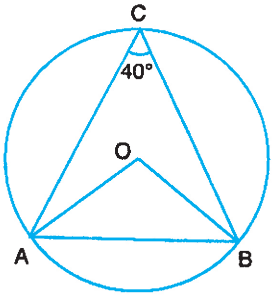


Fig. 16.32

5. In Fig. 16.33, O is the centre of a circle and ∠PQR = 115°. Find ∠POR.

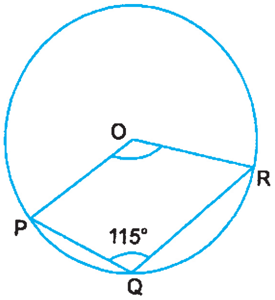


Fig. 16.33

6. In Fig. 16.34, O is the centre of a circle, ∠AOB = 80° and ∠PQB = 70°. Find ∠PBO.

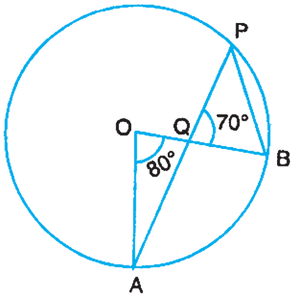


Fig. 16.34

ANSWERS TO CHECK YOUR PROGRESS

16.1

1. 70^o 2. Yes, angle in a semi-circle is a right angle

3. 35^o 4. 30^o

16.2

1. 55^o 2. 80^o 3. 20^o 4. Yes

ANSWERS TO TERMINAL EXERCISE

1. 90^o 4. 50^o 5. 130^o 6. 70^o

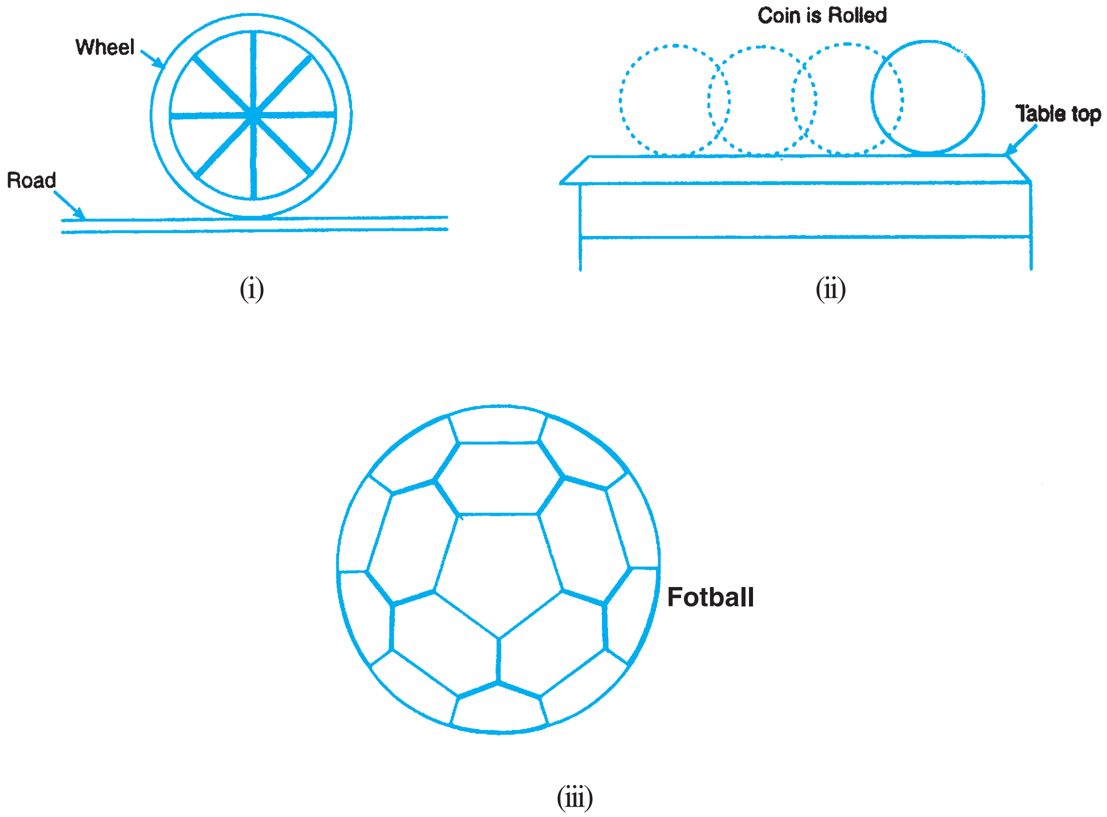
17

SECANTS, TANGENTS AND THEIR PROPERTIES

Look at the moving cycle. You will observe that at any instant of time, the wheels of the moving cycle touch the road at a very limited area, more correctly a point.

If you roll a coin on a smooth surface, say a table or floor, you will find that at any instant of time, only one point of the coin comes in contact with the surface it is rolled upon.

What do you observe from the above situations?



(i) (ii) (iii)

Fig. 17.1

If you consider a wheel or a coin as a circle and the touching surface (road or table) as a line, the above illustrations show that a line touches a circle. In this lesson, we shall study about the possible contacts that a line and a circle can have and try to study their properties.

OBJECTIVES

After studying this lesson, you will be able to

- *define a secant and a tangent to the circle;*
- *differentitate between a secant and a tangent;*
- *prove that the tangents drawn from an external point to a circle are of equal length;*
- *verify the un-starred results (given in the curriculum) related to tangents and secants to circle experimentally.*

EXPECTED BACKGROUND KNOWLEDGE

- Measurement of angles and line segments
- Drawing circles of given radii
- Drawing lines perpendicular and parallel to given lines
- Knowledge of previous results about lines and angles, congruence and circles
- Knowledge of Pythagoras Theorem

17.1 SECANTS AND TANGENTS—AN INTRODUCTION

You have read about lines and circles in your earlier lessons. Recall that a circle is the locus of a point in a plane which moves in such a way that its distance from a fixed point in the plane always remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle. You also know that a line is a collection of points, extending indefinitely to both sides, whereas a line segment is a portion of a line bounded by two points.

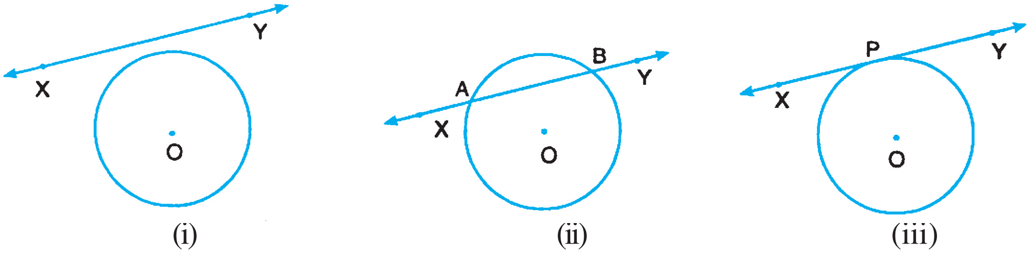


Fig. 17.2

Now consider the case when a line and a circle co-exist in the same plane. There can be three distinct possibilities as shown in Fig. 17.2.

You can see that in Fig. 17.2(i), the XY does not intersect the circle, with centre O. In other words, we say that the line XY and the circle have no common point. In Fig. 17.2 (ii), the line XY intersects the circle in two distinct point A and B, and in Fig. 17.2 (iii), the line XY intersects the circle in only one point and is said to touch the circle at the point P.

Thus, we can say that in case of intersection of a line and a circle, the following three possibilities are there:

(i) The line does not intersect the circle at all, i.e., the line lies in the exterior of the circle.

(ii) The line intersects the circle at two distinct points. In that case, a part of the line lies in the interior of the circle, the two points of intersection lie on the circle and the remaining portion of the line lies in the exterior of the circle.

(iii) The line touches the circle in exactly one point. We therefore define the following:

Tangent:

A line which touches a circle at exactly one point is called a tangent line and the point where it touches the circle is called the point of contact

Thus, in Fig. 17.2 (iii), XY is a tangent of the circle at P, which is called the point of contact.

Secant:

A line which interesects the circle in two distinct points is called a secant line (usually referred to as a secant).

In Fig. 17.2 (ii), XY is a secant line to the circle and A and B are called the points of intersection of the line XY and the circle with centre O.

17.2 TANGENT AS A LIMITING CASE

Consider the secant XY of the circle with centre O, intersecting the circle in the points A and B. Imagine that one point A, which lies on the circle, of the secant XY is fixed and the secant rotates about A, intersecting the circle at B', B'', B''', B'''' as shown in Fig. 17.3 and ultimately attains the position of the line XAY, when it becomes tangent to the circle at A.

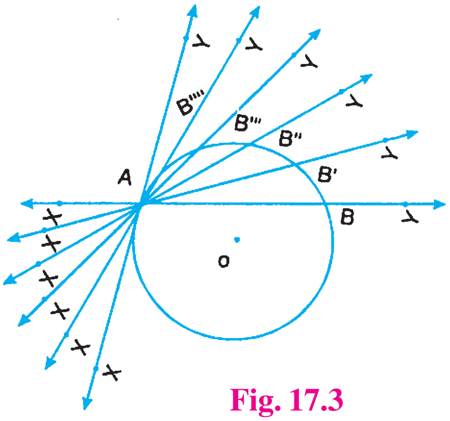


Fig. 17.3

Thus, we say that:

A tangent is the limiting position of a secant when the two points of intersection coincide.

17.3 TANGENT AND RADIUS THROUGH THE POINT OF CONTACT

Let XY be a tangent to the circle, with centre O, at the point P. Join OP.

Take points Q, R, S and T on the tangent XY and join OQ, OR, OS and OT.

As Q, R, S and T are points in the exterior of the circle and P is on the circle.

$\therefore$ OP is less than each of OQ, OR, OS and OT.

From our, "previous study of Geometry, we know that of all the segments that can be drawn from a point (not on the line) to the line, the perpendicular segment is the shortest":

As OP is the shortest distance from O to the line XY

$$\therefore \quad OP \perp XY$$

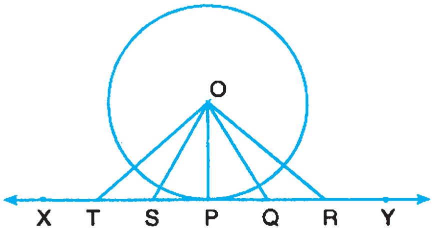


Fig. 17.4

Thus, we can state that

A radius, though the point of contact of tangent to a circle, is perpendicular to the tangent at that point.

The above result can also be verified by measuring angles OPX and OPY and finding each of them equal to 90^o.

17.4 TANGENTS FROM A POINT OUTSIDE THE CIRCLE

Take any point P in the exterior of the circle with centre O. Draw lines through P. Some of these are shown as PT, PA, PB, PC, PD and PT′ in Fig. 17.5

How many of these touch the circle? Only two.

Repeat the activity with another point and a circle. You will again find the same result.

Thus, we can say that

From an external point, two tangents can be drawn to a circle.

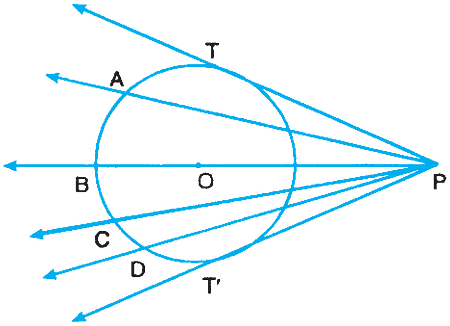


Fig. 17.5

If the point P lies on the circle, can there still be two tangents to the circle from that point? You can see that only one tangent can be drawn to the circle in that case. What about the case when P lies in the interior of the circle? Note that any line through P in that case will intersect the circle in two points and hence no tangent can be drawn from an interior point to the circle.

(A) Now, measure the lengths of PT and PT′. You will find that

$$PT = PT' \qquad ....(i)$$

(B) **Given:** A circle with centre O. PT and PT′ are two tangents from a point P outside the circle.

To Prove: PT = PT′

Construciton: Join OP, OT and OT′ (see Fig. 17.6)

Proof: In Δ's OPT and OPT′

$\angle OTP = \angle OT'P$ (Each being right angle)

OT = OT′

OP = OP (Common)

$\Delta OPT \cong \Delta OPT'$ (RHS criterion)

$\therefore$ PT = PT′

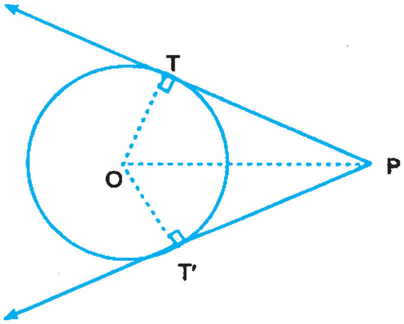


Fig. 17.6

The lengths of two tangents from an external point are equal

Also, from Fig. 17.6, $\angle OPT = \angle OPT'$ (As $\Delta OPT \cong \Delta OPT'$)

The tangents drawn from an external point to a circle are equally inclined to the line joining the point to the centre of the circle.

Let us now take some examples to illustrate:

Example 17.1: In Fig. 17.7, OP = 5 cm and radius of the circle is 3 cm. Find the length of the tangent PT from P to the circle, with centre O.

Solution: $\angle OTP = 90^o$, Let PT = x

In right triangle OTP, we have

$$OP^2 = OT^2 + PT^2$$

or $5^2 = 3^2 + x^2$

or $x^2 = 25 - 9 = 16$

$\therefore$ x = 4

i.e. the length of tangent PT = 4 cm

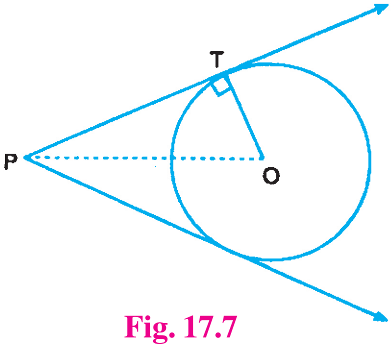


Fig. 17.7

Example 17.2: In Fig. 17.8, tangents PT and PT′ are drawn from a point P at a distance of 25 cm from the centre of the circle whose radius is 7 cm. Find the lengths of PT and PT′.

Solution: Here OP = 25 cm and OT = 7 cm

We also know that

$\angle OTP = 90^o$

$\therefore$ $PT^2 = OP^2 - OT^2$

$= 625 - 49 = 576 = (24)^2$

$\therefore$ PT = 24 cm

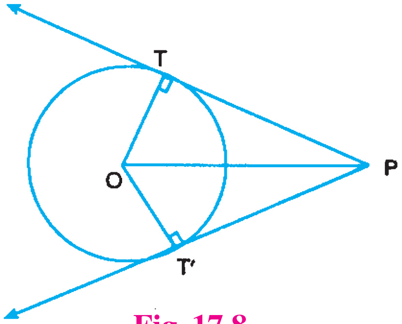


Fig. 17.8

We also know that

$$PT = PT'$$

$$\therefore \qquad PT' = 24 \text{ cm}$$

Example 17.3: In Fig. 17.9, A, B and C are three exterior points of the circle with centre O. The tangents AP, BQ and CR are of lengths 3 cm, 4 cm and 3.5 cm respectively. Find the perimeter of ΔABC.

Solution: We know that the lengths of two tangents from an external point to a circle are equal

$$\therefore \qquad AP = AR$$

$$BP = BQ,$$

$$CQ = CR$$

$$\therefore \qquad AP = AR = 3 \text{ cm}$$

$$BP = BQ = 4 \text{ cm}$$

$$\text{and} \qquad CR = CQ = 3.5 \text{ cm}$$

$$AB = AP + PB;$$

$$= (3 + 4) \text{ cm} = 7 \text{ cm}$$

$$BC = BQ + QC;$$

$$= (4 + 3.5) \text{ cm} = 7.5 \text{ cm}$$

$$CA = AR + CR$$

$$= (3 + 3.5) \text{ cm}$$

$$\therefore \qquad = 6.5 \text{ cm}$$

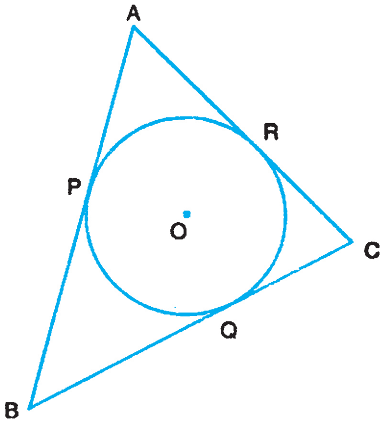


Fig. 17.9

$$\therefore \qquad \text{Perimeter of } \Delta ABC = (7 + 7.5 + 6.5) \text{ cm} = 21 \text{ cm}$$

Example 17.4: In Fig. 17.10, $\angle AOB = 50^{o}$. Find $\angle ABO$ and $\angle OBT$.

Solution: We know that $OA \perp XY$

$$\Rightarrow \qquad \angle OAB = 90^{o}$$

$$\therefore \qquad \angle ABO = 180^{o} - (\angle OAB + \angle AOB)$$

$$= 180^{o} - (90^{o} + 50^{o}) = 40^{o}$$

We know that $\angle OAB = \angle OBT$

$$\Rightarrow \qquad \angle OBT = 40^{o}$$

$$\therefore \qquad \angle ABO = \angle OBT = 40^{o}$$

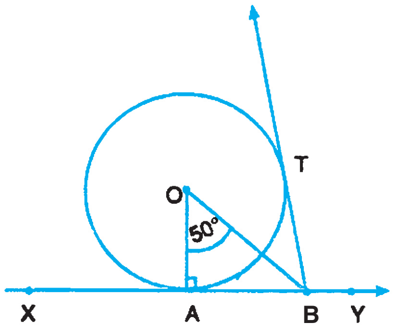


Fig. 17.10

CHECK YOUR PROGRESS 17.1

1. Fill in the blanks:

   (i) A tangent is __________ to the radius through the point of contact.

   (ii) The lengths of tangents from an external point to a circle are __________

   (iii) A tangent is the limiting position of a secant when the two ______ coincide.

   (iv) From an external point ________ tangents can be drawn to a circle.

   (v) From a point in the interior of the circle, ______ tangent(s) can be drawn to the circle.

2. In Fig. 17.11, ∠POY = 40°, Find the ∠OYP and ∠OYT.

3. In Fig. 17.12, the incircle of ΔPQR is drawn. If PX = 2.5 cm, RZ = 3.5 cm and perimeter of ΔPQR = 18 cm, find the lenght of QY.

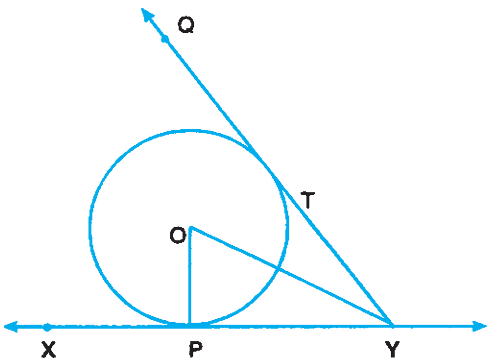


Fig. 17.11

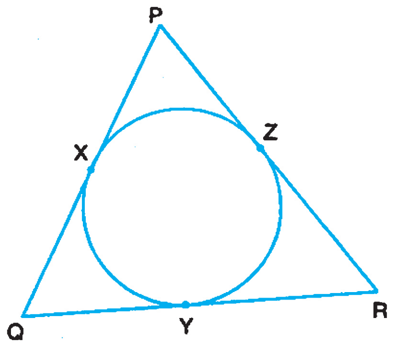


Fig. 17.12

4. Write an experiment to show that the lengths of tangents from an external point to a circle are equal.

17.5 INTERSECTING CHORDS INSIDE AND OUTSIDE A CIRCLE

You have read various results about chords in the previous lessons. We will now verify some results regarding chords intersecting inside a circle or outside a circle, when produced.

Let us perform the following activity:

Draw a circle with centre O and any radius. Draw two chords AB and CD intersecting at P inside the circle.

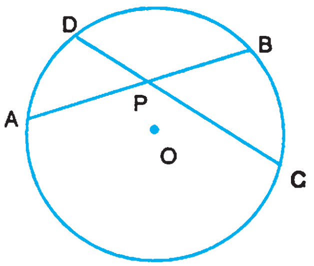


Fig. 17.13

Measure the lenghts of the line-segments PD, PC, PA and PB. Find the products PA × PB and PC × PD.

You will find that they are equal.

Repeat the above activity with another circle after drawing chrods intersecting inside. You will again find that

$$PA \times PB = PC \times PD$$

Let us now consider the case of chrods intersecting outside the circle. Let us perform the following activity:

Draw a circle of any radius and centre O. Draw two chords BA and DC intersecting each other outside the circle at P. Measure the lengths of line segments PA, PB, PC and PD. Find the products PA × PB and PC × PD.

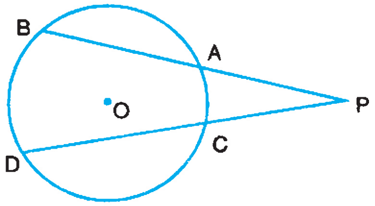


Fig. 17.14

You will see that the product PA × PB is equal to the product PC × PD, i.e.,

$$PA \times PB = PC \times PD$$

Repeat this activity with two circles with chords intersecting outside the circle. You will again find that

$$PA \times PB = PC \times PD.$$

Thus, we can say that

If two chords AB and CD of a circle intersect at a point P (inside or outside the circle), then

$$\mathbf{PA \times PB = PC \times PD}$$

17.6 INTERSECTING SECANT AND TANGENT OF A CIRCLE

To see if there is some relation beween the intersecting secant and tangent outside a circle, we conduct the following activity.

Draw a circle of any radius with centre O. From an external point P, draw a secant PAB and a tangent PT to the circle.

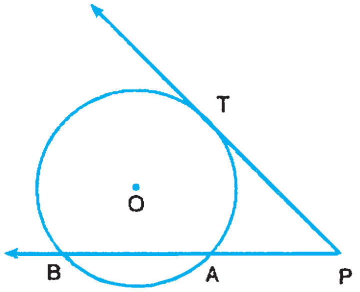


Fig. 17.15

Measure the length of the line-segment PA, PB and PT. Find the products PA × PB and PT × PT or PT^2. What do you find?

You will find that

$$PA \times PB = PT^2$$

Repeat the above activity with two other circles. You will again find the same result.

Thus, we can say

If PAB is a secant to a circle intersecting the circle at A and B, and PT is a tangent to the circle at T, then

$PA \times PB = PT^2$

Let us illustrate these with the help of examples:

Example 17.5: In Fig. 17.16, AB and CD are two chords of a circle intersecting at a point P inside the circle. If PA = 3 cm, PB = 2 cm and PC = 1.5 cm, then find the length of PD.

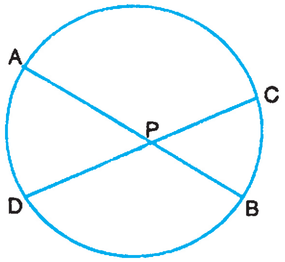


Fig. 17.16

Solution: It is given that PA = 3 cm, PB = 2 cm and PC = 1.5 cm.

Let $PD = x$

We know that $PA \times PB = PC \times PD$

$\Rightarrow \quad 3 \times 2 = (1.5) \times x$

$$\Rightarrow \quad x = \frac{3 \times 2}{1.5} = 4$$

∴ Length of the line-segment PD = 4 cm.

Example 17.6: In Fig. 17.17, PAB is a secant to the circle from a point P outside the circle. PAB passes through the centre of the circle and PT is a tangent. If PT = 8 cm and OP = 10 cm, find the radius of the circle, using $PA \times PB = PT^2$

Solution: Let x be the radius of the circle.

It is given that $OP = 10$ cm

∴ $PA = PO - OA = (10 - x)$ cm

and $PB = OP + OB = (10 + x)$ cm

$PT = 8$ cm

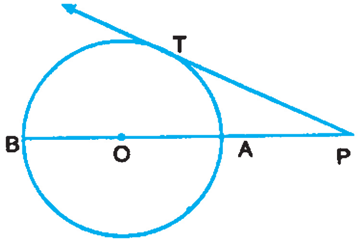


Fig. 17.17

We know that $PA \times PB = PT^2$

∴ $(10 - x)(10 + x) = 8^2$

or $100 - x^2 = 64$

or $x^2 = 36$ or $x = 6$

i.e., radius of the circle is 6 cm.

Example 17.7: In Fig. 17.18, BA and DC are two chords of a circle intersecting each other at a point P outside the circle. If PA = 4 cm, PB = 10 cm, CD = 3 cm, find PC.

Solution: We are given that PA = 4 cm, PB = 10 cm, CD = 3 cm

Let PC =x

We know that $PA \times PB = PC \times PD$

or $4 \times 10 = (x + 3)\,x$

or $x^2 + 3x - 40 = 0$

$(x + 8)(x - 5) = 0$

$\Rightarrow$ $x = 5$

$\therefore$ PC = 5 cm

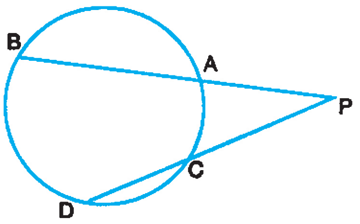


Fig. 17.18

CHECK YOUR PROGRESS 17.2

1. In Fig. 17.19, if PA = 3 cm, PB = 6 cm and PD = 4 cm then find the length of PC.
2. In Fig. 17.19, PA = 4 cm, PB = x + 3, PD = 3 cm and PC = x + 5, find the value of x.

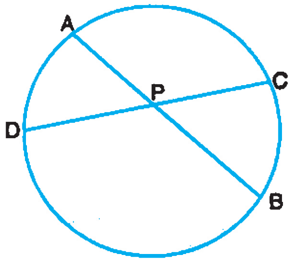


Fig. 17.19

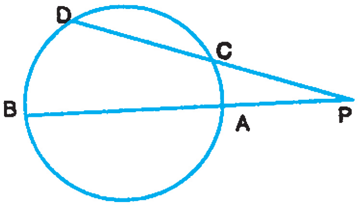


Fig. 17.20

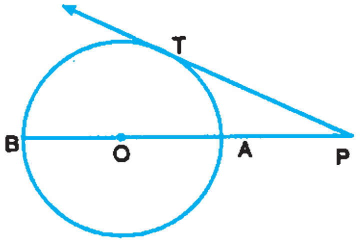


Fig. 17.21

3. If Fig. 17.20, if PA = 4cm, PB = 10 cm, PC = 5 cm, find PD.
4. In Fig. 17.20, if PC = 4 cm, PD = (x + 5) cm, PA = 5 cm and PB = (x + 2) cm, find x.
5. In Fig. 17.21, PT = $2\sqrt{7}$ cm, OP = 8 cm, find the radius of the circle, if O is the centre of the circle.

17.7 ANGLES MADE BY A TANGENT AND A CHORD

Let there be a circle with centre O and let XY be a tangent to the circle at point P. Draw a chord PQ of the circle through the point P as shown in the Fig. 17.22. Mark a point R on the major arc PRQ and let S be a point on the minor arc PSQ.

The segment formed by the major arc PRQ and chord PQ is said to be the alternate segment of ∠QPY and the segment formed by the minor PSQ and chord PQ is said to be the alternate segment to ∠QPX.

Let us see if there is some relationship between angles in the alternate segment and the angle between tangent and chord.

Join QR and PR.

Measure ∠PRQ and ∠QPY (See Fig. 17.22)

What do you find? You will see that ∠PRQ = ∠QPY

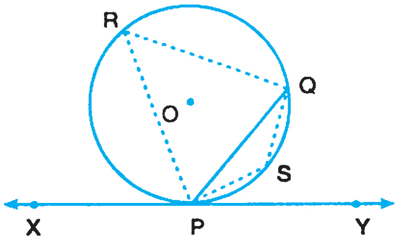


Fig. 17.22

Repeat this activity with another circle and same or different radius. You will again find that ∠QPY = ∠PRQ

Now measure ∠QPX and ∠QSP. You will again find that these angles are equal.

Thus, we can state that

> **The angles formed in the alternate segments by a chord through the point of contact of a tangent to a circle is equal to the angle between the chord and the tangent.**

This result is more commonly called as "Angles in the Alternate Segment".

Let us now check the converse of the above result.

Draw a circle, with centre O, and draw a chord PQ and let it form ∠PRQ in alternate segment as shown in Fig. 17.23.

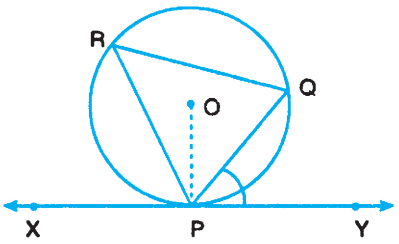


Fig. 17.23

At P, draw ∠QPY = ∠QRP. Extend the line segment PY to both sides to form line XY. Join OP and measure ∠OPY.

What do you observe? You will find that ∠OPY = 90^o showing thereby that XY is a tangent to the circle.

Repeat this activity by taking different circles and you find the same result. Thus, we can state that

> **If a line makes with a chord angles which are equal respectively to the the angles formed by the chord in alternate segments, then the line is a tangent to the circle.**

Let us now take some examples to illustrate:

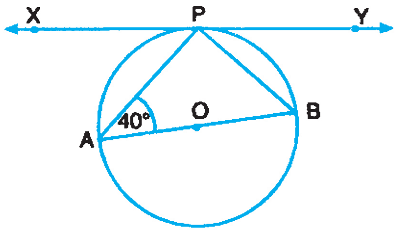


Fig. 17.24

Example 17.8: In Fig. 17.24, XY is tangent to a circle with centre O. If AOB is a diameter and ∠PAB = 40^o, find ∠APX and ∠BPY.

Solution: By the Alternate Segment theorem, we know that

$\angle BPY = \angle BAP$

$\therefore \quad \angle BPY = 40^\circ$

Again, $\angle APB = 90^\circ$ (Angle in a semi-circle]

And, $\angle BPY + \angle APB + \angle APX = 180^\circ$ (Angles on a line)

$\therefore \quad \angle APX = 180^\circ - (\angle BPY + \angle APB)$

$= 180^\circ - (40^\circ + 90^\circ) = 50^\circ$

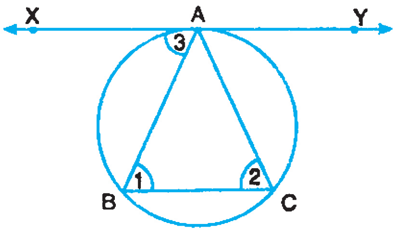


Fig. 17.25

Example 17.9: In Fig. 17.25, ABC is an isoceles triangle with AB = AC and XY is a tangent to the circumcircle of ΔABC. Show that XY is parallel to base BC.

Solution: In ΔABC, AB = AC

$\therefore \quad \angle 1 = \angle 2$

Again XY is tangent to the circle at A.

$\therefore \quad \angle 3 = \angle 2$ (Angles in the alternate segment)

$\therefore \quad \angle 1 = \angle 3$

But these are alternate angles

$\therefore \quad XY \parallel BC$

CHECK YOUR PROGRESS 17.3

1. Explain with the help of a diagram, the angle formed by a chord in the alternate segment of a circle.
2. In Fig. 17.26, XY is a tangent to the circle with centre O at a point P. If $\angle OQP = 40^\circ$, find the value of a and b.

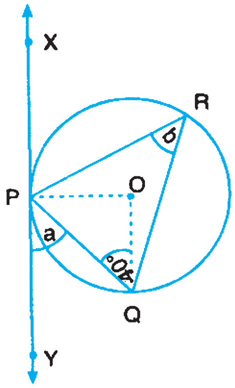


Fig. 17.26

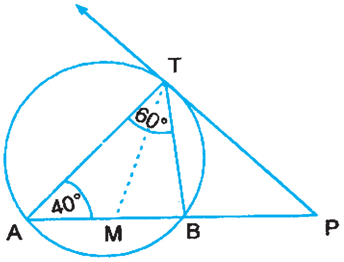


Fig.17.27

3. In Fig. 17.27, PT is a tangent to the circle from an external point P. Chord AB of the circle, when produced meets TP in P. TA and TB are joined and TM is the angle bisector of ∠ATB.

   If ∠PAT = 40° and ∠ATB = 60°, show that PM = PT.

LET US SUM UP

- A line which intersects the circle in two points is called a secant of the circle.
- A line which touches the circle at a point is called a tangent to the circle.
- A tangent is the limiting position of a secant when the two points of intersection coincide.
- A tangent to a circle is perpendicular to the radius through the point of contact.
- From an external point, two tangents can be drawn to a circle, which are of equal length.
- If two chords AB and CD of a circle intersect at a point P (inside or outside the circle), then

  PA × PB = PC × PD
- If PAB is a secant to a circle intersecting the circle at A and B, and PT is a tangent to the circle at T, then

  $PA \times PB = PT^2$
- The angles formed in the alternate segments by a chord through the point of contact of a tangent to a circle are equal to the angles between the chord and the tangent.
- If a line makes with a chord angles which are respectively equal to the angles formed by the chord in alternate segments, then the line is a tangent to the circle.

TERMINAL EXERCISE

1. Differentiate between a secant and a tangent to a circle with the help of a figure.
2. Show that a tangent is a line perpendicular to the radius through the point of contact, with the help of an activity.
3. In Fig. 17.28, if AC = BC and AB is a diameter of circle, find ∠x, ∠y and ∠z.

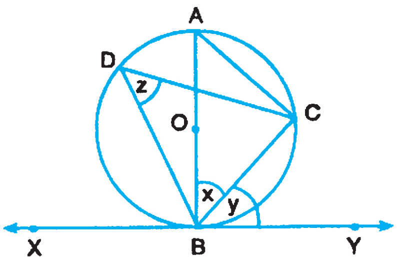


Fig. 17.28

4. In Fig. 17.29, OT = 7 cm and OP = 25 cm, find the length of PT. If PT′ is another tangent to the circle, find the length of PT′ and $\angle POT$.

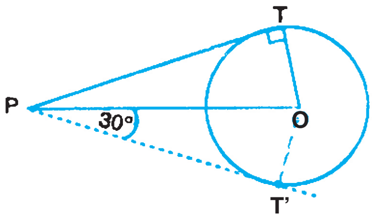


Fig. 17.29

5. In Fig. 17.30, the perimeter of ΔABC equals 27 cm. If PA = 4 cm, QB = 5 cm, find the length of QC.

6. In Fig. 17.30, if $\angle ABC = 70^o$, find $\angle BOC$.

   [Hint: $\angle OBC + \angle OCB = \frac{1}{2}(\angle ABC + \angle ACB)$]

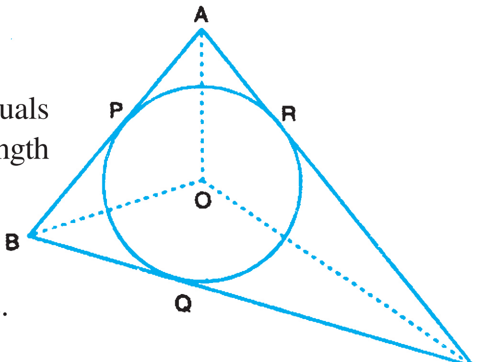


Fig. 17.30

7. In Fig. 17.31, AB and CD are two chords of a circle intersecting at the interior point P of a circle. If PA = (x + 3) cm, PB = (x – 3) cm, PD = 3 cm and PC = $5\frac{1}{3}$ cm, find x.

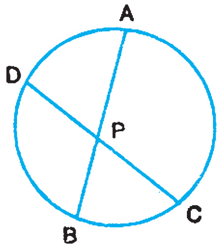


Fig. 17.31

8. In Fig. 17.32, chords BA and DC of the circle, with centre O, intersect at a point P outside the circle. If PA = 4 cm and PB = 9 cm, PC = x and PD = 4x, find the value of x.

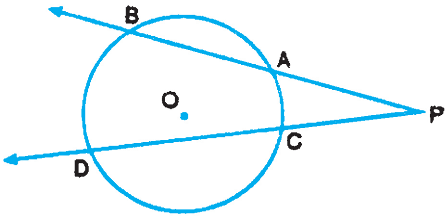


Fig. 17.32

9. In Fig. 17.33, PAB is a secant and PT is a tangent to the circle from an external point. If PT = x cm, PA = 4 cm and AB = 5 cm, find x.

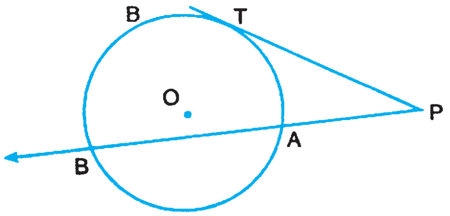


Fig. 17.33

10. In Fig. 17.34, O is the centre of the circle and $\angle PBQ = 40^o$, find

    (i) $\angle QPY$

    (ii) $\angle POQ$

    (iii) $\angle OPQ$

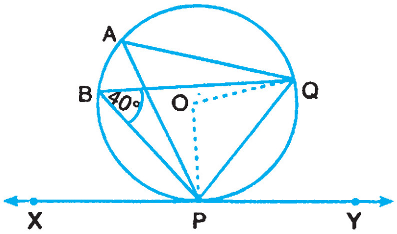


Fig. 17.34

ANSWERS TO CHECK YOUR PROGRESS

17.1

1. (i) Perpendicular (ii) equal (iii) points of intersection

   (iv) two (v) no

2. 50°, 50°

3. 3 cm

17.2

1. 4.3 cm 2. 3 cm 3. 8 cm

4. 10 cm 4. 6 cm

17.3

2. $\angle a = \angle b = 50^o$

ANSWERS TO TERMINAL EXERCISE

1. $\angle x = \angle y = \angle z = 45^o$

4. PT = 24 cm; PT’ = 24 cm, $\angle POT' = 60^o$

5. QC = 4.5 6. $\angle BOC = 125^o$

7. x = 5 8. x = 3

9. x = 6

10. (i) 40° (ii) 80° (iii) 50°

18

CONSTRUCTIONS

One of the aims of studying Geometry is to acquire the skill of drawing figures accurately. You have learnt how to construct geometrical figures namely triangles, squares and circles with the help of ruler and compasses. You have constructed angles of 30°, 60°, 90°, 120° and 45°. You have also drawn perpendicular bisector of a line segment and bisector of an angle.

In this lesson we will extend our learning to construct some other important geometrical figures.

OBJECTIVES

After studying this lesson, you will be able to

- *divide a given line segment internally in a given ratio;*
- *construct a triangle from the given data;*

  (i) SSS

  (ii) SAS

  (iii) ASA

  (iv) RHS

  (v) perimeter and base angles

  (vi) base, sum/difference of the other two sides and one base angle.

  (vii) two sides and a median corresponding to one of these sides.
- *construct a triangle, similar to a given triangle; and;*
- *Construct tangents to a circle from a point:*

  (i) on it using the centre of the circle.

  (i) outside it.

EXPECTED BACKGROUND KNOWLEDGE

We assume that the learner already knows how to use a pair of compasses and ruler to construct

- angles of 30°, 45°, 60°, 90°, 105°, 120°
- the right bisector of a line segment
- bisector of a given angle.

18.1 DIVISION OF A LINE SEGMENT IN THE GIVEN RATIO INTERNALLY

Construction 1: To divide a line segment internally in a given ratio.

Given a line segment AB. You are required to divide it internally in the ratio 2 : 3. We go through the following steps.

Step 1: Draw a ray AC making an acute angle with AB.

Step 2: Starting with A, mark off 5 points C_1, C_2, C_3, C_4 and C_5 on AC at equal distances from the point A.

Step 3: Join C_5 and B.

Step 4: Through C_2 (i.e. the second point), draw C_2D parallel to C_5B meeting AB in D.

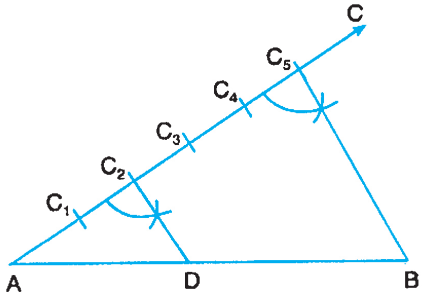


Fig. 18.1

Then D is the required point which divides AB internally in the ratio 2 : 3 as shown in Fig. 18.1.

CHECK YOUR PROGRESS 18.1

1. Draw a line segment 7 cm long. Divide it internally in the ratio 3 : 4. Measure each part. Also write the steps of construction.

2. Draw a line segment PQ = 8 cm. Find point R on it such that PR = $\frac{3}{4}$ PQ.

[**Hint:** Divide the line segment PQ internally in the ratio 3 : 1]

18.2 CONSTRUCTION OF TRIANGLES

Construction 2: To construct a triangle when three sides are given (SSS)

Suppose you are required to construct ΔABC in which AB = 6 cm, AC = 4.8 cm and BC = 5 cm.

We go through the following steps:

Step 1: Draw AB = 6 cm.

Step 2: With A as centre and radius 4.8 cm, draw an arc.

Step 3: With B as centre and radius 5 cm draw another arc intersecting the arc of Step 2 at C.

Step 4: Join AC and BC.

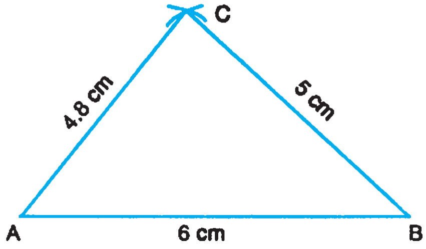


Fig. 18.2

Then ΔABC is the required triangle.

[**Note:** You may take BC or AC as a base]

Construction 3: To construct a triangle, when two sides and the included angle is given (SAS).

Suppose you are required to construct a triangle PQR in which PQ = 5.6 cm, QR = 4.5 cm and ∠PQR = 60°.

Step 1: Draw PQ = 5.6 cm

Step 2: At Q, construct an angle ∠PQX = 60°

Step 3: With Q as centre and radius 4.5 cm draw an arc cutting QX at R.

Step 4: Join PR

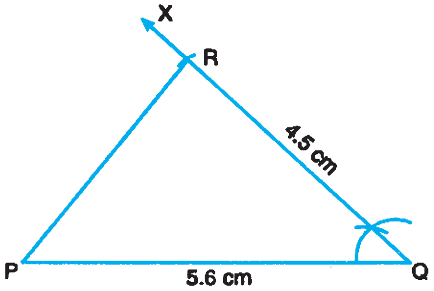


Fig. 18.3

Then ΔPQR is the required triangle.

[**Note:** You may take QR = 4.5 cm as the base instead of PQ]

Construction 4: To construct a triangle when two angles and the included side are given (ASA).

Let us construct a ΔABC in which ∠B = 60°, ∠C = 45° and BC = 4.7 cm.

To construct the triangle we go through the following steps:

Step 1: Draw BC = 4.7 cm.

Step 2: At B, construct ∠CBQ = 60°

Step 3: At C, construct ∠BCR = 45° meeting BQ at A.

Then ΔABC is the required triangle.

Note: To construct a triangle when two angles and any side (other than the included side) are given, we find the third angle (using angle sum property of the triangle) and then use the above method for constructing the triangle.

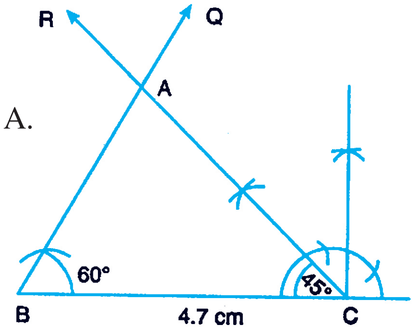


Fig. 18.4

Construction 5: To construct a right triangle, when its hypotenuse and a side are given.

Let us construct a right triangle ABC, right angled at B, side BC = 3 cm and hypotenuse AC = 5 cm

To construct the triangle, we go through the following steps:

Step 1: Draw BC = 3 cm

Step 2: At B, construct ∠CBP = 90°

Step 3: With C as centre and radius 5 cm draw an arc cutting BP in A.

Step 4: Join AC

ΔABC is the required triangle.

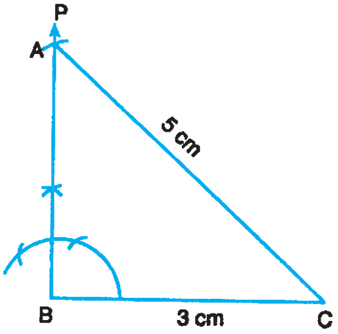


Fig. 18.5

Construction 6: To construct a triangle when its perimeter and two base angles are given.

Suppose we have to construct a triangle whose perimeter is 9.5 cm and base angles are 60° and 45°

To construct the triangle, we go through the following steps:

Step 1: Draw XY = 9.5 cm

Step 2: At X, construct ∠YXP = 30° [which is 1/2 × 60°]

Step 3: At Y, construct ∠XYQ = 22½° [which is 1/2 × 45°]

Let XP and YQ intersect A.

Step 4: Draw right bisector of XA intersecting XY at B.

Step 5: Draw right bisector of YA intersecting XY at C.

Step 6: Join AB and AC.

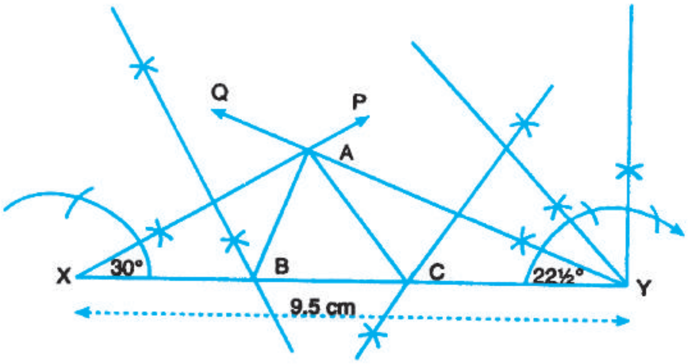


Fig. 18.6

ΔABC is the required triangle.

Construction 7: To construct a triangle when sum of two sides, third side and one of the angles on the third side are given.

Suppose you are required to construct a triangle ABC in which

AB + AC = 8.2 cm, BC = 3.6 cm and ∠B = 45°

To construct the triangle, we go through the following steps:

Step 1: Draw BC = 3.6 cm

Step 2: At B, construct ∠CBK = 45°

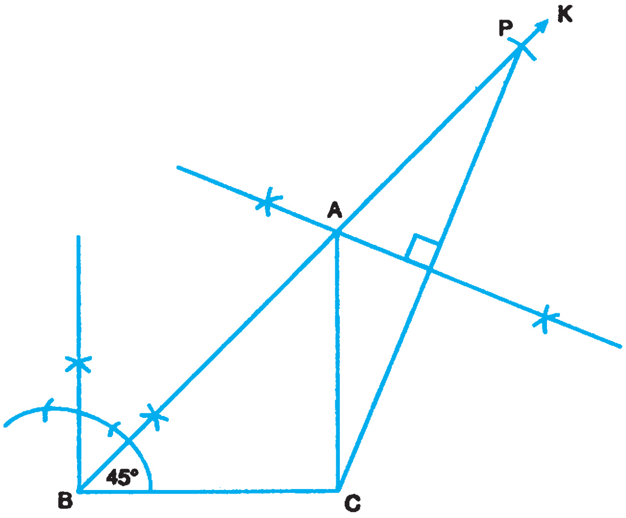


Fig. 18.7

Step 3: From BK, cut off BP = 8.2 cm.

Step 4: Join CP.

Step 5: Draw right bisector of CP intersecting BP at A.

Step 6: Join AC

ΔABC is required triangle.

Construction 8: To construct a triangle when difference of two sides, the third side and one of the angles on the third side are given.

Suppose we have to construct a ΔABC, in which BC = 4 cm, ∠B = 60^o, AB – AC = 1.2 cm.

To construct the triangle we go through the following steps:

Step 1: Draw BC = 4 cm.

Step 2: Construct ∠CBP = 60^o

Step 3: From BP cut off BK = 1.2 cm.

Step 4: Join CK.

Step 5: Draw right bisector of CK meeting BP produced at A.

Step 6: Join AC

ΔABC is the required triangle.

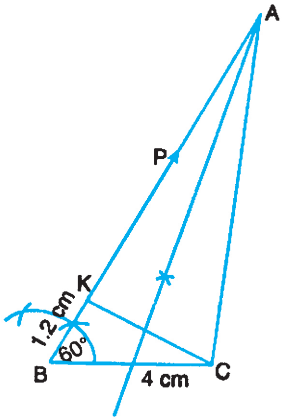


Fig. 18.8

Construction 9: To construct a triangle when its two sides and a median corresponding to one of these sides, are given:

Suppose you have to construct a ΔABC in which AB = 6 cm, BC = 4 cm and median CD = 3.5 cm.

We go through the following steps:

Step 1: Draw AB = 6 cm

Step 2: Draw right bisector of AB meeting AB in D.

Step 3: With D as centre and radius 3.5 cm draw an arc.

Step 4: With B as centre and radius 4 cm draw another arc intersecting the arc of Step 3 in C.

Step 5: Join AC and BC.

Then ΔABC is required triangle.

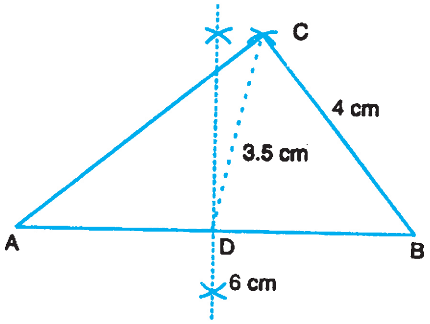


Fig. 18.9

CHECK YOUR PROGRESS 18.2

1. Construct a ΔDEF, given that DE = 5.1 cm, EF = 4 cm and DF = 5.6 cm. Write the steps of construction.

   Note: *You are also required to write the steps of construction in each of the remaining problems.*

2. Construct a ΔPQR, given that PR = 6.5 cm, ∠P = 120° and PQ = 5.2 cm.
3. Construct a ΔABC given that BC = 5.5 cm, ∠B = 75° and ∠C = 45°.
4. Construct a right triangle in which one side is 3 cm and hypotenuse is 7.5 cm.
5. Construct a right angled isoceles triangle in which one of equal sides is 4.8 cm.
6. Construct a ΔABC given that AB + BC + AC = 10 cm, ∠B = 60°, ∠C = 30°.
7. Construct a ΔABC in which AB = 5 cm, ∠A = 60°, BC + AC = 9.8 cm.
8. Construct a ΔLMN, when ∠M = 30°, MN = 5 cm and LM – LN = 1.5 cm.
9. Construct a triangle PQR in which PQ = 5 cm, QR = 4.2 cm and median RS = 3.8 cm.

18.3 TO CONSTRUCT A TRIANGLE SIMILAR TO A GIVEN TRIANGLE, AS PER GIVEN SCALE FACTOR

[Here, **Scale Factor** means the ratio of the sides of the triangle to be constructed, to the corresponding sides of the given triangle.]

Construction 10: Construct a triangle similar to a given triangle ABC with its sides equal to 3/5 of the corresponding sides of the triangle ABC.

Steps of Construction:

1. Let ABC be the given Δ. Draw any ray BX making an acute angle with BC on the side opposite to vertex A.
2. Locate 5 points B_1, B_2, B_3, B_4 and B_5 on BX so that

   $BB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B_5$
3. Join B_5C and draw a line through B_3 parallel to B_5C to meet BC at C′.
4. Draw a line though C′ parallel to CA to meet AB in A′.

   Then ΔA′BC′ is the required Triangle.

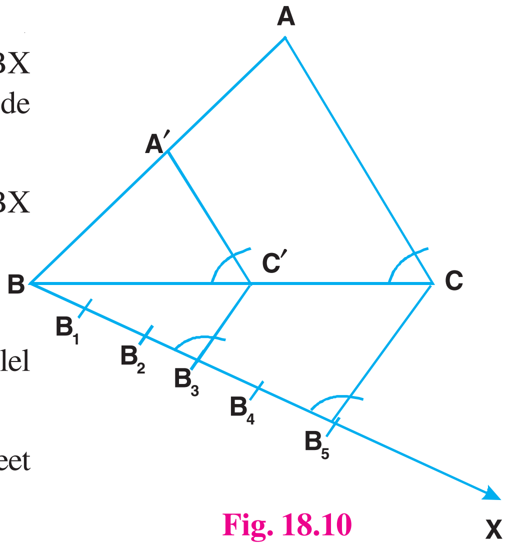


Fig. 18.10

Construction 11: Construct a triangle with sides 5cm, 6 cm and 7 cm. Construct another triangle similar to this triangle with scale factor $\frac{2}{3}$.

Steps of Construction:

1. Draw of a line segment BC = 7 cm
2. Through B draw an arc of radius 6 cm. Through C draw another arc of radius 5 cm to intersect the first arc at A.
3. Join AB and AC to get ΔABC.
4. Draw a ray BX making an acute angle with BC.
5. Locate 3 points B_1, B_2 and B_3 on BX such that $BB_1 = B_1B_2 = B_2B_3$
6. Join B_3C and through B_2 draw a line parallel to B_3C to meet BC in C′.
7. Through C′, draw a line parallel to CA to meet AB at A′.

   Then A′BC′ is the required triangle.

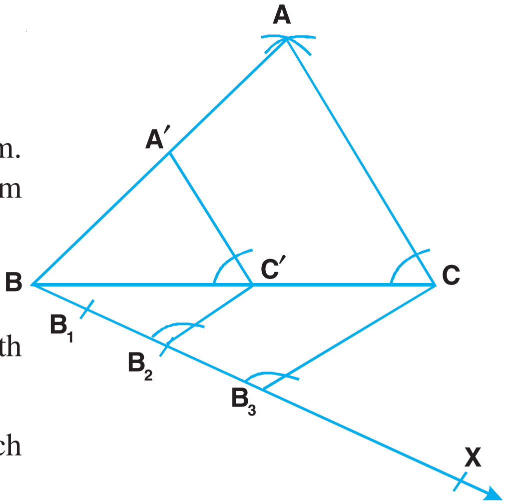


Fig. 18.11

CHECK YOUR PROGRESS 18.3

1. Construct a triangle of sides 4cm, 5 cm and 7 cm and then a triangle similar to it whose sides are $\frac{3}{4}$ of the corresponding sides of the first triangle.
2. Draw a triangle ABC with BC = 7 cm, AB = 5 cm and $\angle ABC = 60^o$. Then construct a triangle whose sides are $\frac{4}{5}$ of the corresponding sides of the triangle ABC.
3. Draw a right triangle with sides (other than hypotenuse) of lenghts 5 cm and 6 cm. Then construct another triangle similar to this triangle with scale factor $\frac{4}{5}$.
4. Draw a ΔABC with base BC = 6 cm, $\angle ABC = 60^o$ and side AB = 4.5 cm. Construct a triangle A′BC′ similar to ABC with scale factor $\frac{5}{6}$.

18.4 CONSTRUCTION OF TANGENTS TO A CIRCLE

Construction 12: To draw a tangent to a given circle at a given point on it using the centre of the circle.

Suppose C be the given circle with centre O and a point P on it. You to draw a tangent to the circle. We go through the following steps:

Step 1: Join OP.

Step 2: At P, draw PT ⊥ OP.

Step 3: Produce TP to Q

Then TPQ is the required tangent.

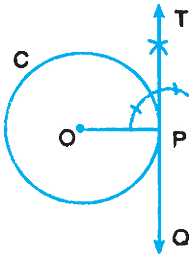


Fig. 18.12

Construction 13: To draw tangents to a circle from a given point outside it.

Suppose C be the given circle with centre O and a point A outside it. You have to draw tangents to the circle from the point A. For that, we go through the following steps:

Step 1: Join OA.

Step 2: Draw the right bisector of OA. Let R be mid point of OA.

Step 3: With R as centre and radius equal to RO, draw a circle intersecting the given circle at P and Q.

Step 4: Join AP and AQ.

Then AP and AQ are the two required tangents.

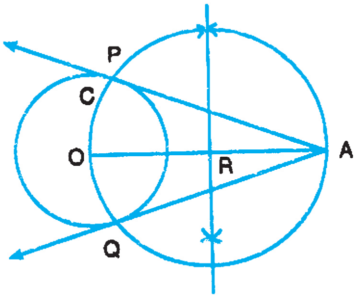


Fig. 18.13

CHECK YOUR PROGRESS 18.4

1. Draw a circle of 3 cm radius. Take a point A on the circle. At A, draw a tangent to the circle by using the centre of the circle. Also write steps of construction.
2. Draw a circle of radius 2.5 cm. From a point P outside the circle, draw two tangents PQ and PR to the circle. Verify that lengths of PQ and PR are equal. Also write steps of construction.

TERMINAL EXERCISE

1. Draw a line segment PQ = 8 cm long. Divide it internally in the ratio 3 : 5. Also write the steps of construction.

   Note: *You are also required to write the steps of construction in each of the following problems.*

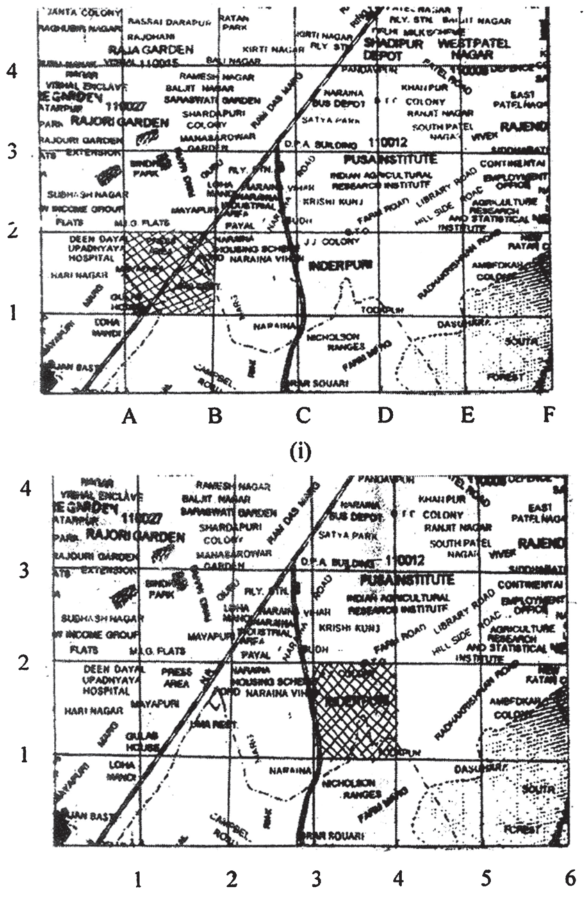
2. Draw a line segment AB = 6 cm. Find a point C on AB such that AC : CB = 3 : 2. Measure AC and CB

3. Construct a triangle with perimeter 14 cm and base angles 60° and 90°.
4. Construct a right angled triangle whose hypotenuse is 8 cm and one of its other two sides is 5.5 cm.
5. Construct a ΔABC in which $BC = 3.5$ cm, $AB + AC = 8$ cm and $\angle B = 60^o$.
6. Construct a ΔABC in which $AB = 4$ cm, $\angle A = 45^o$, and $AC - BC = 1$ cm.
7. Construct a ΔPQR with $PQ = 5$ cm, $PR = 5.5$ cm and the base $QR = 6.5$ cm. Construct another triangle $P'QR'$ similar to ΔPQR such that each of its sides are $\frac{5}{7}$ times the corresponding sides of ΔPQR.
8. Construct a right triangle with sides 5 cm, 12 cm and 13 cm. Construct another triangle similar to it with scale factor 5/6.
9. Draw a circle of diameter 6 cm. From a point P outside the circle at a distance of 6 cm from the centre, draw two tangents to the circle.
10. Draw a line segment AB of length 8 cm. Taking A as centre, draw a circle of radius 4 cm and taking B as centre, draw another circle of radius 3 cm. Construct tangents to each circle from the centre of the other circle.

19

CO-ORDINATE GEOMETRY

The problem of locating a village or a road on a large map can involve a good deal of searching. But the task can be made easier by dividing it into squares of managable size. Each square is identified by a combination of a letter and a number, or of two numbers, one of which refers to a vertical division of the map into columns, and the other to a horizontal division into rows.



(i)

(ii)

Fig. 19.1

In the above Fig. 19.1 (i), we can identify the shaded square on the map by the coding, (B,2)or (4, 2) [See Fig. 191 (ii))]. The pair of numbers used for coding is called ordered pair. If we know the coding of a particular city, roughly we can indicate it's location inside the shaded square on the map. But still we do not know its precise location. The method of finding the , position of a point in a plane very precisely was introduced by the French Mathematician and Philosopher, Rene Descartes (1596-1650).

In this, a point in the plane is represented by an ordered pair of numbers, called the Cartesian co-ordinates of a point.

In this lesson, we will learn more about cartesian co-ordinates of a point, distance between two points in a plane, section formula and co-ordinates of the centroid of a triangle.

OBJECTIVES

After studying this lesson, you will be able to

- *fix the position of different points in a plane, whose coordinates are given, using rectangular system of coordinates and vice-versa;*
- *find the distance between two different points whose co-ordinates are given;*
- *find the co-ordinates of a point, which divides the line segment joining two points in a given ratio internally;*
- *find the co-ordinates of the mid-point of the join of two points;*
- *find the co-ordinates of the centroid of a triangle with given vertices;*
- *solve problems based on the above concepts.*

EXPECTED BACKGROUND KNOWLEDGE

- Idea of number line
- Fundamental operations on numbers
- Properties of a right triangle

19.1 CO-ORDINATE SYSTEM

Recall that you have learnt to draw the graph of a linear equation in two variables in Lesson 5.

The position of a point in a plane is fixed w.r.t. to its distances from two axes of reference, which are usually drawn by the two graduated number lines XOX′ and YOY′, at right angles to each other at O (See Fig, 19.2)

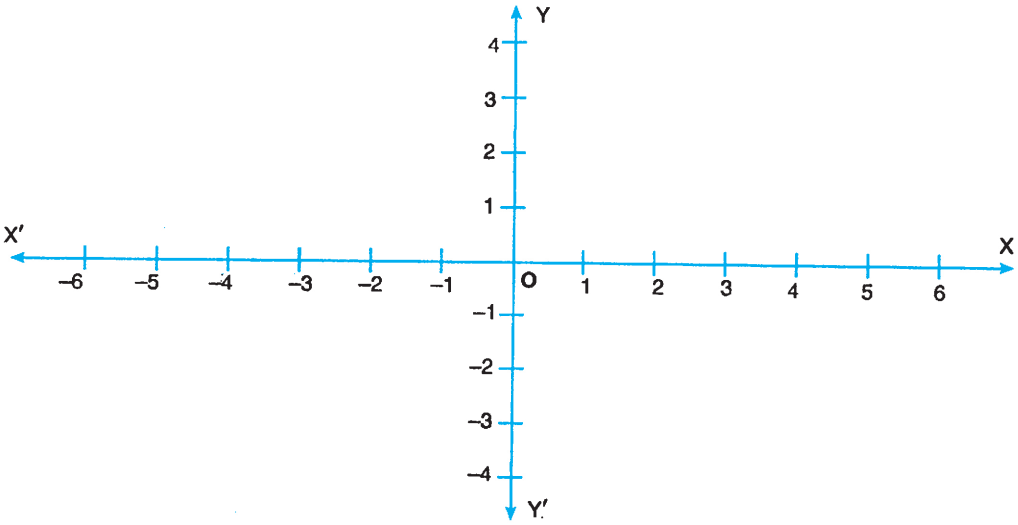


Fig. 19.2

The horizontal number line XOX′ is called **x-axis** and the vertical number line YOY′ is called **y-axis.** The point O, where both axes intersect each other is called the **origin**. The two axes together are called rectangular coordinate system.

It may be noted that, the positive direction of x-axis is taken to the right of the origin O, OX and the negative direction is taken to the left of the origin O, i.e., the side OX′.

Similarly, the portion of y-axis above the origin O, i.e., the side OY is taken as positive and the portion below the origin O, i.e., the side OY′ is taken as negative.

19.2 CO-ORDINATES OF A POINT

The position of a point is given by two numbers, called co-ordinates which refer to the distances of the point from these two axes. By convention the first number, the x-co-ordinate (or abscissa), always indicates the distance from the y-axis and the second number, the y-coordinate (or ordinate) indicates the distance from the x-axis.

In the above Fig. 19.3, the co-ordinates of the points A and B are (3, 2) and (–2, –4) respectively.

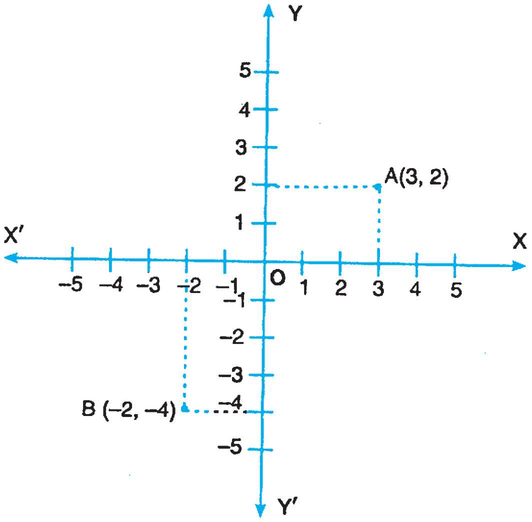


Fig. 19.3

You can say that the distance of the point A(3, 2) from the y-axis is 3 units and from the x-axis is 2 units. It is customary to write the co-ordinates of a point as an ordered pair i.e., **(x co-ordinate, y co-ordinate).**

It is clear from the point A(3, 2) that its x co-ordinate is 3 and the y co-ordinate is 2. Similarly x co-ordinate and y co-ordinate of the point B(–2, –4) are –2 and –4 respectively.

In general, co-ordinates of a point P(x, y) imply that distance of P from the y-axis is x units and its distance from the x-axis is y units.

You may note that the co-ordinates of the origin O are (0, 0). The y co-ordinate of every point on the x-axis is 0 and the x co-ordinate of every point on the y-axis is 0.

In general, co-ordinates of any point on the x-axis to the right of the origin is (a, 0) and that to left of the origin is (–a, 0), where 'a' is a non-zero positive number.

Similarly, y co-ordinates of any point on the y-axis above and below the x-axis would be (0, b) and (0, –b) respectively where 'b' is a non-zero positive number.

You may also note that the position of points (x, y) and (y, x) in the rectangular, co-ordinate system is not the same. For example position of points (3, 4) and (4, 3) are shown in Fig 19.4.

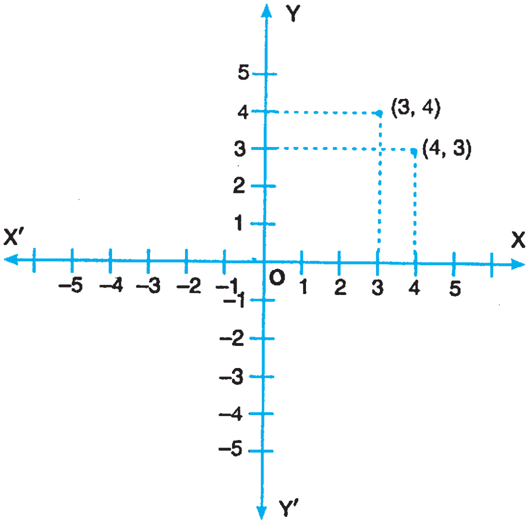


Fig. 19.4

Example 19.1: Write down x and y co-ordinates for each of the following points

(a) (l, 1) (b) (–3, 2) (c) (–7, –5) (d) (2, –6)

Solution : (a) x co-ordinate is l (b) x co-ordinate is –3

y co-ordinate is l y co-ordinate is 2.

(c) x co-ordinate is –7 (d) x co-ordinate is 2

y co-ordinate is –5. y co-ordinate is –6.

Example 19.2 : Write down distances from y and x axes respectively for each of the following points :

(a) A(3, 4) (b) B(–5, 1) (c) C(–3, –3) (d) D(8, –9)

Solution : (a) The distance of the point A from the y-axis is 3 units to the right of origin and from the x-axis is 4 units above the origin.

(b) The distance of the point B from the y-axis is 5 units to the left of the origin and from the x-axis is 1 unit above the origin.

(c) The distance of the point C from the y-axis is 3 units to the left of the origin and from the x-axis is also 3 units below the origin.

(d) The distance of the point D from the y-axis is 8 units to the right of the origin and from the x-axis is 9 units below the origin.

19.3 QUADRANTS

The two axes XOX′ and YOY′ divide the plane into four parts called quadrants.

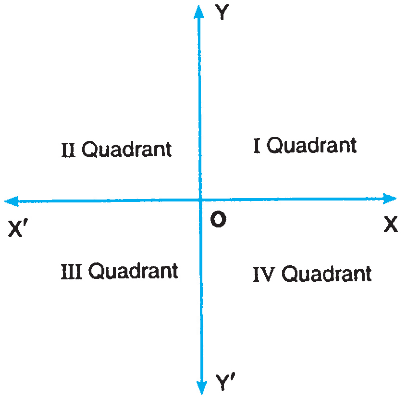


Fig. 19.5

The four quadrants (See Fig. 19.5) are named as follows :

XOY : I Quadrant ; YOX’ : II Quadrant;

X’OY’ : III Quadrant ; Y’OX : IV Quadrant.

We have discussed in Section 19.4 that

(i) along x-axis, the positive direction is taken to the right of the origin and negative direction to its left.

(ii) along y-axis, portion above the x-axis is taken as positive and portion below the x-axis is taken as negative (See Fig. 19.6)

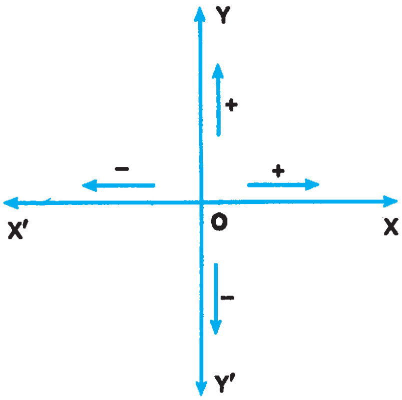


Fig. 19.6

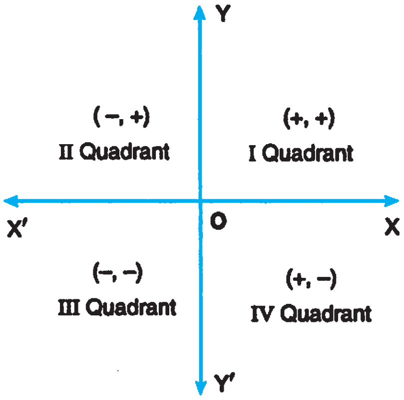


Fig. 19.7

Therefore, co-ordinates of all points in the first quadrant are of the type (+, +) (See Fig. 19.7)

Any point in the II quadrant has x co-ordinate negative and y co-ordinate positive (–, +), Similarly, in III quadrant, a point has both x and y co-ordinates negative (–,–) and in IV quadrant, a point has x co-ordinate positive and y co-ordinate negative (+,–).

For example :

(a) P(5, 6) lies in the first quadrant as both x and y co-ordinates are positive.

(b) Q(–3, 4) lies in the second quadrant as its x co-ordiante is negative and y co-ordinate is positive.

(c) R (–2, –3) lies in the third quadrant as its both x and y co-ordinates are negative.

(d) S(4, –1) lies in the fourth quadrant as its x co-ordinate is positive and y coordinate is negative.

CHECK YOUR PROGRESS 19.1

1. Write down x and y co-ordinates for each of the following points :

   (a) (3, 3) (b) (–6, 5) (c) (–1, –3) (d) (4, –2)

2. Write down distances of each of the following points from the y and x axis respectively.

   (a) A(2, 4) (b) B(–2, 4) (c) C(–2, –4) (d) D(2, –4)

3. Group each of the following points quadrantwise ;

   A(–3, 2), B (2, 3), C(7, -6), D(l, 1), E(–9, –9),

   F (–6, 1), G (–4, –5),. H(11, –3), P(3, 12), Q(–13, 6),

19.4 PLOTTING OF A POINT WHOSE CO-ORDINATES ARE GIVEN

The point can be plotted by measuring its distances from the axes. Thus, any point (h, k) can be plotted as follows:

(i) Measure OM equal to h along the x-axis (See Fig. 19.8).

(ii) Measure MP perpendicular to OM and equal to k.

Follow the rule of sign in both cases.

For example points (–3, 5) and (4, –6) would be plotted as shown in Fig. 19.9.

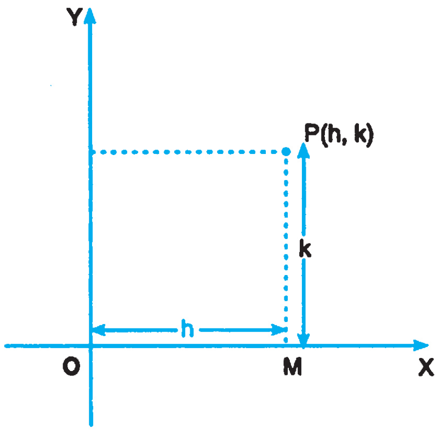


Fig. 19.8

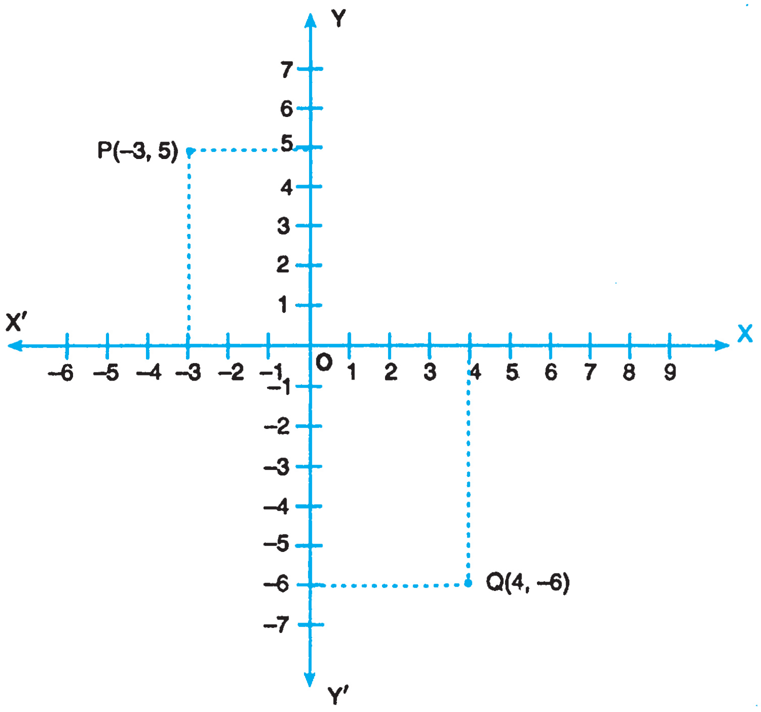


Fig. 19.9

19.5 DISTANCE BETWEEN TWO POINTS

The distance between any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the plane is the length of the line segment PQ.

From P, Q draw PL and QM perpendicular on the x-axis and PR perpendicular on QM.

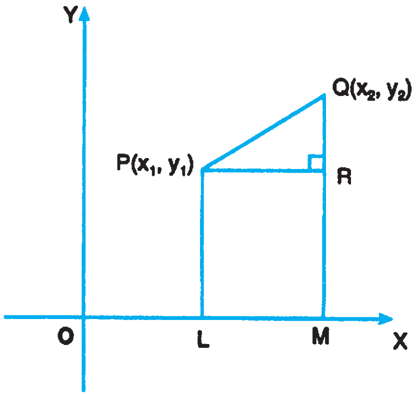


Fig. 19.10

Then, OL = x_1, Om = x_2, PL = y_1 and QM = y_2

$\therefore$ PR = LM = OM – OL = $x_2 - x_1$

QR = QM – RM = QM – PL = $y_2 - y_1$

Since PQR is a right angled triangle

$$\therefore \quad PQ^2 = PR^2 + QR^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 \qquad \text{(By Pythagoras Theorem)}$$

$$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Therefore,

$$\textbf{Distance between two points} = \sqrt{(\textbf{difference of abscissae})^2 + (\textbf{difference of ordinates})^2}$$

The result will be expressed in Units in use.

Corollary: The distance of the point (x_1, y_1) from the origin (0, 0) is

$$\sqrt{(x_1 - 0)^2 + (y_1 - 0)^2} = \sqrt{x_1{}^2 + y_1{}^2}$$

Let us consider some examples to illustrate.

Example 19.3: Find the distance between each of the following points:

(a) P(6, 8) and Q(–9, –12)

(b) A(–6, –1) and B(–6, 11)

Solution: (a) Here the points are P(6, 8) and Q(–9, –12)

By using distance formula, we have

$$PQ = \sqrt{(-9-6)^2 + \{(-12-8)\}^2}$$

$$= \sqrt{15^2 + 20^2} = \sqrt{225 + 400} = \sqrt{625} = 25$$

Hence, PQ = 25 units.

(b) Here the points are A(–6, –1) and B(–6, 11)

By using distance formula, we have

$$AB = \sqrt{\{-6-(-6)\}^2+\{11-(-1)\}^2}$$

$$= \sqrt{0^2+12^2} = 12$$

Hence, AB = 12 units

Example 19.4: The distance between two points (0, 0) and (x, 3) is 5. Find x.

Solution: By using distance formula, we have the distance between (0, 0) and (x, 3) is

$$\sqrt{(x-0)^2+(3-0)^2}$$

It is given that

$$\sqrt{(x-0)^2+(3-0)^2} = 5$$

or $\sqrt{x^2+3^2} = 5$

Squaring both sides,

$$x^2 + 9 = 25$$

or $x^2 = 16$

or $x = \pm 4$

Hence $x = +4$ or -4 units

Example 19.5: Show that the points (1, 1), (3, 0) and (–1, 2) are collinear.

Solution: Let P(1, 1), Q(3, 0) and R(–1, 2) be the given points

$$\therefore \quad PQ = \sqrt{(3-1)^2+(0-1)^2} = \sqrt{4+1} \text{ or } \sqrt{5} \text{ units}$$

$$QR = \sqrt{(-1-3)^2+(2-0)^2} = \sqrt{16+4} \text{ or } 2\sqrt{5} \text{ units}$$

$$RP = \sqrt{(-1-1)^2+(2-1)^2} = \sqrt{4+1} \text{ or } \sqrt{5} \text{ units}$$

Now, $PQ + RP = \left(\sqrt{5}+\sqrt{5}\right)$ units $= 2\sqrt{5}$ units $= QR$

$\therefore$ P, Q and R are collinear points.

Example 19.6: Find the radius of the circle whose centre is at (0, 0) and which passes through the point (–6, 8).

Solution: Let O(0, 0) and B(–6, 8) be the given points of the line segment OB.

$$\therefore \qquad OB = \sqrt{(-6-0)^2 + (8-0)^2}$$

$$= \sqrt{36+64} = \sqrt{100}$$

$$= 10$$

Hence radius of the circle is 10 units.

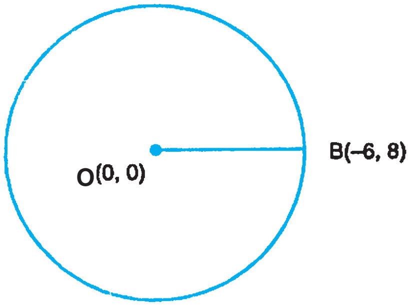


Fig. 19.11

CHECK YOUR PROGRESS 19.2

1. Find the distance between each of the following pair of points:

   (a) (3, 2) and (11, 8) (b) (–1, 0) and (0, 3)

   (c) (3, –4) and (8, 5) (d) (2, –11) and (–9, –3)

2. Find the radius of the circle whose centre is at (2, 0) and which passes through the point (7, –12).

3. Show that the points (–5, 6), (–1, 2) and (2, –1) are collinear

19.6 SECTION FORMULA

To find the co-ordinates of a point, which divides the line segment joining two points, in a given ratio internally.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points and P(x, y) be a point on AB which divides it in the given ratio m : n. We have to find the co-ordinates of P.

Draw the perpendiculars AL, PM, BN on OX, and, AK, PT on PM and BN respectively. Then, from similar triangles APK and PBT, we have

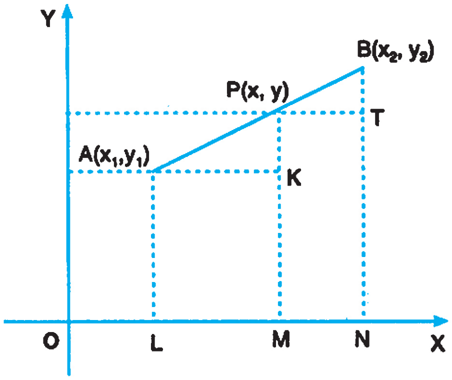


Fig. 19.12

$$\frac{AP}{PB} = \frac{AK}{PT} = \frac{KP}{TB} \qquad \text{...(i)}$$

Now, $AK = LM = OM - OL = x - x_1$

$PT = MN = ON - OM = x_2 - x$

$KP = MP - MK = MP - LA = y - y_1$

$TB = NB - NT = NB - MP = y_2 - y$

∴ From (i), we have

$$\frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

From the first two relations we get,

$$\frac{m}{n} = \frac{x - x_1}{x_2 - x}$$

or $\qquad mx_2 - mx = nx - nx_1$

or $\qquad x(m + n) = mx_2 + nx_1$

or $\qquad x = \frac{mx_2 + nx_1}{m + n}$

Similarly, from the raltion $\frac{AP}{PB} = \frac{KP}{TB}$, we get

$$\frac{m}{n} = \frac{y - y_1}{y_2 - y}$$ which gives on simplification.

$$y = \frac{my_2 + ny_1}{m + n}$$

$$\therefore \qquad x = \frac{mx_2 + nx_1}{m + n}, \text{ and } y = \frac{my_2 + ny_1}{m + n} \qquad \text{...(i)}$$

Hence co-ordiantes of a point which divides the line segment joining the points (x_1, y_1) and (x_2, y_2) in the ratio m : n internally are :

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right)$$

19.6.1 Mid- Point Formula

The co-ordinates of the mid-point of the line segment joining two points (x_1, y_1) and (x_2, y_2) can be obtained by taking $m = n$ in the section formula above.

Putting $m = n$ in (1) above, we have

$$x = \frac{nx_2 + nx_1}{n + n} = \frac{x_2 + x_1}{2}$$

and $\qquad y = \frac{ny_2 + ny_1}{n + n} = \frac{y_2 + y_1}{2}$

The co-ordinates of the mid-point joining two points (x_1, y_1) and (x_2, y_2) are:

$$\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$

Let us take some examples to illustrate:

Example 19.7: Find the co-ordinates of a point which divides the line segment joining each of the following points in the given ratio:

(a) (2, 3) and (7, 8) in the ratio 2 : 3 internally.

(b) (–1, 4) and (0, –3) in the ratio 1 : 4 internally.

Solution: (a) Let A(2, 3) and B(7, 8) be the given points.

Let P(x, y) divide AB in the ratio 2 : 3 internally.

Using section formula, we have

$$x = \frac{2\times7+3\times2}{2+3} = \frac{20}{5} = 4$$

and $$y = \frac{2\times8+3\times3}{2+3} = \frac{25}{5} = 5$$

∴ P(4, 5) divides AB in the ratio 2 : 3 internally.

(b) Let A(–1, 4) and B(0, –3) be the given points.

Let P(x, y) divide AB in the ratio 1 : 4 internally.

Using section formula, we have

$$x = \frac{1\times0+4\times(-1)}{1+4} = -\frac{4}{5}$$

and $$y = \frac{1\times(-3)+4\times4}{1+4} = \frac{13}{5}$$

∴ $P\left(-\frac{4}{5}, \frac{13}{5}\right)$ divides AB in the ratio 1 : 4 internally.

Example 19.8: Find the mid-point of the line segment joining two points (3, 4) and (5, 12).

Solution: Let A(3, 4) and B(5, 12) be the given points.

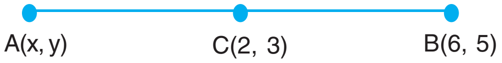
Let C(x, y) be the mid-point of AB. Using mid-point formula, we have,

$$x = \frac{3+5}{2} = 4$$

and $$y = \frac{4+12}{2} = 8$$

$\therefore$ C(4, 8) are the co-ordinates of the mid-point of the line segment joining two points (3, 4) and (5, 12).

Example 19.9: The co-ordinates of the mid-point of a segment are (2, 3). If co-ordinates of one of the end points of the line segment are (6, 5), find the co-ordinates of the other end point.



Solution: Let other end point be A(x, y)

It is given that C(2, 3) is the mid point

$\therefore$ We can write,

$$2 = \frac{x+6}{2} \quad \text{and} \quad 3 = \frac{y+5}{2}$$

$$\text{or} \quad 4 = x + 6 \quad \text{or} \quad 6 = y + 5$$

$$\text{or} \quad x = -2 \quad \text{or} \quad y = 1$$

$\therefore$ (–2, 1) are the coordinates of the other end point.

19.7 CENTROID OF A TRIANGLE

To find the co-ordinates of the centroid of a triangle whose vertices are given.

Definition: The centroid of a triangle is the point of concurrency of its medians and divides each median in the ratio of 2 : 1.

Let A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) be the vertices of the triangle ABC. Let AD be the median bisecting its base BC. Then, using mid-point formula, we have

$$D = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$

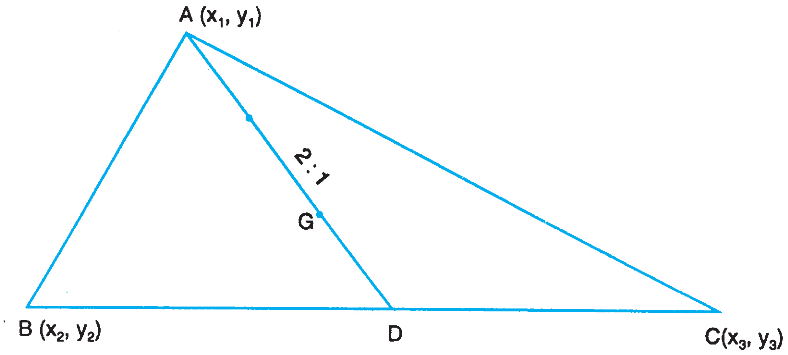


Fig. 19.14

Now, the point G on AD, which divides it internally in the ratio 2 : 1, is the centroid. If (x, y) are the co-ordinates of G, then

$$x = \frac{2 \times \frac{x_2 + x_3}{2} + 1 \times x_1}{2+1} = \frac{x_1 + x_2 + x_3}{3}$$

$$y = \frac{2 \times \frac{y_2 + y_3}{2} + 1 \times y_1}{2+1} = \frac{y_1 + y_2 + y_3}{3}$$

Hence, the co-ordiantes of the centroid are given by

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

Example 19.10: The co-ordinates of the vertices of a triangle are (3, –1), (10, 7) and (5, 3). Find the co-ordinates of its centroid.

Solution: Let A(3, –1), B(10, 7) and C(5, 3) be the vertices of a triangle.

Let G(x, y) be its centroid.

Then, $$x = \frac{3+10+5}{3} = 6$$

and $$y = \frac{-1+7+3}{3} = 3$$

Hence, the coordinates of the Centroid are (6, 3).

CHECK YOUR PROGRESS 19.3

1. Find the co-ordinates of the point which divides internally the line segment joining the points:

   (a) (1, –2) and (4, 7) in the ratio 1 : 2

   (b) (3, –2) and (–4, 5) in the ratio 1 : 1

2. Find the mid-point of the line joining:

   (a) (0, 0) and (8, –5)

   (b) (–7, 0) and (0, 10)

3. Find the centroid of the triangle whose vertices are (5, –1), (–3, –2) and (–1, 8).

LET US SUM UP

- If (2, 3) are the co-ordinates of a point, then x co-ordiante (or abscissa) is 2 and the y co-ordinate (or ordinate) is 3.
- In any co-ordiante (x, y), 'x' indicates the distance from the y-axis and ý' indicates the distance from the x-axis.
- The co-ordinates of the origin are (0, 0)
- The y co-ordinate of every point on the x-axis is 0 and the x co-ordiante of every point on the y-axis is 0.
- The two axes XOX' and YOY' divide the plance into four parts called quadrants.
- The distance of the line segment joining two points (x_1, y_1) and (x_2, y_2) is given by:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- The distance of the point (x_1, y_1) from the origin (0, 0) is $\sqrt{x_1{}^2 + y_1{}^2}$
- The co-ordinates of a point, which divides the line segment joining two points (x_1, y_1) and (x_2, y_2) in a ratio m : n internally are given by:

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right)$$

- The co-ordinates of the mid-point of the line segment joining two points (x_1, y_1) and (x_2, y_2) are given by:

$$\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

- The co-ordiantes of the centroid of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) are given by

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

TERMINAL EXERCISE

1. In Fig. 19.15, AB = AC. Find x.

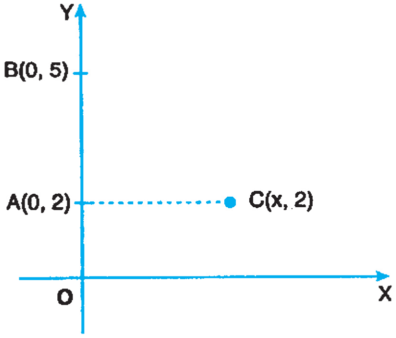


Fig. 19.15

2. The length of the line segment joining two points (2, 3) and (4, x) is $\sqrt{13}$ units. Find x.
3. Find the lengths of the sides of the triangle whose vertices are A(3, 4), B(2, –1) and C(4, –6).
4. Prove that the points (2, –2), (–2, 1) and (5, 2) are the vertices of a right angled triangle.
5. Find the co-ordinates of a point which divides the join of (2, –1) and (–3, 4) in the ratio of 2 : 3 internally.
6. Find the centre of a circle, if the end points of a diameter are P(–5, 7) and Q(3, –11).
7. Find the centroid of the triangle whose vertices are P(–2, 4), Q(7, –3) and R(4, 5).

ANSWERS TO CHECK YOUR PROGRESS

19.1

1. (a) 3; 3 (b) –6; 5 (c) –1; –3 (d) 4; –2
2. (a) 2 units; 4 units

   (b) 2 units to the left of the origin; 4 units above the x-axis

   (c) 2 units to the left of the origin; 4 units below the origin.

   (d) 2 units; 4 units below the origin.
3. Quadrant I: B(2, 3), D(1, 1) and P(3, 12)

   Quadrant II: A(ß, 2), F(–6, 1) and Q(–13, 6)

Quadrant III: E(–9, –9) and G(–4, –5)

Quadrant IV: C(7, –6) and H(11, –3)

19.2

1. (a) 10 units (b) $\sqrt{10}$ units (c) $\sqrt{106}$ units (d) $\sqrt{185}$ units

2. 13 units

19.3

1. (a) (2, 1) (b) (–1, 1)

2. (a) $\left(4, -\frac{5}{2}\right)$ (b) $\left(-\frac{7}{2}, 5\right)$

3. $\left(\frac{1}{3}, \frac{5}{3}\right)$

ANSWERS TO TERMINAL EXERCISE

1. 3 units

2. 0 or 6

3. AB = $\sqrt{26}$ units, BC = $\sqrt{29}$ units and CA = $\sqrt{101}$ units

5. (0, 1)

6. (–1, –2)

7. (3, 2)

Secondary Course Mathematics

Practice Work-Geometry

Maximum Marks: 25 **Time : 45 Minutes**

Instructions:

1. Answer all the questions on a separate sheet of paper.
2. Give the following informations on your answer sheet

   Name

   Enrolment number

   Subject

   Topic of practice work

   Address
3. Get your practice work checked by the subject teacher at your study centre so that you get positive feedback about your performance.

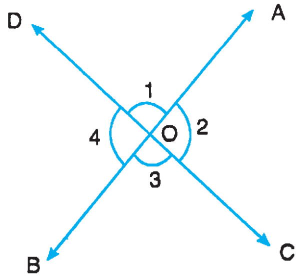
1. Lines AB and CD intersect each other at O as shown in the adjacent figure. A pair of vertically opposite angles is: 1

   (A) 1, 2

   (B) 2, 3

   (C) 3, 4

   (D) 2, 4



2. Which of the following statements is true for a ΔABC? 1

   (A) AB + BC = AC

   (B) AB + BC < AC

   (C) AB + BC > AC

   (D) AB + BC + AC = 0

3. The quadrilateral formed by joining the mid points of the pair of adjacent sides of a rectangle is a: 1

   (A) rectangle

   (B) square

   (C) rhombus

   (D) trapezium

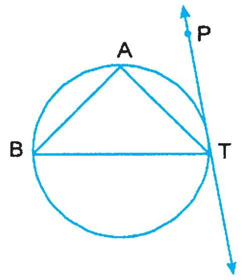
4. In the adjacent figure, PT is a tangent to the circle at T. If $\angle BTA = 45^o$ and $\angle PTB = 70^o$, Then $\angle ABT$ is: 1

   (A) 110^o

   (B) 70^o

   (C) 45^o

   (D) 23^o



5. Two points A, B have co-ordinates (2, 3) and (4, x) respectively. If $AB^2 = 13$, the possible value of x is: 1

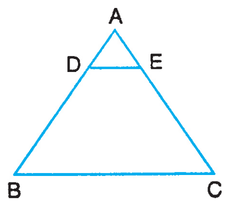
   (A) –6

   (B) 0

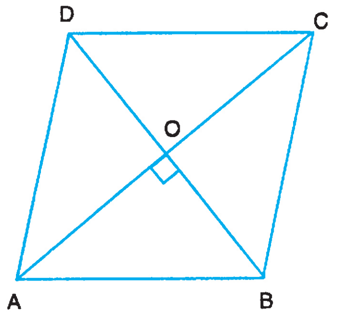
   (C) 9

   (D) 12

6. In ΔABC, AB = 10 cm and DE is parallel to BC such that $AE = \frac{1}{4}AC$. Find AD. 2



7. If ABCD is a rhombus, then prove that $4AB^2 = AC^2 + BD^2$ 2



8. Find the co-ordinates of the point on x-axis which is equidistant from the points whose co-ordinates are (3, 8) and (9, 5). 2

9. The co-ordiantes of the mid-point of a line segment are (2, 3). If co-ordinates of one of the end points of the segment are (6, 5), then find the co-ordinates of the other end point. 2

10. The co-ordinates of the vertices of a triangle are (3, –1), (10, 7) and (5, 3). Find the co-ordinates of its centroid. 2

11. In an acute angled triangle ABC, $AD \perp BC$. Prove that

    $AC^2 = AB^2 + BC^2 - 2BC.\ BD$ 4

12. Prove that parallelograms on equal (or same) bases and between the same parallels are equal in area. 6

MODULE 4

Mensuration

All the mathematical ideas have emerged out of daily life experiences. The first ever need of human being was counting objects. This gave rise to the idea of ***numbers.*** *When the man learnt to grow crops, following types of problems had to be handled:*

(i) Fencing or construcing some kind of a boundary around the field, where the crops were to be grown.

(ii) Allotting lands of different sizes for growing different crops.

(iii) Making suitable places for storing different products grown under different crops.

These problems led to the need of measurement of perimeters (lengths), areas and volumes, which in turn gave rise to a branch of mathematics known as ***Mensuration****. In it, we deal with problems such as finding the cost of putting barbed wire around a field, finding the number of tiles required to floor a room, finding the number of bricks, required for creating a wall, finding the cost of ploughing a given field at a given rate, finding the cost of constructing a water tank for supplying water in a colony, finding the cost of polishing a table-top or painting a door and so on. Due to the above type of problems, sometimes mensuration is referred to as the science of "Furnitures and Walls".*

For solving above type of problems, we need to find the perimeters and areas of simple closed plane figures (figure which lie in a plane) and surface areas and volumes of solid figures (figures which do not lie wholly in a plane). You are already familiar with the concepts of perimeters, areas, surface areas and volumes. In this module, we shall discuss these in details, starting with the results and formulas with which you are already familiar.

20

PERIMETERS AND AREAS OF PLANE FIGURES

You are already familair with a number of plane figures such as rectangle, square, parallelogram, triangle, circle, etc. You also know how to find perimeters and areas of these figures using different formulae. In this lesson, we shall consolidate this knowledge and learn something more about these, particularly the Heron's formula for finding the area of a triangle and formula for finding the area of a sector of a circle.

OBJECTIVES

After studying this lesson, you will be able to

- *find the perimeters and areas of some triangles and quadrilaterals, using formulae learnt earlier;*
- *use Heron's formula for finding the area of a triangle;*
- *find the areas of some rectilinear figures (including rectangular paths) by dividing them into known figures such as triangles, squares, trapeziums, rectangles, etc.;*
- *find the circumference and area of a circle;*
- *find the areas of circular paths;*
- *derive and understand the formulae for perimeter and area of a sector of a circle;*
- *find the perimeter and the area of a sector, using the above formulae;*
- *find the areas of some combinations of figures involving circles, sectors as well as triangles, squares and rectangles;*
- *solve daily life problems based on perimeters and areas of various plane figures.*

EXPECTED BACKGROUND KNOWLEDGE

- Simple closed figures like triangles, quadrilaterals, parallelograms, trapeziums, squares, rectangles, circles and their properties.
- Different units for perimeter and area such as m and m^2, cm and cm^2, mm and mm^2 and so on.
- Conversion of one unit into other units.
- Bigger units for areas such as **acres** and **hectares**.
- Following formulae for perimeters and areas of varioius figures:

(i) Perimeter of a rectangle = 2 (length + breadth)

(ii) Area of a rectangle = length × breadth

(iii) Perimeter of a square = 4 × side

(iv) Area of a square = $(\text{side})^2$

(v) Area of a parallelogram = base × corresponding altitude

(vi) Area of a triangle = $\frac{1}{2}$ base × corresponding altitude

(vii) Area of a rhombus = $\frac{1}{2}$ product of its diagonals

(viii) Area of a trapezium = $\frac{1}{2}$ (sum of the two parallel sides) × distance between them

(ix) circumference of a circle = $2\,\pi \times$ radius

(x) Area of a circle = $\pi \times (\text{radius})^2$

20.1 PERIMETERS AND AREAS OF SOME SPECIFIC QUADRILATEALS AND TRIANGLES

You already know that the distance covered to walk along a plane closed figure (boundary) is called its **perimeter** and the measure of the region enclosed by the figure is called its **area.** You also know that perimeter or length is measured in linear units, while area is measured in square units. For example, units for perimeter (or length) are m or cm or mm and that for area are m^2 or cm^2 or mm^2 (also written as sq.m or sq.cm or sq.mm).

You are also familiar with the calculations of the perimeters and areas of some specific quadrilaterals (such as squares, rectangles, parallelograms, etc.) and triangles, using certain formulae. Lets us consolidate this knowledge through some examples.

Example 20.1: Find the area of square whose perimeter is 80 m.

Solution: Let the side of the square be a m.

So, perimeter of the square = 4 × a m.

Therefore, $4a = 80$

or $a = \frac{80}{4} = 20$

That is, side of the square = 20 m

Therefore, area of the square = $(20\text{m})^2 = 400\ \text{m}^2$

Example 20.2: Length and breadth of a rectangular field are 23.7 m and 14.5 m respectively. Find:

(i) barbed wire required to fence the field

(ii) area of the field.

Solution: (i) Barbed wire for fencing the field = perimeter of the field

$= 2\,(\text{length} + \text{breadth})$

$= 2(23.7 + 14.5)\ \text{m} = 76.4\ \text{m}$

(ii) Area of the field = length × breadth

$= 23.7 \times 14.5\ \text{m}^2$

$= 343.65\ \text{m}^2$

Example 20.3: Find the area of a parallelogram of base 12 cm and corresponding altitude 8 cm.

Solution: Area of the parallelogram = base × corresponding altitude

$= 12 \times 8\ \text{cm}^2$

$= 96\ \text{cm}^2$

Example 20.4: The base of a triangular field is three times its corresponding altitude. If the cost of ploughing the field at the rate of ₹ 15 per square metre is ₹ 20250, find the base and the corresponding altitutde of the field.

Solution: Let the corresponding altitude be x m.

Therefore, base = $3x$ m.

So, area of the field = $\frac{1}{2}$ base × corresponding altitude

$$= \frac{1}{2}\ 3x \times x\ \text{m}^2 = \frac{3x^2}{2}\ \text{m}^2 \qquad ....(1)$$

Also, cost of ploughing the field at ₹ 15 per m^2 = ₹ 20250

Therefore, area of the field $= \dfrac{20250}{15}\ m^2$

$$= 1350\ m^2 \qquad \text{...(2)}$$

From (1) and (2), we have:

$$\frac{3x^2}{2} = 1350$$

or $$x^2 = \frac{1350 \times 2}{3} = 900 = (30)^2$$

or $$x = 30$$

Hence, corresponding altitutde is 30 m and the base is 3 × 30 m i.e., 90 m.

Example 20.5: Find the area of a rhombus whose diagonals are of lengths 16 cm and 12 cm.

Solution: Area of the rhombus $= \dfrac{1}{2}$ product of its diagonals $= \dfrac{1}{2} \times 16 \times 12\ cm^2$

$$= 96\ cm^2$$

Example 20.6: Length of the two parallel sides of a trapezium are 20 cm and 12 cm and the distance between them is 5 cm. Find the area of the trapezium.

Solution: Area of a trapezium $= \dfrac{1}{2}$ (sum of the two parallel sides)×distance between them

$$= \frac{1}{2}(20 + 12) \times 5\ cm^2 = 80\ cm^2$$

CHECK YOUR PROGRESS 20.1

1. Area of a square field is 225 m^2. Find the perimeter of the field.
2. Find the diagonal of a square whose perimeter is 60 cm.
3. Length and breadth of a rectangular field are 22.5 m and 12.5 m respectively. Find:

   (i) Area of the field

   (ii) Length of the barbed wire required to fence the field

4. The length and breadth of rectangle are in the ratio 3 : 2. If the area of the rectangle is 726 m^2, find its perimeter.
5. Find the area of a parallelogram whose base and corresponding altitude are respectively 20 cm and 12 cm.
6. Area of a triangle is 280 cm^2. If base of the triangle is 70 cm, find its corresponding altitude.
7. Find the area of a trapezium, the distance between whose parallel sides of lengths 26 cm and 12 cm is 10 cm.
8. Perimeter of a rhombus is 146 cm and the length of one of its diagonals is 48 cm. Find the length of its other diagonal.

20.2 HERON'S FORMULA

If the base and corresponding altitude of a triangle are known, you have already used the formula:

$$\text{Area of a triangle} = \frac{1}{2}\ \text{base} \times \text{corresponding altitude}$$

However, sometimes we are not given the altitude (height) corresponding to the given base of a triangle. Instead of that we are given the three sides of the triangle. In this case also, we can find the height (or altitude) corresponding to a side and calculate its area. Let us explain it through an example.

Example 20.7: Find the area of the triangle ABC, whose sides AB, BC and CA are respectively 5 cm, 6 cm and 7 cm.

Solution: Draw AD $\perp$ BC as shown in Fig. 20.1.

Let BD = x cm

So, CD = $(6 - x)$ cm

Now, from right triangle ABD, we have:

$AB^2 = BD^2 + AD^2$ (Pythagoras Theorem)

i.e. $25 = x^2 + AD^2$ ...(1)

Similarly, from right triangle ACD, we have:

$AC^2 = CD^2 + AD^2$

i.e. $49 = (6 - x)^2 + AD^2$ ...(2)

From (1) and (2), we have:

$49 - 25 = (6 - x)^2 - x^2$

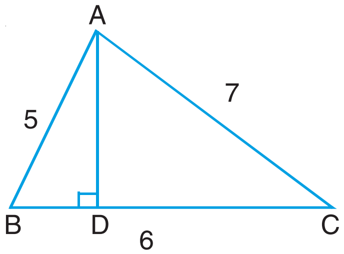


Fig. 20.1

i.e. $24 = 36 - 12x + x^2 - x^2$

or $12\,x = 12$, i.e., $x = 1$

Putting this value of x in (1), we have:

$$25 = 1 + AD^2$$

i.e. $AD^2 = 24$ or $AD = \sqrt{24} = 2\sqrt{6}$ cm

Thus, area of $\Delta ABC = \frac{1}{2}\,BC \times AD = \frac{1}{2} \times 6 \times 2\sqrt{6}\ \text{cm}^2 = 6\sqrt{6}\ \text{cm}^2$

You must have observed that the process involved in the solution of the above example is lengthy. To help us in this matter, a formula for finding the area of a triangle with three given sides was provided by a Greek mathematician Heron (75 B.C. to 10 B.C.). It is as follows:

$$\text{Area of a triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$

where, a, b and c are the three sides of the triangle and $s = \frac{a+b+c}{2}$. This formula can be proved on similar lines as in Example 20.7 by taking a, b and c for 6, 7 and 5 respectively.

Let us find the area of the triangle of Example 20.7 using this formula.

Here, $a = 6$ cm, $b = 7$ cm and $c = 5$ cm

So, $s = \frac{6+7+5}{2} = 9$ cm

Therefore, area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{9(9-6)(9-7)(9-5)}\ \text{cm}^2$$

$$= \sqrt{9 \times 3 \times 2 \times 3}\ \text{cm}^2$$

$= 6\sqrt{6}\ \text{cm}^2$, which is the same as obtained earlier.

Let us take some more examples to illustrate the use of this formula.

Example 20.8: The sides of a triangular field are 165 m, 154 m and 143 m. Find the area of the field.

Solution: $s = \frac{a+b+c}{2} = \frac{(165+154+143)}{2}\ \text{m} = 231\ \text{m}$

So, area of the field $= \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{231\times(231-165)(231-154)(231-143)}\ \text{m}^2$$

$$= \sqrt{231\times66\times77\times88}\ \text{m}^2$$

$$= \sqrt{11\times3\times7\times11\times2\times3\times11\times7\times11\times2\times2\times2}\ \text{m}^2$$

$$= 11 \times 11 \times 3 \times 7 \times 2 \times 2\ \text{m}^2 = 10164\ \text{m}^2$$

Example 20.9: Find the area of a trapezium whose parallel sides are of lengths 11 cm amd 25 cm and whose non-parallel sides are of lengths 15 cm and 13 cm.

Solution: Let ABCD be the trapezium in which AB = 11 cm, CD = 25 cm, AD = 15 cm and BC =13 cm (See Fig. 20.2)

Through B, we draw a line parallel to AD to intersect DC at E. Draw BF ⊥ DC.

Now, clearly BE = AD = 15 cm

BC = 13 cm (given)

and EC = (25 – 11) cm = 14 cm

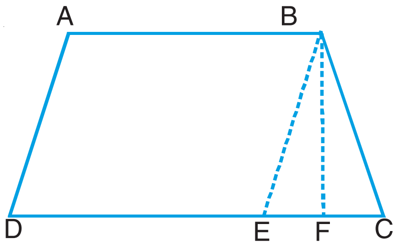


Fig. 20.2

So, for ΔBEC, $s = \dfrac{15+13+14}{2}$ cm = 21 cm

Therefore area of ΔBEC $= \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{21\times(21-15)(21-13)(21-14)}\ \text{cm}^2$$

$$= \sqrt{21\times6\times8\times7}\ \text{cm}^2$$

$$= 7 \times 3 \times 4\ \text{cm}^2 = 84\ \text{cm}^2 \qquad ...(1)$$

Again, area of ΔBEC $= \dfrac{1}{2}$ EC × BF

$$= \frac{1}{2} \times 14 \times \text{BF} \qquad ...(2)$$

So, from (1) and (2), we have:

$$\frac{1}{2} \times 14 \times \text{BF} = 84$$

i.e., $\text{BF} = \dfrac{84}{7}$ cm = 12 cm

Therefore, area of trapezium ABCD = $\frac{1}{2}$ (AB + CD) × BF

$$= \frac{1}{2}(11 + 25) \times 12 \text{ cm}^2$$

$$= 18 \times 12 \text{ cm}^2 = 216 \text{ cm}^2$$

CHECK YOUR PROGRESS 20.2

1. Find the area of a triangle of sides 15 cm, 16 cm and 17 cm.
2. Using Heron's formula, find the area of an equilateral triangle whose side is 12 cm. Hence, find the altitude of the triangle.

20.3 AREAS OF RECTANGULAR PATHS AND SOME RECTILINEAR FIGURES

You might have seen different types of rectangular paths in the parks of your locality. You might have also seen that sometimes lands or fields are not in the shape of a single figure. In fact, they can be considered in the form of a shape made up of a number of polygons such as rectangles, squares, triangles, etc. We shall explain the calculation of areas of such figures through some examples.

Example 20.10: A rectangular park of length 30 m and breadth 24 m is surrounded by a 4 m wide path. Find the area of the path.

Solution: Let ABCD be the park and shaded portion is the path surrounding it (See Fig. 20.3).

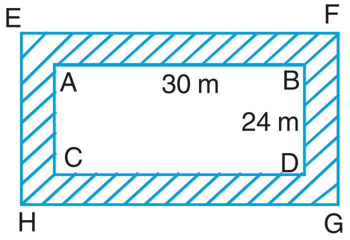


Fig. 20.3

So, length of rectangle EFGH = (30 + 4 + 4) m = 38 m

and breadth of rectangle EFGH = (24 + 4 + 4) m = 32 m

Therefore, area of the path = area of rectangle EFGH – area of rectangle ABCD

$$= (38 \times 32 - 30 \times 24) \text{ m}^2$$

$$= (1216 - 720) \text{ m}^2$$

$$= 496 \text{ m}^2$$

Example 20.11: There are two rectangular paths in the middle of a park as shown in Fig. 20.4. Find the cost of paving the paths with concrete at the rate of ₹ 15 per m^2. It is given that AB = CD = 50 m, AD = BC = 40 m and EF = PQ = 2.5 m.

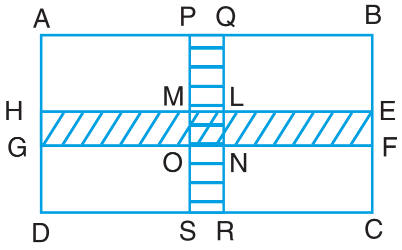


Fig. 20.4

Solution: Area of the paths = Area of PQRS + Area of EFGH – area of square MLNO

$$= (40 \times 2.5 + 50 \times 2.5 - 2.5 \times 2.5)\ \text{m}^2$$

$$= 218.75\ \text{m}^2$$

So, cost of paving the concrete at the rate of ₹ 15 per m^2 = ₹ 218.75 × 15

= ₹ 3281.25

Example 20.12: Find the area of the figure ABCDEFG (See Fig. 20.5) in which ABCG is a rectangle, AB = 3 cm, BC = 5 cm, GF = 2.5 cm = DE = CF., CD = 3.5 cm, EF = 4.5 cm, and CD || EF.

Solution: Required area = area of rectangle ABCG + area of isosceles triangle FGC

+ area of trapezium DCEF ...(1)

Now, area of rectangle ABCG = $l \times b$ = $5 \times 3\ \text{cm}^2 = 15\ \text{cm}^2$ ...(2)

For area of ΔFGC, draw FM $\perp$ CG.

As FG = FC (given), therefore

M is the mid point of GC.

That is, GM = $\frac{3}{2}$ = 1.5 cm

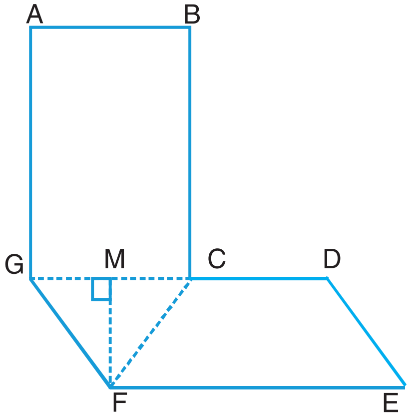


Fig. 20.5

Now, from ΔGMF,

$$GF^2 = FM^2 + GM^2$$

or $$(2.5)^2 = FM^2 + (1.5)^2$$

or $$FM^2 = (2.5)^2 - (1.5)^2 = 4$$

So, FM = 2, i.e., length of FM = 2 cm

So, area of ΔFGC = $\frac{1}{2}$ GC × FM

$$= \frac{1}{2} \times 3 \times 2\ \text{cm}^2 = 3\ \text{cm}^2 \quad ...(3)$$

Also, area of trapezium CDEF = $\frac{1}{2}$ (sum of the parallel sides) × distance between them

$$= \frac{1}{2}\ (3.5 + 4.5) \times 2\ \text{cm}^2$$

$$= \frac{1}{2} \times 8 \times 2\ \text{cm}^2 = 8\ \text{cm}^2 \quad ...(4)$$

So, area of given figure

$$= (15 + 3 + 8)\ \text{cm}^2 \qquad [\text{From (1), (2), (3) and (4)}]$$

$$= 26\ \text{cm}^2$$

CHECK YOUR PROGRESS 20.3

1. There is a 3 m wide path on the inside running around a rectangular park of length 48 m and width 36 m. Find the area of the path.
2. There are two paths of width 2 m each in the middle of a rectangular garden of length 80 m and breadth 60 m such that one path is parallel to the length and the other is parallel to the breadth. Find the area of the paths.
3. Find the area of the rectangular figure ABCDE given in Fig. 20.6, where EF, BG and DH are perpendiculars to AC, AF = 40 m, AG = 50 m, GH = 40 m and CH = 50 m.

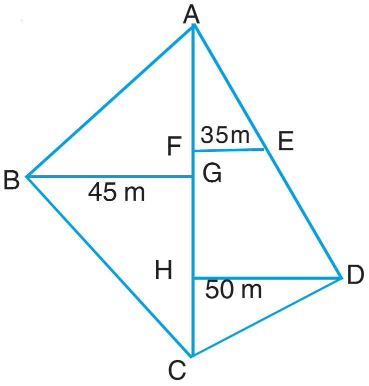


Fig. 20.6

4. Find the area of the figure ABCDEFG in Fig. 20.7, where ABEG is a trapezium, BCDE is a rectangle, and distance between AG and BE is 2 cm.

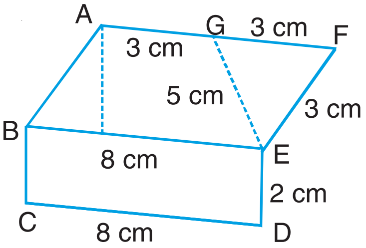


Fig. 20.7

20.4 AREAS OF CIRCLES AND CIRCULAR PATHS

So far, we have discussed about the perimeters and areas of figures made up of line segments only. Now we take up a well known and very useful figure called circle, which is not made up of line segments. (See. Fig. 20.8). You already know that **perimeter (circumference) of a circle is $2\pi r$ and its area is πr^2,** where r is the circle and π is a constant equal to the ratio of circumference of a circle to its diameter. You also know that π is an irrational number.

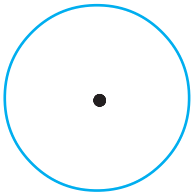


Fig. 20.8

A great Indian mathematician Aryabhata (476 - 550 AD) gave the value of π as $\frac{62832}{20000}$, which is equal to 3.1416 correct to four places of decimals. However, for practical purposes, the value of π is generally taken as $\frac{22}{7}$ or 3.14 approximately. Unless, stated otherwise, we shall take the value of π as $\frac{22}{7}$.

Example 20.13: The radii of two circles are 18 cm and 10 cm. Find the radius of the circle whose circumference is equal to the sum of the circumferences of these two circles.

Solution: Let the radius of the circle be r cm.

Its circumference = $2\pi r$ cm ....(1)

Also, sum of the circumferences of the two circles = $(2\pi \times 18 + 2\pi \times 10)$ cm

$$= 2\pi \times 28 \text{ cm} \quad ...(2)$$

Therefore, from (1) and (2), $2\pi r = 2\pi \times 28$

or $r = 28$

i.e., radius of the circle is 28 cm.

Example 20.14: There is a circular path of width 2 m along the boundary and inside a circular park of radius 16 m. Find the cost of paving the path with bricks at the rate of ₹ 24 per m^2. (Use $\pi = 3.14$)

Solution: Let OA be radius of the park and shaded portion be the path (See. Fig. 20.9)

So, OA = 16 m

and OB = 16 m – 2 m = 14 m.

Therefore, area of the path

$$= (\pi \times 16^2 - \pi \times 14^2) \text{ m}^2$$

$$= \pi(16 + 14)(16 - 14) \text{ m}^2$$

$$= 3.14 \times 30 \times 2 = 188.4 \text{ m}^2$$

So, cost of paving the bricks at ₹ 24 per m^2

= ₹ 24 × 188.4

= ₹ 4521.60

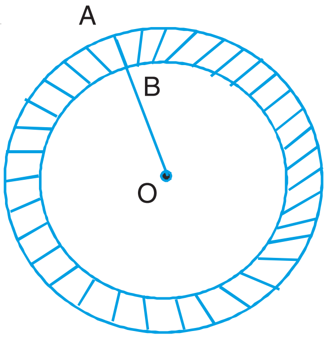


Fig. 20.9

CHECK YOUR PROGRESS 20.4

1. The radii of two circles are 9 cm and 12 cm respectively. Find the radius of the circle whose area is equal to the sum of the areas of these two circles.

2. The wheels of a car are of radius 40 cm each. If the car is travelling at a speed of 66 km per hour, find the number of revolutions made by each wheel in 20 minutes.

3. Around a circular park of radius 21 m, there is circular road of uniform width 7 m outside it. Find the area of the road.

20.5 PERIMETER AND AREA OF A SECTOR

You are already familar with the term **sector of a circle**. Recall that a part of a circular region enclosed between two radii of the corresponding circle is called a sector of the circle. Thus, in Fig. 20.10, the shaded region OAPB is a sector of the circle with centre O. ∠AOB is called the **central angle** or simply the angle of the sector. Clearly, APB is the corresponding arc of this sector. You may note that the part OAQB (unshaded region) is also a sector of this circle. For obvious reasons, OAPB is called the **minor sector** and OAQB is called the **major sector** of the circle (with major arc AQB).

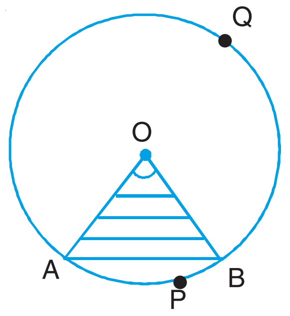


Fig. 20.10

Note: *unless stated otherwise, by sector, we shall mean a minor sector.*

(i) **Perimeter of the sector:** Clearly, perimeter of the sector OAPB is equal to OA + OB + length of arc APB.

Let radius OA (or OB) be r, length of the arc APB be *l* and ∠AOB be θ.

We can find the length *l* of the arc APB as follows:

We know that circumference of the circle = 2 πr

Now, for total angle 360° at the centre, length = 2πr

$$\text{So, for angle } \theta\text{, length } l = \frac{2\pi r}{360^\circ} \times \theta$$

$$\text{or} \qquad l = \frac{\pi r \theta}{180^\circ} \qquad \ldots(1)$$

Thus, perimeter of the sector OAPB = OA + OB + *l*

$$= r + r + \frac{\pi r \theta}{180^\circ} = 2\,r + \frac{\pi r \theta}{180^\circ}$$

(ii) **Area of the sector**

Area of the circle = πr^2

Now, for total angle 360°, area = πr^2

$$\text{So, } \textbf{for angle } \boldsymbol{\theta}, \textbf{ area} = \frac{\pi r^2}{360^\circ} \times \theta$$

Thus, area of the sector OAPB = $\frac{\pi r^2\theta}{360^o}$

Note: *By taking the angle as* $360^o - \theta$*, we can find the perimeter and area of the major sector OAQB as follows*

$$\text{Perimeter} = 2r + \frac{\pi r(360^o - \theta)}{180^o}$$

$$\text{and} \qquad \text{area} = \frac{\pi r^2}{360^o} \times (360^o - \theta)$$

Example 20.15: Find the perimeter and area of the sector of a circle of radius 9 cm with central angle 35°.

Solution: Perimeter of the sector $= 2r + \frac{\pi r\theta}{180^o}$

$$= \left(2\times 9 + \frac{22}{7}\times\frac{9\times 35^o}{180^o}\right)\text{cm}$$

$$= \left(18 + \frac{11\times 1}{2}\right)\text{cm} = \frac{47}{2}\text{ cm}$$

Area of the sector $= \frac{\pi r^2\times\theta}{360^o}$

$$= \left(\frac{22}{7}\times\frac{81\times 35^o}{360^o}\right)\text{cm}^2$$

$$= \left(\frac{11\times 9}{4}\right)\text{cm}^2 = \frac{99}{4}\text{cm}^2$$

Example 20.16: Find the perimeter and area of the sector of a circle of radius 6 cm and length of the arc of the sector as 22 cm.

Solution: Perimeter of the sector $= 2r + \text{length of the arc}$

$$= (2 \times 6 + 22)\text{ cm} = 34\text{ cm}$$

For area, let us first find the central angle θ.

So, $\frac{\pi r\theta}{180^o} = 22$

or $$\frac{22}{7}\times 6\times\frac{\theta}{180^{\circ}}=22$$

or $$\theta=\frac{180^{\circ}\times 7}{6}=210^{\circ}$$

So, area of the sector $$=\frac{\pi r^{2}\theta}{360^{\circ}}$$

$$=\frac{22}{7}\times\frac{36\times 210^{\circ}}{360^{\circ}}$$

$$=66 \text{ cm}^2$$

Alternate method for area:

Circumference of the circle $= 2\pi r$

$$= 2\times\frac{22}{7}\times 6 \text{ cm}$$

and area of the circle $= \pi r^2 = \frac{22}{7}\times 6\times 6 \text{ cm}^2$

For length $2\times\frac{22}{7}\times 6$ cm, area $= \frac{22}{7}\times 6\times 6 \text{ cm}^2$

So, for length 22 cm, area $= \frac{22}{7}\times\frac{6\times 6\times 7\times 22}{2\times 22\times 6} \text{ cm}^2$

$$= 66 \text{ cm}^2$$

CHECK YOUR PROGRESS 20.5

1. Find the perimeter and area of the sector of a circle of radius 14 cm and central angle 30°.
2. Find the perimeter and area of the sector of a circle of radius 6 cm and length of the arc as 11 cm.

20.6 AREAS OF COMBINATIONS OF FIGURES INVOLVING CIRCLES

So far, we have been discussing areas of figures separately. We shall now try to calculate areas of combinations of some plane figures. We come across these type of figures in daily life in the form of various designs such as table covers, flower beds, window designs, etc. Let us explain the process of finding their areas through some examples.

Example 20.17: In a round table cover, a design is made leaving an equilateral triangle ABC in the middle as shown in Fig. 20.11. If the radius of the cover is 3.5 cm, find the cost of making the design at the rate of ₹ 0.50 per cm^2 (use $\pi = 3.14$ and $\sqrt{3} = 1.7$)

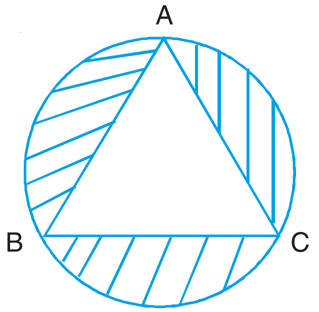


Fig. 20.11

Solution: Let the centre of the cover be O.

Draw OP ⊥ BC and join OB, OC. (Fig. 20.12)

Now, $\angle BOC = 2\ \angle BAC = 2 \times 60^o = 120^o$

Also, $\angle BOP = \angle COP = \frac{1}{2}\ \angle BOC = \frac{1}{2} \times 120^o = 60^o$

Now, $\frac{BP}{OB} = \sin\angle BOP = \sin 60^o = \frac{\sqrt{3}}{2}$ [See Lessons 22-23]

i.e., $\frac{BP}{3.5} = \frac{\sqrt{3}}{2}$

So, $BC = 2 \times \frac{3.5\sqrt{3}}{2}\ cm = 3.5\sqrt{3}\ cm$

Therefore, area of $\Delta ABC = \frac{\sqrt{3}}{4} BC^2$

$= \frac{\sqrt{3}}{4} \times 3.5 \times 3.5 \times 3\ cm^2$

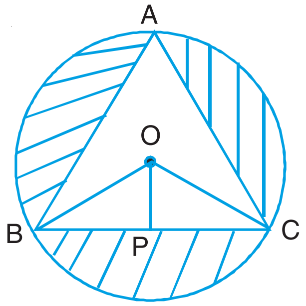


Fig. 20.12

Now, area of the design = area of the circle – area of ΔABC

$= (3.14 \times 3.5 \times 3.5 - \frac{\sqrt{3}}{4} \times 3.5 \times 3.5 \times 3)\ cm^2$

$= (3.14 \times 3.5 \times 3.5 - \frac{1.7 \times 3.5 \times 3.5 \times 3}{4})\ cm^2$

$$= 3.5 \times 3.5\left(\frac{12.56 - 5.10}{4}\right) \text{cm}^2$$

$$= 12.25\left(\frac{7.46}{4}\right) \text{cm}^2 = 12.25 \times 1.865 \text{ cm}^2$$

Therefore, cost of making the design at ₹ 0.50 per cm^2

$$= ₹\ 12.25 \times 1.865 \times 0.50 = ₹\ 114.23 \text{ (approx)}$$

Example 20.18: On a square shaped handkerchief, nine circular designs, each of radius 7 cm, are made as shown in Fig. 20.13. Find the area of the remainig portion of the handkerchief.

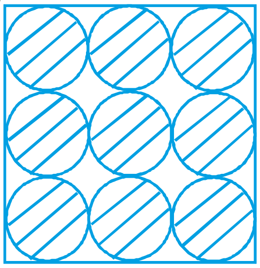


Fig. 20.13

Solution: As radius of each circular design is 7 cm, diameter of each will be 2 × 7 cm = 14 cm

So, side of the square handkerchief = 3 × 14 = 42 cm ...(1)

Therefore, area of the square = 42 × 42 cm^2

Also, area of a circle $= \pi r^2 = \frac{22}{7} \times 7 \times 7 \text{ cm}^2 = 154 \text{ cm}^2$

So, area of 9 circles = 9 × 154 cm^2 ...(2)

Therefore, from (1) and (2), area of the remaining portion

$$= (42 \times 42 - 9 \times 154) \text{ cm}^2$$

$$= (1764 - 1386) \text{ cm}^2 = 378 \text{ cm}^2$$

CHECK YOUR PROGRESS 20.6

1. A square ABCD of side 6 cm has been inscribed in a quadrant of a circle of radius 14 cm (See Fig. 20.14). Find the area of the shaded region in the figure.

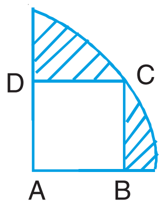


Fig. 20.14

2. A shaded design has been formed by drawing semicircles on the sides of a square of side length 10 cm each as shown in Fig. 20.15. Find the area of the shaded region in the design.

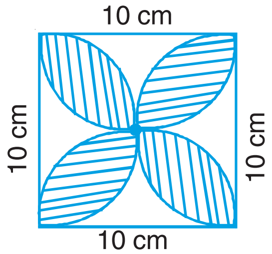


Fig. 20.14

LET US SUM UP

- Perimeter of a rectangle = 2 (length + breadth)
- Area of a rectangle = length × breadth
- Perimeter of a square = 4 × side
- Area of a square = (side)^2
- Area of a parallelogram = base × corresponding altitude
- Area of a triangle = $\frac{1}{2}$ base × corresponding altitude

  and also $\sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the three sides of the triangle

  and $s = \dfrac{a+b+c}{2}$.
- Area of a rhombus = $\frac{1}{2}$ product of its diagonals
- Area of a trapezium = $\frac{1}{2}$ (sum of the two parallel sides) × distance between them
- Area of rectangular path = area of the outer rectangle – area of inner rectangle
- Area of cross paths in the middle = Sum of the areas of the two paths – area of the common portion
- circumference of a circle of radius r = 2 πr
- Area of a circle of radius r = πr^2
- Area of a circular path = Area of the outer circle – area of the inner circle
- Length l of the arc of a sector of a circle of radius r with central angle θ is $l = \dfrac{\pi r\theta}{180^\circ}$
- Perimeter of the sector a circle with radius r and central angle $\theta = 2r + \dfrac{\pi r\theta}{180^\circ}$
- Area of the sector of a circle with radius r and central and $\theta = \dfrac{\pi r^2\theta}{360^\circ}$

- Areas of many rectilinear figures can be found by dividing them into known figures such as squares, rectangles, triangles and so on.
- Areas of various combinations of figures and designs involving circles can also be found by using different known formulas.

TERMINAL EXERCISE

1. The side of a square park is 37.5 m. Find its area.
2. The perimeter of a square is 480 cm. Find its area.
3. Find the time taken by a person in walking along the boundary of a square field of area 40 000 m^2 at a speed of 4 km/h.
4. Length of a room is three times its breadth. If its breadth is 4.5 m, find the area of the floor.
5. The length and breadth of a rectangle are in the ratio of 5 : 2 and its perimeter is 980 cm. Find the area of the rectangle.
6. Find the area of each of the following parallelograms:

   (i) one side is 25 cm and corresponding altitude is 12 cm

   (ii) Two adjacent sides are 13 cm and 14 cm and one diagonal is 15 cm.
7. The area of a rectangular field is 27000 m^2 and its length and breadth are in the ratio 6:5. Find the cost of fencing the field by four rounds of barbed wire at the rate of ₹ 7 per 10 metre.
8. Find the area of each of the following trapeziums:

S. No.	Lengths of parellel sides	Distance between the parallel sides
(i)	30 cm and 20 cm	15 cm
(ii)	15.5 cm and 10.5 cm	7.5 cm
(iii)	15 cm and 45 cm	14.6 cm
(iv)	40 cm and 22 cm	12 cm

9. Find the area of a plot which is in the shape of a quadrilateral, one of whose diagonals is 20 m and lengths of the perpendiculars from the opposite corners on it are of lengths 12 m and 18 m respectively.
10. Find the area of a field in the shape of a trapezium whose parallel sides are of lengths 48 m and 160 m and non-parallel sides of lengths 50 m and 78 m.

11. Find the area and perimeter of a quadrilateral ABCCD in which AB = 8.5 cm, BC = 14.3 cm, CD = 16.5 cm, AD = 8.5 cm and BD = 15.4 cm.

12. Find the areas of the following triangles whose sides are

    (i) 2.5 cm, 6 cm and 6.5 cm

    (ii) 6 cm, 11.1 cm and 15.3 cm

13. The sides of a triangle are 51 cm, 52 cm and 53 cm. Find:

    (i) Area of the triangle

    (ii) Length of the perpendicular to the side of length 52 cm from its opposite vertex.

    (iii) Areas of the two triangles into which the given triangle is divided by the perpendicular of (ii) above.

14. Find the area of a rhombus whose side is of length 5 m and one of its diagonals is of length 8 m.

15. The difference between two parallel sides of a trapezium of area 312 cm^2 is 8 cm. If the distance between the parallel sides is 24 cm, find the length of the two parallel sides.

16. Two perpendicular paths of width 10 m each run in the middle of a rectangular park of dimensions $200 \text{ m} \times 150 \text{ m}$, one parallel to length and the other parallel ot the breadth. Find the cost of constructing these paths at the rate of ₹ 5 per m^2

17. A rectangular lawn of dimensions 65 m × 40 m has a path of uniform width 8 m all around inside it. Find the cost of paving the red stone on this path at the rate of ₹ 5.25 per m^2.

18. A rectangular park is of length 30 m and breadth 20 m. It has two paths, each of width 2 m, around it (one inside and the other outside it). Find the total area of these paths.

19. The difference between the circumference and diameter of a circle is 30 cm. Find its radius.

20. A path of uniform width 3 m runs outside around a circular park of radius 9 m. Find the area of the path.

21. A circular park of radius 15 m has a road 2 m wide all around inside it. Find the area of the road.

22. From a circular piece of cardboard of radius 1.47 m, a sector of angle 60^o has been removed. Find the area of the remaining cardboard.

23. Find the area of a square field, in hectares, whose side is of length 360 m.

24. Area of a triangular field is 2.5 hectares. If one of its sides is 250 m, find its corresponding altitude.

25. A field is in the shape of a trapezium of parallel sides 11 m and 25 m and of non-parallel sides 15 m and 13 m. Find the cost of watering the field at the rate of 5 paise per 500 cm^2.

26. From a circular disc of diameter 8 cm, a square of side 1.5 cm is removed. Find the area of the remaining poriton of the disc. (Use $\pi = 3.14$)

27. Find the area of the adjoining figure with the measurement, as shown. (Use $\pi = 3.14$)

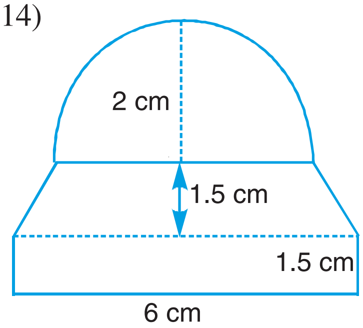


Fig. 20.16

28. A farmer buys a circular field at the rate of ₹ 700 per m^2 for ₹ 316800. Find the perimeter of the field.

29. A horse is tied to a pole at a corner of a square field of side 12 m by a rope of length 3.5 m. Find the area of the part of the field in which the horse can graze.

30. Find the area of the quadrant of a circle whose circumference is 44 cm.

31. In Fig. 20.17, OAQB is a quadrant of a circle of radius 7 cm and APB is a semicircle. Find the area of the shaded region.

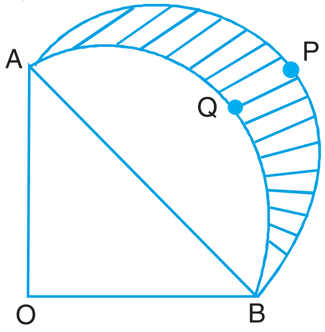


Fig. 20.17

32. In Fig 20.18, radii of the two concentric circles are 7 cm and 14 cm and $\angle AOB = 45^o$, Find the area of the shaded region ABCD.

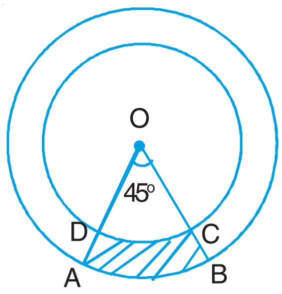


Fig. 20.18

33. In Fig. 20.19, four congruent circles of radius 7 cm touch one another and A, B, C, and D are their centres. Find the area of the shaded region.

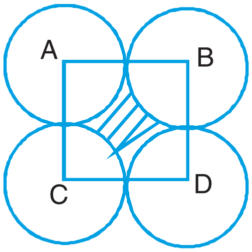


Fig. 20.19

34. Find the area of the flower bed with semicircular ends of Fig. 20.20, if the diameters of the ends are 14cm, 28 cm, 14 cm and 28 cm respectively.

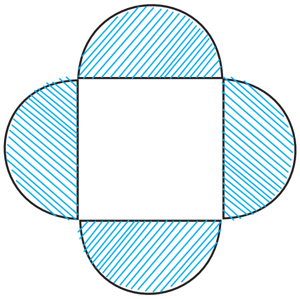


Fig. 20.20

35. In Fig 20.21, two semicircles have been drawn inside the square ABCD of side 14 cm. Find the area of the shaded region as well as the unshaded region.

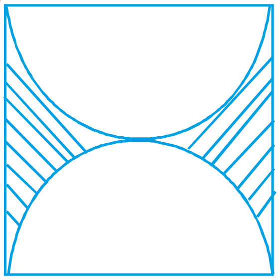


Fig. 20.21

In each of the questions 36 to 42, write the correct answer from the four given options:

36. The perimeter of a square of side a is

(A) a^2 (B) $4a$ (C) $2a$ (D) $\sqrt{2}\,a$

37. The sides of a triangle are 15 cm, 20 cm, and 25 cm. Its area is

(A) 30 cm^2 (B) 150 cm^2 (C) 187.5 cm^2 (D) 300 cm^2

38. The base of an isosceles triangle is 8 cm and one of its equal sides is 5 cm. The corresponding height of the triangle is

(A) 5 cm (B) 4 cm (C) 3 cm (D) 2 cm

39. If a is the side of an equilateral triangle, then its altitude is

(A) $\frac{\sqrt{3}}{2}a^2$ (B) $\frac{\sqrt{3}}{2a^2}$ (C) $\frac{\sqrt{3}}{2}a$ (D) $\frac{\sqrt{3}}{2a}$

40. One side of a parallelogram is 15 cm and its corresponding altitude is 5 cm. Area of the parallelogram is

(A) 75 cm^2 (B) 37.5 cm^2 (C) 20 cm^2 (D) 3 cm^2

41. Area of a rhombus is 156 cm^2 and one of its diagonals is 13 cm. Its other diagonal is

(A) 12 cm (B) 24 cm (C) 36 cm (D) 48 cm

42. Area of a trapezium is 180 cm^2 and its two parallel sides are 28 cm and 12 cm. Distance between these two parallel sides is

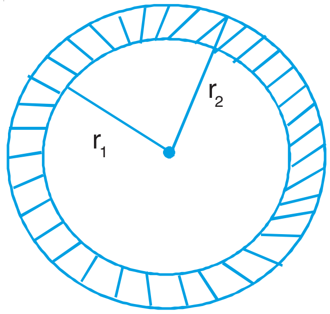
(A) 9 cm (B) 12 cm (C) 15 cm (D) 18 cm

43. Which of the following statements are true and which are false?

(i) Perimeter of a rectangle is equal to length + breadth.

(ii) Area of a circle of radus r is πr^2.

(iii) Area of the circular shaded path of the adjoining figure is $\pi r_1^2 - \pi r_2^2$.



(iv) Area of a triangle of sides a, b and c is $\sqrt{s(s-a)(s-b)(s-c)}$, where s is the perimeter of the triangle.

(v) Area of a sector of circle of radius r and central angle 60° is $\frac{\pi r^2}{6}$.

(vi) Perimeter of a sector of circle of radius 5 cm and central angle 120° is 5 cm + $\frac{10\pi}{3}$ cm

44. Fill in the blanks:

(i) Area of a rhombus = $\frac{1}{2}$ product of its ____________________

(ii) Area of a trapezium = $\frac{1}{2}$ (sum of its ________) × distance between ______

(iii) The ratio of the areas of two sectors of two circles of radii 4 cm and 8 cm and central angles 100° and 50° respectively is __________

(iv) The ratio of the lengths of the arcs of two sectors of two circles of radii 10 cm and 5 cm and central angles 75° and 150° is _____________.

(v) Perimeter of a rhombus of diagonals 16 cm and 12 cm is __________

ANSWERS TO CHECK YOUR PROGRESS

20.1

1. 60 m

2. $15\sqrt{2}$ cm

3. (i) 281.25 m^2 (ii) 70 m

4. 110 m [Hint $3x \times 2x = 726 \Rightarrow x = 11$ m]

5. 240 cm^2

6. 80 cm

7. 190 cm^2

8. 55 cm, 1320 cm^2

20.2

1. $24\sqrt{21}$ cm^2

2. $36\sqrt{3}$ cm^2; $6\sqrt{3}$ cm

20.3

1. 648 m^2

2. 276 m^2

3. 7225 m^2

4. $\left(27+\frac{5}{4}\sqrt{11}\right)$ cm^2

20.4

1. 15 cm

2. 8750

3. 10.78 m^2

20.5

1. Perimeter = $35\frac{1}{2}$ cm; Area = $\frac{154}{3}$ cm^2

2. Perimeter = 23 cm, Area = 33 cm^2

20.6

1. 118 cm^2

2. $4\times\frac{1}{2}\pi\times5^2$ – 10 × 10 cm^2

= (50π – 100) cm^2

ANSWERS TO TERMINAL EXERCISE

1. 1406.25 m^2 2. 14400 cm^2 3. 12 minutes

4. 60.75 m^2 5. 49000 cm^2 6. (i) 300 cm^2 (ii) 168 cm^2

7. ₹ 1848

8. (i) 375 cm^2 (ii) 97.5 cm^2 (iii) 438 m^2 (iv) 372 cm^2

9. 300 m^2 10. 3120 m^2 11. 129.36 cm^2

12. (i) 7.5 cm^2 (ii) 27.54 cm^2

13. (i) 1170 cm^2 (ii) 45 cm (iii) 540 cm^2, 630 cm^2

14. 24 m^2 15. 17 cm and 9 cm 16. ₹ 17000

17. ₹ 7476 18. 400 m^2 19. 7 cm

20. 198 m^2 21. 176 m^2 22. 1.1319 m^2

23. 12.96 ha 24. 200 m 25. ₹ 216

26. 47.99 cm^2 27. 22.78 cm^2 28. $75\frac{3}{7}$ m

29. $\frac{77}{8}$ m^2 30. $\frac{77}{2}$ cm^2 31. $\frac{49}{2}$ cm^2

32. $\frac{231}{4}$ cm^2 33. 42 cm^2 34. 1162 cm^2

35. 42 cm^2, 154 cm^2 36. (B) 37. (B)

38. (C) 39. (C) 40. (A)

41. (B) 42. (A)

43. (i) False (ii) True (iii) False
(iv) False (v) True (vi) False

44. (i) diagonals (ii) parallel sides, them (iii) 1 : 2
(iv) 1 : 1 (v) 40 cm.

21

SURFACE AREAS AND VOLUMES OF SOLID FIGURES

In the previous lesson, you have studied about perimeters and areas of plane figures like rectangles, squares, triangles, trapeziums, circles, sectors of circles, etc. These are called plane figures because each of them lies wholly in a plane. However, most of the objects that we come across in daily life do not wholly lie in a plane. Some of these objects are bricks, balls, ice cream cones, drums, and so on. These are called solid objects or three dimensional objects. The figures representing these solids are called **three dimensional** or **solid figures**. Some common solid figures are cuboids, cubes, cylinders, cones and spheres. In this lesson, we shall study about the surface areas and volumes of all these solids.

OBJECTIVES

After studying this lesson, you will be able to

- *explain the meanings of surface area and volume of a solid figure,*
- *identify situations where there is a need of finding surface area and where there is a need of finding volume of a solid figure;*
- *find the surface areas of cuboids, cubes, cylinders, cones spheres and hemispheres, using their respective formulae;*
- *find the volumes of cuboids, cubes, cylinders, cones, spheres and hemispheres using their respective formulae;*
- *solve some problems related to daily life situations involving surface areas and volumes of above solid figures.*

EXPECTED BACKGROUND KNOWLEDGE

- Perimeters and Areas of Plane rectilinear figures.
- Circumference and area of a circle.

- Four fundamental operations on numbers
- Solving equations in one or two variables.

21.1 MEANINGS OF SURFACE AREA AND VOLUME

Look at the following objects given in Fig. 21.1.

Fig. 21.1

Geometrically, these objects are represented by three dimensional or solid figures as follows:

Objects	**Solid Figure**
Bricks, Almirah	Cuboid
Die, Tea packet	Cube
Drum, powder tin	Cylinder
Jocker's cap, Icecream cone,	Cone
Football, ball	Sphere
Bowl.	Hemisphere

You may recall that a rectangle is a figure made up of only its sides. You may also recall that the sum of the lengths of all the sides of the rectangle is called its permeter and the measure of the region enclosed by it is called its **area**. Similarly, the sum of the lengths of the three sides of a triangle is called its **permeters**, while the measure of the region enclosed by the triangle is called its area. In other words, the measure of the plane figure, i.e., the boundary triangle or rectangle is called its perimeter, while the measure of the plane region enclosed by the figure is called its **area**.

Following the same analogy, a solid figure is made up of only its boundary (or outer surface). For example, cuboid is a solid figure made up of only its six rectangular regions (called its faces). Similarly, a sphere is made up only of its outer surface or boundary. Like plane figures, solid figures can also be measured in two ways as follows:

(1) Measuring the surface (or boundary) constituting the solid. It is called the **surface area** of the solid figure.

(2) Measuring the space region enclosed by the solid figure. It is called the **volume** of the solid figure.

Thus, it can be said that the surface area is the measure of the solid figure itself, while volume is the measure of the space region enclosed by the solid figure. Just as area is measured in square units, volume is measured in **cubic units**. If the unit is chosen as a **unit cube** of side 1 cm, then the unit for volume is cm^3, if the unit is chosen as a **unit cube** of side 1m, then the unit for volume is m^3 and so on.

In daily life, there are many situations, where we have to find the surface area and there are many situations where we have to find the volume. For example, if we are interested in white washing the walls and ceiling of a room, we shall have to find the surface areas of the walls and ceiling. On the other hand, if we are interested in storing some milk or water in a container or some food grains in a godown, we shall have to find the volume.

21.2 CUBOIDS AND CUBES

As already stated, a brick, chalk box, geometry box, match box, a book, etc are all examples of a cuboid. Fig. 21.2 represents a cuboid. It can be easily seen from the figure that a cuboid has six rectangular regions as its faces. These are ABCD, ABFE, BCGF, EFGH, ADHE and CDHG.. Out of these, opposite faces ABFE and CDHG; ABCD and EFGH and ADHE and BCGH are respectively congruent and parallel to each other. The two adjacent faces meet in a line segment called an **edge** of the cuboid. For example, faces ABCD and ABFE meet in the **edge** AB. There are in all 12 edges of a cuboid. Points A,B,C,D,E,F,G and H are called the **corners** or **vertices** of the cuboid. So, there are 8 **corners** or **vertices** of a cuboid.

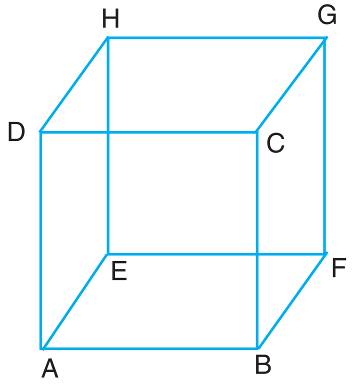
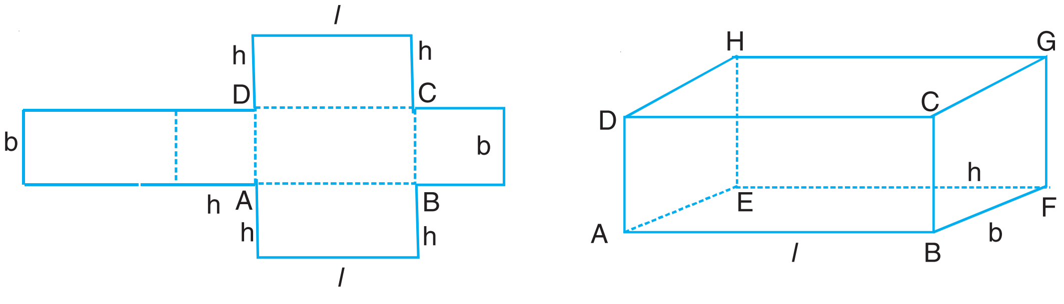


Fig. 21.2

It can also be seen that at each vertex, three edges meet. One of these three edges is taken as the length, the second as the breadth and third is taken as the height (or thickness or depth) of the cuboid. These are usually denoted by l, b and h respectively. Thus, we may say that A B (= EF = CD = GH) is the **length**, AE (=BF = CG = DH) is the **breadth** and AD (= EH = BC = FG) is the **height** of the cuboid.

Note that three faces ABFE, AEHD and EFGH meet at the vertex E and their opposite faces DCGH, BFGC and ABCD meet at the point C. Therefore, E and C are called the **opposite corners** or **vertices** of the cuboid. The line segment joing E and C. i.e., EC is called a **diagonal** of the cuboid. Similarly, the diagonals of the cuboid are AG, BH and FD. In all there are **four diagonals** of cuboid.

Surface Area



(i) (ii)

Fig. 21.3

Look at Fig. 21.3 (i). If it is folded along the dotted lines, it will take the shape as shown in Fig. 21.3 (ii), which is a cuboid. Clearly, the length, breadth and height of the cuboid obtained in Fig. 21.3 (ii) are l, b and h respectively. What can you say about its surface area. Obviously, surface area of the cuboid is equal to the sum of the areas of all the six rectangles shown in Fig. 21.3 (i).

Thus, surface area of the cuboid

$$= l\times b + b\times h + h\times l + l\times b + b\times h + h\times l$$

$$= 2(lb + bh + hl)$$

In Fig. 21.3 (ii), let us join BE and EC (See Fig. 21.4)

We have :

$BE^2 = AB^2 + AE^2$ (As $\angle EAB = 90^o$)

or $BE^2 = l^2 + b^2$ -(1)

Also, $EC^2 = BC^2 + BE^2$ (As $\angle CBE = 90^o$)

or $EC^2 = h^2 + l^2 + b^2$ [From (i)]

So, $EC = \sqrt{l^2 + b^2 + h^2}.$

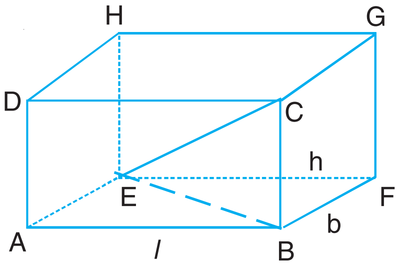


Fig. 21.4

Hence, **diagonal of a cuboid** = $\sqrt{l^2 + b^2 + h^2}.$

We know that cube is a special type of cuboid in which length = breadth = height, i.e., l = b = h.

Hence,

surface area of a cube of side or edge a

$$= 2\,(a \times a + a \times a + a \times a)$$

$$= 6a^2$$

and its **diagonal** = $\sqrt{a^2 + a^2 + a^2}.$ = a $\sqrt{3}.$

Note: Fig. 21.3 (i) is usually referred to as a **net** of the cuboid given in Fig. 21.3 (ii).

Volume:

Take some unit cubes of side 1 cm each and join them to form a cuboid as shown in Fig. 21.5 given below:

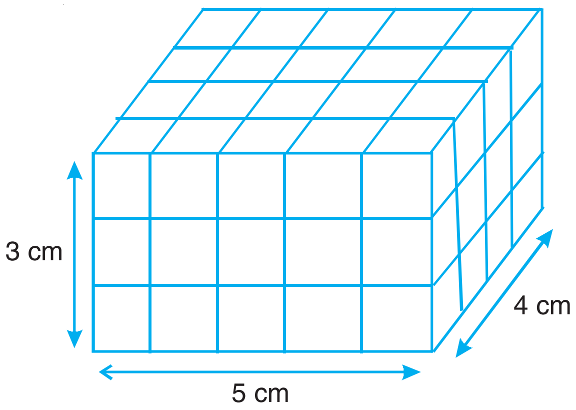


Fig. 21.5

By actually counting the unit cubes, you can see that this cuboid is made up of 60 unit cubes.

So, its volume = 60 cubic cm or 60cm^3 (Because volume of 1 unit cube, in this case, is 1 cm^3)

Also, you can observe that length × breadth × height $= 5 \times 4 \times 3$ cm^3

$$= 60 \text{ cm}^3$$

You can form some more cuboids by joining different number of unit cubes and find their volumes by counting the unit cubes and then by the product of length, breadth and height. Everytime, you wll find that

Volume of a cuboid = length × breadth × height

or volume of a cuboid = lbh

Further, as cube is a special case of cuboid in which $l = b = h$, we have;

volume of a cube of side $a = a \times a \times a \times = a^3$.

We now take some examples to explain the use of these formulae.

Example 21.1: Length, breadth and height of cuboid are 4 cm, 3 cm and 12 cm respectively. Find

(i) surface area (ii) volume and (iii) diagonal of the cuboid.

Solution: (i) Surface area of the cuboid

$$= 2\ (\text{lb} + \text{bh} + \text{hl})$$

$$= 2\ (4{\times}3{+}3{\times}12{+}12{\times}4)\text{cm}^2$$

$$= 2\ (12 + 36 + 48)\text{cm}^2 = 192\ \text{cm}^2$$

(ii) Volume of cuboid $= lbh$

$$= 4 \times 3 \times 12\ \text{cm}^3 = 144\ \text{cm}^2$$

(iii) Diagonal of the cuboid $= \sqrt{l^2 + b^2 + h^2}.$

$$= \sqrt{4^2 + 3^2 + 12^2}\ \text{cm.}$$

$$= \sqrt{16 + 9 + 144}\ \text{cm.}$$

$$= \sqrt{169}\ \text{cm} = 13\ \text{cm}$$

Example 21.2: Find the volume of a cuboidal stone slab of length 3m, breadth 2m and thickness 25cm.

Solution : Here, $l = 3$m, $b = 2$m and

$$h = 25\text{cm} = \frac{25}{100} = \frac{1}{4}m$$

(Note that here we have thickness as the third dimension in place of height)

So, required volume $= lbh$

$$= 3{\times}2{\times}\frac{1}{4}\ \text{m}^3 = 1.5\text{m}^3$$

Example 21.3 : Volume of a cube is 2197 cm^3. Find its surface area and the diagonal.

Solution: Let the edge of the cube be a cm.

So, its volume $= a^3$ cm^3

Therefore, from the question, we have :

$$a^3 = 2197$$

or $a^3 = 13{\times}13{\times}13$

So, $a = 13$

i.e., edge of the cube = 13 cm

Now, surface area of the cube = $6a^2$

$$= 6 \times 13 \times 13 \text{ cm}^2$$

$$= 1014 \text{ cm}^2$$

Its diagonal = $a\sqrt{3}\text{ cm} = 13\sqrt{3}\text{ cm}$

Thus, surface area of the cube is 1014 cm^2 and its diagonal is $13\sqrt{3}\text{ cm}$.

Example 21.4 : The length and breadth of a cuboidal tank are 5m and 4m respectively. If it is full of water and contains 60 m^3 of water, find the depth of the water in the tank.

Solution: let the depth be d metres

So, volume of water in the tank

$$= l \times b \times \text{h}$$

$$= 5 \times 4 \times \text{d m}^3$$

Thus, according to the question,

$$5 \times 4 \times d = 60$$

or $$d = \frac{60}{5 \times 4}\text{m} = 3\text{m}$$

So, depth of the water in the tank is 3m.

Note: Volume of a container is usually called its **capacity**. Thus, here it can be said that capacity of the tank is 60m^3. Capacity is also expressed in terms of litres, where 1 litre = $\frac{1}{1000}\text{m}^3$, i.e., 1m^3 = 1000 litres.

So, it can be said that capacity of the tank is 60 × 1000 litre = 60 kilolitres.

Example 21.5 : A wooden box 1.5m long, 1.25 m broad, 65 cm deep and open at the top is to be made. Assuming the thickness of the wood negligible, find the cost of the wood required for making the box at the rate of ₹ 200 per m^2.

Solution : Surface area of the wood required

$$= lb + 2bh + 2hl \text{ (Because the box is open at the top)}$$

$$= (1.5 \times 1.25 + 2 \times 1.25 \times \frac{65}{100} + 2 \times \frac{65}{100} \times 1.5)\text{m}^2$$

$$= (1.875 + \frac{162.5}{100} + \frac{195}{100}\ )\ \text{m}^2$$

$$= (1.875 + 1.625 + 1.95)\text{m}^2 = 5.450\ \text{m}^2$$

So, cost of the wood at the rate of ₹ 200 per m^2

$$= ₹\ 200 \times 5.450$$

$$= ₹\ 1090$$

Example 21.6 : A river 10m deep and 100m wide is flowing at the rate of 4.5 km per hour. Find the volume of the water running into the sea per second from this river.

Solution : Rate of flow of water = 4.5 km/h

$$= \frac{4.5 \times 1000}{60 \times 60} \text{ metres per second}$$

$$= \frac{4500}{3600} \text{ metres per second}$$

$$= \frac{5}{4} \text{ metres per second}$$

Threfore, volume of the water running into the sea per second = volume of the cuboid

$$= l \times b \times h$$

$$= \frac{5}{4} \times 100 \times 10\ \text{m}^3$$

$$= 1250\ \text{m}^3$$

Example 21.7: A tank 30m long, 20m wide and 12 m deep is dug in a rectangular field of length 588 m and breadth 50m. The earth so dug out is spread evenly on the remaining part of the field. Find the height of the field raised by it.

Solution: Volume of the earth dug out = volume of a cuboid of dimensions 30 m × 20 m × 12 m

$$= 30 \times 20 \times 12\ \text{m}^3 = 7200\ \text{m}^3$$

Area of the remaining part of the field

= Area of the field - Area of the top surface of the tank

$$= 588 \times 50\ \text{m}^2 - 30 \times 20\ \text{m}^2$$

$$= 29400\ \text{m}^2 - 600\ \text{m}^2$$

$$= 28800\ \text{m}^2$$

Therefore, height of the field raised

$$= \frac{\text{Volume of earth dug out}}{\text{Area of the remaining part of the field}}$$

$$= \frac{7200}{28800}\text{ m} = \frac{1}{4}\text{ m} = 25\text{ cm}$$

Example 21.8: Length, breadth and height of a room are 7m, 4m and 3m respectively. It has a door and a window of dimensions $2\text{ m} \times 1\frac{1}{2}\text{ m}$ and $1\frac{1}{2}\text{ m} \times 1\text{ m}$ respectively. Find the cost of white washing the walls and ceiling of the room at the rate of ₹ 4 per m^2.

Solution: Shape of the room is that of a cuboid.

Area to be white washed = Area of four walls

+ Area of the ceiling

– Area of the door – Area of the window.

Area of the four walls $= l\times h + b\times h + l\times h + b\times h$

$$= 2(l+b)\times h$$

$$= 2(7+4)\times 3\text{ m}^2 = 66\text{m}^2$$

Area of the ceiling $= l\times b$

$$= 7\times 4\text{ m}^2 = 28\text{m}^2$$

So, area to be white washed $= 66\text{ m}^2 + 28\text{ m}^2 - 2\times 1\frac{1}{2}\text{ m}^2 - 1\frac{1}{2}\times 1\text{m}^2$

$$= 94\text{ m}^2 - 3\text{ m}^2 - \frac{3}{2}\text{ m}^2$$

$$= \frac{(188-6-3)}{2}\text{ m}^2$$

$$= \frac{179}{2}\text{ m}^2$$

Therefore, cost of white-washing at the rate of ₹ 4 per m^2

$$= ₹\ 4 \times \frac{179}{2} = ₹\ 358$$

Note: You can directly use the relation area of four walls $= 2\,(l+b)\times h$ as a formula]

CHECK YOUR PROGRESS 21.1

1. Find the surface area and volume of a cuboid of length 6m, breadth 3m and height 2.5m.
2. Find the surface area and volume of a cube of edge 3.6 cm
3. Find the edge of a cube whose volume is 3375 cm^3. Also, find its surface area.
4. The external dimensions of a closed wooden box are 42 cm × 32 cm × 27 cm. Find the internal volume of the box, if the thickness of the wood is 1cm.
5. The length, breadth and height of a godown are 12m, 8m and 6 metres respectively. How many boxes it can hold if each box occupies 1.5 m^3 space?
6. Find the length and surface area of a wooden plank of width 3m, thickness 75 cm and volume 33.75m^3.
7. Three cubes of edge 8 cm each are joined end to end to form a cuboid. Find the surface area and volume of the cuboid so formed.
8. A room is 6m long, 5m wide and 4m high. The doors and windows in the room occupy 4 square metres of space. Find the cost of papering the remaining portion of the walls with paper 75cm wide at the rate of ₹ 2.40 per metre.
9. Find the length of the longest rod that can be put in a room of dimensions 6m × 4m ×3m.

21.3 RIGHT CIRCULAR CYLINDER

Let us rotate a rectangle ABCD about one of its edges say AB. The solid generated as a result of this rotation is called a **right circular cylinder** (See Fig. 21.6). In daily life, we come across many solids of this shape such as water pipes, tin cans, drumes, powder boxes, etc.

It can be seen that the two ends (or bases) of a right circular cylinder are congruent circles. In Fig. 21.6, A and B are the centres of these two circles of radii AD (= BC). Further, AB is perpendicular to each of these circles.

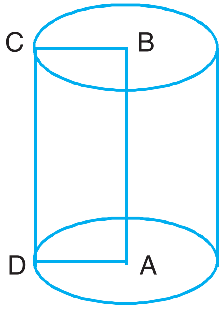


Fig. 21.6

Here, AD (or BC) is called the **base radius** and AB is called the **height** of the cylinder.

It can also be seen that the surface formed by two circular ends are **flat** and the remaining surface is **curved**.

Surface Area

Let us take a hollow cylinder of radius *r* and height *h* and cut it along any line on its curved surface parallel to the line segment joining the centres of the two circular ends (see Fig. 21.7(i)].We obtain a rectangle of length 2π r and breadth *h* as shown in Fig. 21.7 (ii). Clearly, area of this rectangle is equal to the area of the curved surface of the cylinder.

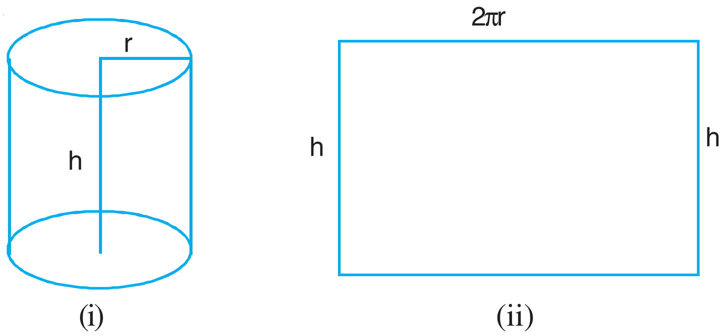


Fig. 21.7

So, curved surface area of the cylinder

$$= \text{area of the rectangle}$$

$$= 2\pi r \times h = 2\pi rh.$$

In case, the cylinder is closed at both the ends, then the total surface area of the cylinder

$$= 2\pi rh + 2\pi r^2$$

$$= 2\pi\ r\,(r + h)$$

Volume

In the case of a cuboid, we have seen that its volume = l × b × h

$$= \text{area of the base} \times \text{height}$$

Extending this rule for a right circular cylinder (assuming it to be the sum of the infininte number of small cuboids), we get : **Volume of a right circular cylinder**

$$= \text{Area of the base} \times \text{height}$$

$$= \pi r^2 \times h$$

$$= \pi r^2 h$$

We now take some examples to illustrate the use of these formulae; (In all the problems in this lesson, we shall take the value of π = 22/7, unless stated otherwise)

Example 21.9: The radius and height of a right circular cylinder are 7cm and 10cm respectively. Find its

(i) curved surface area

(ii) total surface area, and the

(iii) volume

Solution : (i) curved surface area = $2\pi rh$

$$= 2\times\frac{22}{7}\times7\times10\text{ cm}^2 = 440\text{ cm}^2$$

(ii) total surface area $= 2\pi rh + 2\pi r^2$

$$= (2\times\frac{22}{7}\times7\times10+2\times\frac{22}{7}\times7\times7)\text{ cm}^2$$

$$= 440\text{ cm}^2 + 308\text{ cm}^2 = 748\text{ cm}^2$$

(iii) volume $= \pi r^2h$

$$= \frac{22}{7}\times7\times7\times10\text{cm}^3$$

$$= 1540\text{ cm}^3$$

Example 21.10: A hollow cylindrical metallic pipe is open at both the ends and its external diameter is 12 cm. If the length of the pipe is 70 cm and the thickness of the metal used is 1 cm, find the volume of the metal used for making the pipe.

Solution: Here, external radius of the pipe

$$= \frac{12}{2}\text{ cm} = 6\text{cm}$$

Therefore, internal radius = (6–1) = 5 cm (As thickness of metal = 1 cm)

Note that here virtually two cylinders have been formed and the volume of the metal used in making the pipe.

= Volume of the external cylinder – Volume of the internal cylinder

$= \pi r_1{}^2h - \pi r_2{}^2h$ (where r_1 and r_2 are the external and internal radii and h is the length of each cylinder .

$$= \left(\frac{22}{7}\times6\times6\times70-\frac{22}{7}\times5\times5\times70\right)\text{cm}^3$$

$$= 22\times10\times(36-25)\text{cm}^3$$

$$= 2420\text{ cm}^3$$

Example 21.11: Radius of a road roller is 35 cm and it is 1 metre long. If it takes 200 revolutions to level a playground, find the cost of levelling the ground at the rate of ₹ 3 per m^2.

Solution: Area of the playground levelled by the road roller in one revolution

= curved surface area of the roller

$$= 2\pi rh = 2 \times \frac{22}{7} \times 35 \times 100 \text{ cm}^2 \text{ (r = 35 cm, h = 1 m = 100 cm)}$$

$$= 22000 \text{ cm}^2$$

$$= \frac{22000}{100 \times 100} \text{ m}^2$$

(since 100 cm = 1 m, so 100 cm × 100 cm = 1 m × 1 m)

$$= 2.2 \text{ m}^2$$

Therefore, area of the playground levelled in 200 revolutions = $2.2 \times 200 \text{ m}^2 = 440 \text{ m}^2$

Hence, cost of levelling at the rate of ₹ 3 per m^2 = ₹ 3 × 440 = ₹ 1320.

Example 21.12: A metallic solid of volume 1 m^3 is melted and drawn into the form of a wire of diameter 3.5 mm. Find the length of the wire so drawn.

Solution: Let the length of the wire be x mm

You can observe that wire is of the shape of a right circular cylinder.

Its diameter = 3.5 mm

$$\text{So, its radius} = \frac{3.5}{2} \text{mm} = \frac{35}{20} = \frac{7}{4} \text{mm}$$

Also, length of wire will be treated as the height of the cylinder.

So, volume of the cylinder = $\pi r^2 h$

$$= \frac{22}{7} \times \frac{7}{4} \times \frac{7}{4} \times x \text{ mm}^3$$

But the wire has been drawn from the metal of volume 1 m^3

$$\text{Therefore, } \frac{22}{7} \times \frac{7}{4} \times \frac{7}{4} \times \frac{x}{1000000000} = 1 \text{ (since 1 m = 1000 mm)}$$

$$\text{or} \quad x = \frac{1 \times 7 \times 4 \times 4 \times 1000000000}{22 \times 7 \times 7} \text{ mm.}$$

$$= \frac{16000000000}{154}$$

Thus, length of the wire $= \dfrac{16000000000}{154}$ mm

$$= \frac{16000000000}{154000} \text{ m}$$

$$= \frac{16000000}{154} \text{ m} = 103896 \text{ m (approx)}$$

CHECK YOUR PROGRESS 21.2

1. Find the curved surface area, total surface area and volume of a right circular cylinder of radius 5 m and height 1.4 m.
2. Volume of a right circular cylinder is 3080 cm^3 and radius of its base is 7 cm. Find the curved surface area of the cylinder.
3. A cylindrical water tank is of base diameter 7 m and height 2.1 m. Find the capacity of the tank in litres.
4. Length and breadth of a paper is 33 cm and 16 cm respectively. It is folded about its breadth to form a cylinder. Find the volume of the cylinder.
5. A cylindrical bucket of base diameter 28 cm and height 12 cm is full of water. This water is poured in to a rectangular tub of length 66 cm and breadth 28 cm. Find the height to which water will rise in the tub.
6. A hollow metallic cylinder is open at both the ends and is of length 8 cm. If the thickness of the metal is 2 cm and external diameter of the cylinder is 10 cm, find the whole curved surface area of the cylinder (use $\pi = 3.14$).

   [**Hint:** whole curved surface = Internal curved surface + External curved surface]

21.4 RIGHT CIRCULAR CONE

Let us rotate a right triangle ABC right angled at B about one of its side AB containing the right angle. The solid generated as a result of this rotation is called a **right circular cone** (see Fig. 21.8). In daily life, we come across many objects of this shape, such as Joker's cap, tent, ice cream cones, etc.

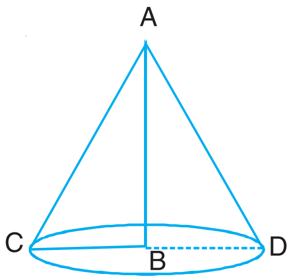


Fig. 21.8

It can be seen that end (or base) of a right circular cone is a circle. In Fig. 21.8, BC is the **radius** of the base with centre B and AB is the **height** of the cone and it is perpendicular to the base. Further, A is called the **vertex** of the cone and AC is called its **slant height**. from the Pythagoras Theorem, we have

$$\text{slant height} = \sqrt{\text{radius}^2 + \text{height}^2}$$

or $l = \sqrt{r^2 + h^2}$, where r, h and l are respectively the base radius, height and slant height of the cone.

You can also observe that surface formed by the base of the cone is **flat** and the remaining surface of the cone is **curved**.

Surface Area

Let us take a hollow right circular cone of radius r and height h and cut it along its slant height. Now spread it on a piece of paper. You obtain a sector of a circle of radius l and its arc length is equal to $2\pi r$ (Fig. 21.9).

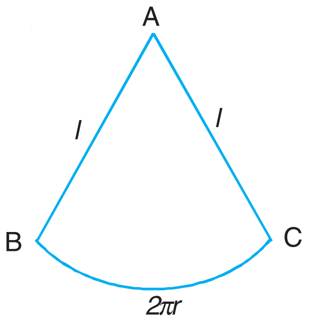


Fig. 21.9

Area of this sector =

$$\frac{\text{Arc length of the sector}}{\text{Circumference of the circle with radius } l} \times \text{Area of circle with radius } l$$

$$= \frac{2\pi r}{2\pi l} \times \pi l^2 = \pi r l$$

Clearly, curved surface of the cone = Area of the sector

$$= \boldsymbol{\pi r l}$$

If the area of the base is added to the above, then it becomes the total surface area.

So, total surface area of the cone = $\pi r l + \pi r^2$

$$= \boldsymbol{\pi r(l + r)}$$

Volume

Take a right circular cylinder and a right circular cone of the same base radius and same height. Now, fill the cone with sand (or water) and pour it in to the cylinder. Repeat the process three times. You will observe that the cylinder is completely filled with the sand (or water). It shows that volume of a cone with radius r and height h is one third the volume of the cylinder with radius r and height h.

So, **volume of a cone** = $\frac{1}{3}$ **volume of the cylinder**

$$= \frac{1}{3}\pi r^2 h$$

Now, let us consider some examples to illustrate the use of these formulae.

Example 21.13: The base radius and height of a right circular cone is 7 cm and 24 cm. Find its curved surface area, total surface area and volume.

Solution: Here, r = 7 cm and h = 24 cm.

So, slant height $l = \sqrt{r^2 + h^2}$

$$= \sqrt{7\times7+24\times24} \text{ cm}$$

$$= \sqrt{49+576} \text{ cm} = 25 \text{ cm}$$

Thus, curved surface area = $\pi r l$

$$= \frac{22}{7} \times 7 \times 25 \text{ cm}^2 = 550 \text{ cm}^2$$

Total surface area = $\pi r l + \pi r^2$

$$= (550 + \frac{22}{7} \times 49) \text{ cm}^2$$

$$= (550 + 154) \text{ cm}^2 = 704 \text{ cm}^2$$

Volume = $\frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times 49 \times 24 \text{ cm}^3$

$$= 1232 \text{ cm}^3$$

Example 21.14: A conical tent is 6 m high and its base radius is 8 m. Find the cost of the canvas required to make the tent at the rate of ₹ 120 per m² (Use $\pi = 3.14$)

Solution: Let the slant height of the tent be x metres.

So, from $l = \sqrt{r^2 + h^2}$ we have,

$$l = \sqrt{36+64} = \sqrt{100}$$

or $l = 10$

Thus, slant height of the tent is 10 m.

So, its curved surface area = $\pi r l$

$$= 3.14 \times 8 \times 10 \text{ cm}^2 = 251.2 \text{ cm}^2$$

Thus, canvas required for making the tent = 251.2 m^2

Therefore, cost of the canvas at ₹ 120 per m^2

= ₹ 120 × 251.2

= ₹ 30144

CHECK YOUR PROGRESS 21.3

1. Find the curved surface area, total surface area and volume of a right circular cone whose base radius and height are respectively 5 cm and 12 cm.
2. Find the volume of a right circular cone of base area 616 cm^2 and height 9 cm.
3. Volume of a right circular cone of height 10.5 cm is 176 cm^3. Find the radius of the cone.
4. Find the length of the 3 m wide canvas required to make a conical tent of base radius 9 m and height 12 m (use $\pi = 3.14$).
5. Find the curved surface area of a right circular cone of volume 12936 cm^3 and base diameter 42 cm.

21.5 SPHERE

Let us rotate a semicircle about its diameter. The solid so generated with this rotation is called a **sphere**. It can also be defined as follows:

The locus of a point which moves in space in such a way that its distance from a fixed point remains the same is called a sphere. The fixed point is called the **centre** of the sphere and the same distance is called the **radius** of the sphere (Fig. 21.10). A football, cricket ball, a marble etc. are examples of spheres that we come across in daily life.

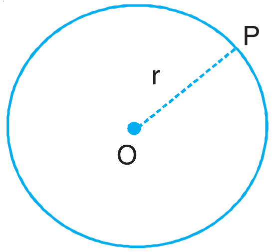


Fig. 21.10

Hemisphere

If a sphere is cut into two equal parts by a plane passing through its centre, then each part is called a hemisphere (Fig. 21.11).

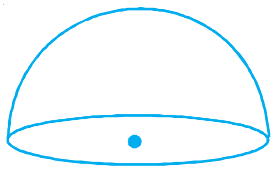
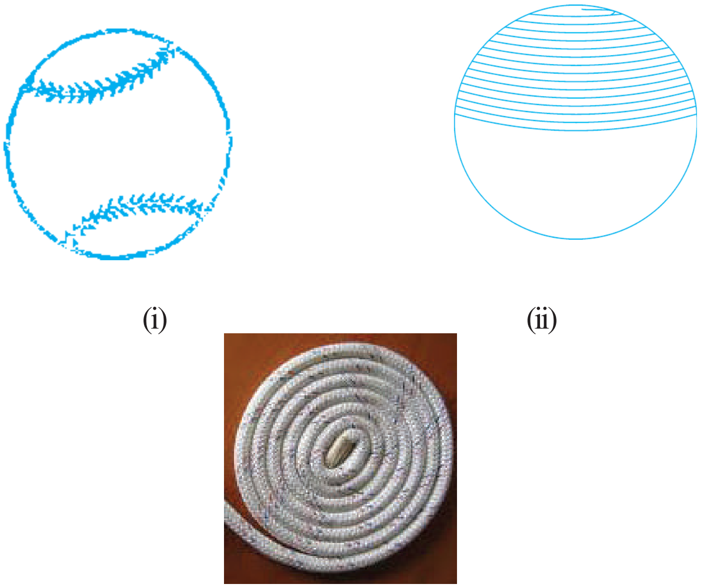


Fig. 21.11

Surface Areas of sphere and hemisphere

Let us take a spherical rubber (or wooden) ball and cut it into equal parts (hemisphere) [See Fig. 21.12(i), Let the radius of the ball be r. Now, put a pin (or a nail) at the top of the ball. starting from this point, wrap a string in a spiral form till the upper hemisphere is completely covered with string as shown in Fig. 21.12(ii). Measure the length of the string used in covering the hemisphere.



(i)

(ii)

(iii)

Fig. 21.12

Now draw a circle of radius r (i.e. the same radius as that of the ball and cover it with a similar string starting from the centre of the circle [See Fig. 21.12 (iii)]. Measure the length of the string used to cover the circle. What do you observe? You will observe that **length of the string used to cover the hemisphere is twice the length of the string used to cover the circle.**

Since the width of the two strings is the same, therefore

surface area of the hemisphere = 2 × area of the circle

$= 2\ \pi r^2$ (Area of the circle is πr^2)

So, surface area of the sphere $= 2 \times 2\pi r^2 = 4\pi r^2$

Thus, we have:

Surface area of a sphere $= 4\pi r^2$

Curved surface area of a solid hemisphere $= 2\pi r^2 + \pi r^2 = 3\pi r^2$

Where r is the radius of the sphere (hemisphere)

Volumes of Sphere and Hemisphere

Take a hollow hemisphere and a hollow right circular cone of the same base radius and same height (say r). Now fill the cone with sand (or water) and pour it into the hemisphere. Repeat the process two times. You will observe that hemisphere is completely filled with the sand (or water). It shows that volume of a hemisphere of radius r is twice the volume of the cone with same base radius and same height.

So, volume of the hemisphere $= 2 \times \frac{1}{3}\pi r^2 h$

$$= \frac{2}{3} \times \pi r^2 \times r \qquad \text{(Because h = r)}$$

$$= \frac{2}{3} \times \pi r^3$$

Therefore, volume of the sphere of radius r

$$= 2 \times \frac{2}{3}\pi r^3 = \frac{4}{3}\pi r^3$$

Thus, we have:

Volume of a sphere $= \frac{4}{3}\pi r^3$

and volume of a hemisphere $= \frac{2}{3}\pi r^3$,

where r is the radius of the sphere (or hemisphere)

Let us illustrate the use of these formulae through some examples:

Example 21.15: Find the surface area and volume of a sphere of diameter 21 cm.

Solution: Radius of the sphere $= \frac{21}{2}$ cm

So, its surface area $= 4\pi r^2$

$$= 4 \times \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2} \text{ cm}^2$$

$$= 1386 \text{ cm}^2$$

Its volume $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2} \times \frac{21}{2} \text{ cm}^3 = 4851 \text{ cm}^3$$

Example 21.16: The volume of a hemispherical bowl is 2425.5 cm^3. Find its radius and surface area.

Solution: Let the radius be r cm.

$$\text{So, } \frac{2}{3}\pi r^3 = 2425.5$$

$$\text{or } \frac{2}{3} \times \frac{22}{7} r^3 = 2425.5$$

$$\text{or } r^3 = \frac{3 \times 2425.5 \times 7}{2 \times 22} = \frac{21 \times 21 \times 21}{8}$$

$$\text{So, } r = \frac{21}{2}, \text{ i.e. radius} = 10.5 \text{ cm.}$$

Now surface area of bowl = curved surface area = $2\pi r^2 = 2 \times \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2}$ cm^2

= 693 cm^2

Note: As the bowl (hemisphere) is open at the top, therefore area of the top, i.e., πr^2 will not be included in its surface area.

CHECK YOUR PROGRESS 21.4

1. Find the surface area and volume of a sphere of radius 14 cm.
2. Volume of a sphere is 38808 cm^3. Find its radius and hence its surface area.
3. Diameter of a hemispherical toy is 56 cm. Find its
   (i) curved surface area
   (ii) total surface area
   (iii) volume
4. A metallic solid ball of radius 28 cm is melted and converted into small solid balls of radius 7 cm each. Find the number of small balls so formed.

LET US SUM UP

- The objects or figures that do not wholly lie in a plane are called solid (or three dimensional) objects or figures.

- The measure of the boundary constituting the solid figure itself is called its surface.
- The measure of the space region enclosed by a solid figure is called its volume.
- some solid figures have only flat surfaces, some have only curved surfaces and some have both flat as well as curved surfaces.
- Surface area of a cuboid = $2(lb + bh + hl)$ and volume of cuboid = lbh, where l, b and h are respectively length, breadth and height of the cuboid.
- Diagonal of the above cuboid is $\sqrt{l^2+b^2+h^2}$
- Cube is a special cuboid whose each edge is of same length.
- Surface area of a cube of edge a is $6a^2$ and its volume is a^3.
- Diagonal of the above cube is a $\sqrt{3}$.
- Area of the four walls of a room of dimensions l, b and $h = 2(l + b)\,h$
- Curved surface area of a right circular cylinder = $2\pi rh$; its total surface area = $2\pi rh+2\pi r^2$ and its volume = πr^2h, where r and h are respectively the base radius and height of the cylinder.
- Curved surface area of a right circular cone is πrl, its total surface area = $\pi rl + \pi r^2$ and its volume = $\frac{1}{3}\pi r^2h$, where r, h and l are respectively the base radius, height and slant height of the cone.
- Surface area of sphere = $4\pi r^2$ and its volume = $\frac{4}{3}\pi r^3$, where r is the radius of the sphere.
- Curved surface area of a hemisphere of radius $r = 2\pi r^2$; its total surface area = $3\pi r^2$ and its volume = $\frac{2}{3}\pi r^3$

TERMINAL EXERCISE

1. Fill in the blanks:

   (i) Surface area of a cuboid of length l, breadth b and height h = __________

   (ii) Diagonal of the cuboid of length l, breadth b and height h = ____________

   (iii) Volume of the cube of side a = ___________

(iv) Surface area of cylinder open at one end = ______, where r and h are the radius and height of the cylinder.

(v) Volume of the cylinder of radius r and height h = __________

(vi) Curved surface area of cone = ____________, where r and l are respectively the ______ and _________ of the cone.

(vii) Surface area of a sphere of radius r = __________

(viii) Volume of a hemisphere of radius r = ___________

2. Choose the correct answer from the given four options:

(i) The edge of a cube whose volume is equal to the volume of a cuboid of dimensions 63 cm × 56 cm × 21 cm is

(A) 21 cm (B) 28 cm (C) 36 cm (D) 42 cm

(ii) If radius of a sphere is doubled, then its volume will become how many times of the original volume?

(A) 2 times (B) 3 times (C) 4 times (D) 8 times

(iii) Volume of a cylinder of the same base radius and the same height as that of a cone is

(A) the same as that of the cone (B) 2 times the volume of the cone

(C) $\frac{1}{3}$ times the volume of the cone (D) 3 times the volume of the cone.

3. If the surface area of a cube is 96 cm^2, then find its volume.

4. Find the surface area and volume of a cuboid of length 3m, breadth 2.5 m and height 1.5 m.

5. Find the surface area and volume of a cube of edge 1.6 cm.

6. Find the length of the diagonal of a cuboid of dimensions 6 cm × 8 cm × 10 cm.

7. Find the length of the diagonal of a cube of edge 8 cm.

8. Areas of the three adjecent faces of cuboid are A, B and C square units respectively and its volume is V cubic units. Prove that $V^2 = ABC$.

9. Find the total surface area of a hollow cylindrical pipe open at the ends if its height is 10 cm, external diameter 10 cm and thickness 12 cm (use $\pi = 3.14$).

10. Find the slant height of a cone whose volume is 12936 cm^3 and radius of the base is 21 cm. Also, find its total surface area.

11. A well of radius 5.6 m and depth 20 m is dug in a rectangular field of dimensions 150 m × 70 m and the earth dug out from it is evenly spread on the remaining part of the field. Find the height by which the field is raised.

12. Find the radius and surface area of a sphere whose volume is 606.375 m^3.

13. In a room of length 12 m, breadth 4 m and height 3 m, there are two windows of dimensions 2m × 1 m and a door of dimensions 2.5 m × 2 m. Find the cost of papering the walls at the rate of ₹ 30 per m^2.

14. A cubic centimetre gold is drawn into a wire of diameter 0.2 mm. Find the length of the wire. (use $\pi = 3.14$).

15. If the radius of a sphere is tripled, find the ratio of the

    (i) Volume of the original sphere to that of the new sphere.

    (ii) surface area of the original sphere to that of the new sphere.

16. A cone, a cylinder and a hemisphere are of the same base and same height. Find the ratio of their volumes.

17. Slant height and radius of the base of a right circular cone are 25 cm and 7 cm respectively. Find its

    (i) curved surface area

    (ii) total surface area, and

    (iii) volume

18. Four cubes each of side 5 cm are joined end to end in a row. Find the surface and the volume of the resulting cuboid.

19. The radii of two cylinders are in the ratio 3 : 2 and their heights are in the ratio 7 : 4. Find the ratio of their

    (i) volumes.

    (ii) curved surface areas.

20. State which of the following statements are true and which are false:

    (i) Surface area of a cube of side a is $6a^2$.

    (ii) Total surface area of a cone is πrl, where r and l are resepctively the base radius and slant height of the cone.

    (iii) If the base radius and height of cone and hemisphere are the same, then volume of the hemisphere is thrice the volume of the cone.

    (iv) Length of the longest rod that can be put in a room of length l, breadth b and height h is $\sqrt{l^2 + b^2 + h^2}$

    (v) Surface area of a hemisphere of radius r is $2\pi r^2$.

ANSWERS TO CHECK YOUR PROGRESS

21.1

1. 81 m^2; 45 m^3
2. 77.76 cm^2; 46.656 cm^3
3. 15 cm, 1350 cm^2
4. 30000 cm^3
5. 384
6. 15 m, 117 m^2
7. 896 cm^2, 1536 cm^3
8. ₹ 460.80
9. $\sqrt{61}$ m

21.2

1. 44 m^2; $201\frac{1}{7}$ m^2; 110 m^3
2. 880 cm^2
3. 80850 litres
4. 1386 cm^3
5. 4 cm
6. 401.92 cm^2

21.3

1. $\frac{1430}{7}$ cm^2; $\frac{1980}{7}$ cm^2; $\frac{2200}{7}$ cm^3
2. 1848 cm^3
3. 2 cm
4. 141.3 m
5. 2310 cm^2

21.4

1. 2464 cm^2; $11498\frac{2}{3}$ cm^3
2. 21 cm, 5544 cm^2
3. (i) 9928 cm^2 (ii) 14892 cm^2 (iii) $92661\frac{1}{3}$ cm^3
4. 64

ANSWERS TO TERMINAL EXERCISE

1. (i) $2(lb + bh + hl)$ (ii) $\sqrt{l^2 + b^2 + h^2}$ (iii) a^3
(iv) $2\pi rh + \pi r^2$ (v) $\pi r^2 h$

(vi) πrl, radius, slant height (vii) $4\pi r^2$ (vii) $\frac{2}{3}\pi r^3$

2. (i) D (ii) D (iii) D

3. 64 cm^3 4. 31.5 m^2; 11.25 m^3 5. 11.76 cm^2; 3.136 cm^3

6. $10\sqrt{2}$ cm 7. $8\sqrt{3}$ cm 8. [**Hint**: A = $l \times h$; B = $b \times h$; and C = $h \times l$]

9. 621.72 cm^2 10. 35 cm, 3696 cm^2 11. 18.95 cm

12. 21 m, 5544 m^2 13. ₹ 2610 14. 31.84 m

15. (i) 1 : 27 (ii) 1 : 9

16. 1: 3: 2

17. (i) 550 cm^2 (ii) 704 cm^2 (iii) 1232 cm^3

18. 350 cm^2; 375 cm^3

19. (i) 63 : 16 (ii) 21 : 8

20. (i) True (ii) False (iii) False

(iv) True (v) False

Secondary Course Mathematics

Practice Work-Mensuration

Maximum Marks: 25 **Time : 45 Minutes**

Instructions:

1. Answer all the questions on a separate sheet of paper.
2. Give the following informations on your answer sheet

   Name

   Enrolment number

   Subject

   Topic of practice work

   Address
3. Get your practice work checked by the subject teacher at your study centre so that you get positive feedback about your performance.

1. The measure of each side of an equilateral triangle whose area is $\sqrt{3}$ cm^2 is 1

   (A) 8 cm

   (B) 4 cm

   (C) 2 cm

   (D) 16 cm

2. The sides of a triangle are in the ratio 3 : 5 : 7. If the perimeter of the triangle is 60 cm, then the area of the triangle is 1

   (A) $60\sqrt{3}$ cm^2

   (B) $30\sqrt{3}$ cm^3

(C) $15\sqrt{3}$ cm^2

(D) $120\sqrt{3}$ cm^2

3. The area of a rhombus is 96 sq cm. If one of its diagonals is 16 cm, then length of its side is 1

(A)5 cm

(B) 6 cm

(C) 8 cm

(D) 10 cm

4. A cuboid having surface areas of three adjacent faces as a, b, c has the volume 1

(A) $\sqrt[3]{abc}$

(B) $\sqrt{abc}$

(C) abc

(D) $a^3b^3c^3$

5. The surface area of a hemispherical bowl of radius 3.5 m is 1

(A) 38.5 m^2

(B) 77 m^2

(C) 115.5 m^2

(D) 154 m^2

6. The parallel sides of a trapezium are 20 metres and 16 metres and the distance between them is 11m. Find its area. 2

7. A path 3 metres wide runs around a circular park whose radius is 9 metres. Find the area of the path. 2

8. The radii of two right circular cylinders are in the ratio 4 : 5 and their heights are in the ratio 5 : 3. Find the ratio of their volumes. 2

9. The circumference of the base of a 9 metre high wooden solid cone is 44 m. Find the volume of the cone. 2

10. Find the surface area and volume of a sphere of diameter 41 cm. 2

11. The radius and height of a right circular cone are in the ratio 5 : 12. If its volume is 314 m^3, find its slant height. (Use $\pi = 3.14$) 4

12. A field is 200 m long and 75 m broad. A tank 40 m long, 20 m broad and 10 m deep is dug in the field and the earth taken out of it, is spread evenly over the field. How much is the level of field raised? 6

MODULE 5

Trigonometry

Imagine a man standing near the base of a hill, looking at the temple on the top of the hill. Before deciding to start climbing the hill, he wants to have an approximation of the distance between him and the temple. We know that problems of this and related problems can be solved only with the help of a science called ***trigonometry.***

The first introduction to this topic was done by ***Hipparcus*** *in* ***140 B.C.,*** *when he hinted at the possibility of finding distances and heights of inaccessible objects. In* ***150 A.D. Tolemy*** *again raised the same possibility and suggested the use of a right triangle for the same. But it was* ***Aryabhatta*** *(476 A.D.) whose introduction to the name "Jaya" lead to the name "sine" of an acute angle of a right triangle. The subject was completed by* ***Bhaskaracharya*** *(1114 A.D.) while writing his work on* ***Goladhayay.*** *In that, he used the words Jaya, Kotijya and "sparshjya" which are presently used for sine, cosine and tangent (of an angle). But it goes to the credit* ***of Neelkanth Somstuvan*** *(1500 A.D.) who developed the science and used terms like elevation, depression and gave examples of some problems on heights and distance.*

In this chapter, we shall define an angle-positive or negative, in terms of rotation of a ray from its initial position to its final position, define trigonometric ratios of an acute angle of a right triangle, in terms of its sides develop some trigonometric identities, trigonometric ratios of complementary angles and solve simple problems on height and distances, using at the most two right triangles, using angles of 30°, 45° and 60°.

22

INTRODUCTION TO TRIGONOMETRY

Study of triangles occupies important place in Mathematics. Triangle being the bounded figure with minimum number of sides serve the purpose of building blocks for study of any figure bounded by straight lines. Right angled triangles get easy link with study of circles as well.

In Geometry, we have studied triangles where most of the results about triangles are given in the form of statements. Here in trigonometry, the approach is quite different, easy and crisp. Most of the results, here, are the form of formulas. In Trigonometry, the main focus is study of right angled triangle. Let us consider some situations, where we can observe the formation of right triangles.

Have you seen a tall coconut tree? On seeing the tree, a question about its height comes to the mind. Can you find out the height of the coconut tree without actually measuring it? If you look up at the top of the tree, a right triangle can be imagined between your eye, the top of the tree, a horizontal line passing through the point of your eye and a vertical line from the top of the tree to the horizontal line.

Let us take another example.

Suppose you are flying a kite. When the kite is in the sky, can you find its height? Again a right triangle can be imagined to form between the kite, your eye, a horizontal line passing through the point of your eye, and a vertical line from the point on the kite to the horizontal line.

Let us consider another situation where a person is standing on the bank of a river and observing a temple on the other bank of the river. Can you find the width of the river if the height of the temple is given? In this case also you can imagine a right triangle.

Finally suppose you are standing on the roof of your house and suddenly you find an aeroplane in the sky. When you look at it, again a right triangle can be imagined. You find the aeroplane moving

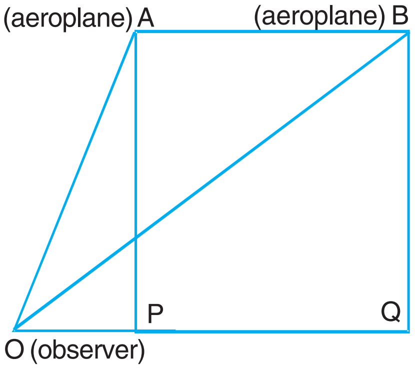


Fig. 22.1

away from you and after a few seconds, if you look at it again, a right triangle can be imagined between your eye, the aeroplane and a horizontal line passing through the point (eye) and a vertical line from the plane to the horizontal line as shown in the figure.

Can you find the distance AB, the aeroplane has moved during this period?

In all the four situations discussed above and in many more such situations, heights or distance can be found (without actually measuring them) by using some mathematical techniques which come under branch of Mathematics called, "Trigonometry".

Trigonometry is a word derived from three Greek words- 'Tri' meaning 'Three' 'Gon' meaning 'Sides' and 'Metron' meaning 'to measure'. Thus Trigonometry literally means measurement of sides and angles of a triangle. Originally it was considered as that branch of mathematics which dealt with the sides and the angles of a triangle. It has its application in astronomy, geography, surveying, engineering, navigation etc. In the past astronomers used it to find out the distance of stars and planets from the earth. Now a day, the advanced technology used in Engineering is based on trigonometrical concepts.

In this lesson, we shall define trigonometric ratios of angles in terms of ratios of sides of a right triangle and establish relationship between different trigonometric ratios. We shall also establish some standard trigonometric identities.

OBJECTIVES

After studying this lesson, you will be able to

- *write the trigonometric ratios of an acute angle of right triangle;*
- *find the sides and angles of a right triangle when some of its sides and trigonometric ratios are known;*
- *write the relationships amongst trigonometric ratios;*
- *establish the trigonometric identities;*
- *solve problems based on trigonometric ratios and identities;*
- *find trigonometric ratios of complementary angles and solve problems based on these.*

EXPECTED BACKGROUND KNOWLEDGE

- Concept of an angle
- Construction of right triangles
- Drawing parallel and perpendiculars lines

- Types of angles- acute, obtuse and right
- Types of triangles- acute, obtuse and right
- Types of triangles- isosceles and equilateral
- Complementary angles.

22.1 TRIGONOMETRIC RATIOS OF AN ACUTE ANGLE OF A RIGHT ANGLED TRIANGLE

Let there be a right triangle ABC, right angled at B. Here ∠A (i.e. ∠CAB) is an acute angle, AC is hypotenuse, side BC is opposite to ∠A and side AB is adjacent to ∠A.

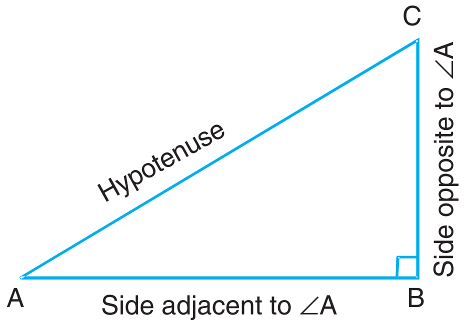


Fig. 22.2

Again, if we consider acute ∠C, then side AB is side opposite to ∠C and side BC is adjacent to ∠C.

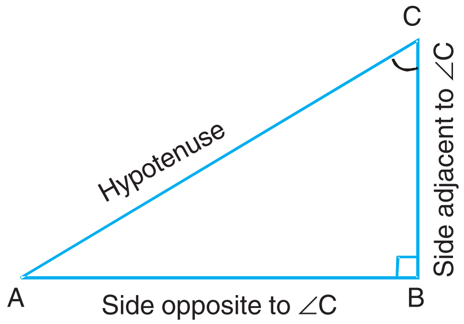


Fig. 22.3

We now define certain ratios involving the sides of a right triangle, called **trigonometric ratios**.

The trigonometric ratios of ∠A in right angled ΔABC are defined as:

(i) $\text{sine } A = \dfrac{\text{side opposite to } \angle A}{\text{Hypotenuse}} = \dfrac{BC}{AC}$

(ii) $\text{cosine } A = \dfrac{\text{side adjacent to } \angle A}{\text{Hypotenuse}} = \dfrac{AB}{AC}$

(iii) $\text{tangent } A = \frac{\text{side opposite to } \angle A}{\text{side adjacent to } \angle A} = \frac{BC}{AB}$

(iv) $\text{cosecant } A = \frac{\text{Hypotenuse}}{\text{side opposite to } \angle A} = \frac{AC}{BC}$

(v) $\text{secant } A = \frac{\text{Hypotenuse}}{\text{side adjacent to } \angle A} = \frac{AC}{AB}$

(vi) $\text{cotangent } A = \frac{\text{side adjacent to } \angle A}{\text{side opposite to } \angle A} = \frac{AB}{BC}$

The above trigonometric ratios are abbreviated as sin A, cos A, tan A, cosec A, sec A and cot A respectively. Trigonometric ratios are abbreviated as **t-ratios**.

If we write $\angle A = \theta$, then the above results are

$$\sin\theta = \frac{BC}{AC}, \qquad \cos\theta = \frac{AB}{AC}, \qquad \tan\theta = \frac{BC}{AB}$$

$$\text{cosec}\,\theta = \frac{AC}{BC}, \qquad \sec\theta = \frac{AC}{AB} \qquad \text{and} \quad \cot\theta = \frac{AB}{BC}$$

Note: Observe here that sin θ and cosec θ are reciprocals of each other. Similarly cot θ and sec θ are respectively reciprocals of tan θ and cos θ.

Remarks

Thus in right ΔABC,

AB = 4cm, BC = 3cm and

AC = 5cm, then

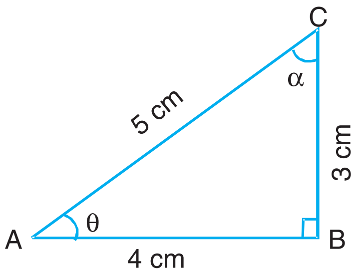


Fig. 22.4

$$\sin\theta = \frac{BC}{AC} = \frac{3}{5}$$

$$\cos\theta = \frac{AB}{AC} = \frac{4}{5}$$

$$\tan\theta = \frac{BC}{AB} = \frac{3}{4}$$

$$\text{cosec}\,\theta = \frac{AC}{BC} = \frac{5}{3}$$

$$\sec\theta = \frac{AC}{AB} = \frac{5}{4}$$

and $$\cot\theta = \frac{AB}{BC} = \frac{4}{3}$$

In the above figure, if we take angle C = α, then

$$\sin\alpha = \frac{\text{side opposite to } \angle\alpha}{\text{Hypotenuse}} = \frac{AB}{AC} = \frac{4}{5}$$

$$\cos\alpha = \frac{\text{side adjacent to } \angle\alpha}{\text{Hypotenuse}} = \frac{BC}{AC} = \frac{3}{5}$$

$$\tan\alpha = \frac{\text{side opposite to } \angle\alpha}{\text{side adjacent to } \angle\alpha} = \frac{AB}{BC} = \frac{4}{3}$$

$$\operatorname{cosec}\alpha = \frac{\text{Hypotenuse}}{\text{side opposite to } \angle\alpha} = \frac{AC}{AB} = \frac{5}{4}$$

$$\sec\alpha = \frac{\text{Hypotenuse}}{\text{side adjacent to } \angle\alpha} = \frac{AC}{BC} = \frac{5}{3}$$

and $$\cot\alpha = \frac{\text{side adjacent to } \angle\alpha}{\text{side opposite to } \angle\alpha} = \frac{BC}{AB} = \frac{3}{4}$$

Remarks :

1. Sin A or sin θ is one symbol and sin cannot be separated from A or θ. It is not equal to sin × θ. The same applies to other trigonometric ratios.
2. Every t-ratio is a real number.
3. For convenience, we use notations $\sin^2\theta$, $\cos^2\theta$, $\tan^2\theta$ for $(\sin\theta)^2$, $(\cos\theta)^2$, and $(\tan\theta)^2$ respectively. We apply the similar notation for higher powers of trigonometric ratios.
4. We have restricted ourselves to t-ratios when A or θ is an acute angle.

Now the question arises: "Does the value of a t-ratio remains the same for the same angle of different right triangles?." To get the answer, let us consider a right triangle ABC, right angled at B. Let P be any point on the hypotenuse AC.

Let $PQ \perp AB$

Now in right ΔABC,

$$\sin A = \frac{BC}{AC} \qquad \text{----(i)}$$

and in right ΔAQP,

$$\sin A = \frac{PQ}{AP} \qquad \text{----(ii)}$$

Now in ΔAQP and ΔABC,

$$\angle Q = \angle B \qquad \text{----(Each = 90°)}$$

and $$\angle A = \angle A \qquad \text{----(Common)}$$

$$\therefore \quad \Delta AQP \sim \Delta ABC$$

$$\therefore \frac{AP}{AC} = \frac{QP}{BC} = \frac{AQ}{AB}$$

or $$\frac{BC}{AC} = \frac{PQ}{AP} \qquad \text{----(iii)}$$

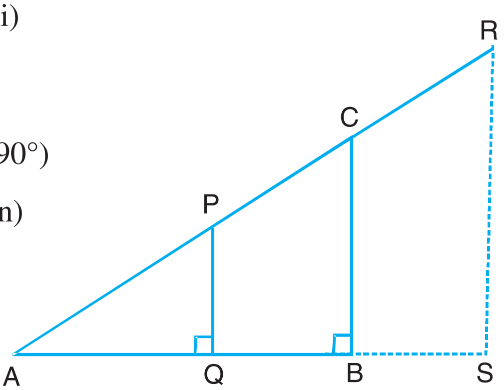


Fig. 22.5

From (i), (ii), and (iii), we find that sin A has the same value in both the triangles.

Similarly, we have $\cos A = \frac{AB}{AC} = \frac{AQ}{AP}$ and $\tan A = \frac{BC}{AB} = \frac{PQ}{AQ}$

Let R be any point on AC produced. Draw RS ⊥ AB produced meeing it at S. You can verify that value of t-ratios remains the same in ΔASR also.

Thus, we conclude that the value of trigonometric ratios of an angle does not depend on the size of right triangle. They only depend on the angle.

Example 22.1: In Fig. 22.6, ΔABC is right angled at B. If AB = 5 cm, BC = 12 cm and AC = 13 cm, find the value of tan C, cosec C and sec C.

Solution: We know that

$$\tan C = \frac{\text{side opposite to } \angle C}{\text{side adjacent to } \angle C} = \frac{AB}{BC} = \frac{5}{12}$$

$$\operatorname{cosec} C = \frac{\text{Hypotenuse}}{\text{side opposite to } \angle C} = \frac{AC}{AB} = \frac{13}{5}$$

and $$\sec C = \frac{\text{Hypotenuse}}{\text{side adjacent to } \angle C} = \frac{AC}{BC} = \frac{13}{12}$$

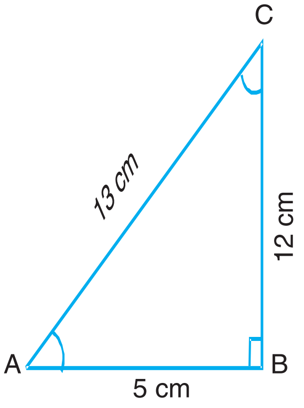


Fig. 22.6

Example 22.2 : Find the value of sin θ, cot θ and Sec θ from Fig. 22.7.

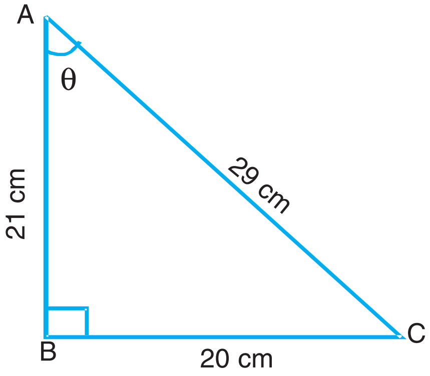


Fig. 22.7

Solution:

$$\sin\theta = \frac{\text{side opposite to } \angle\theta}{\text{Hypotenuse}} = \frac{BC}{AC} = \frac{20}{29}$$

$$\cot\theta = \frac{\text{side adjacent to } \angle\theta}{\text{side opposite to } \angle\theta} = \frac{AB}{BC} = \frac{21}{20}$$

and $$\sec\theta = \frac{\text{Hypotenuse}}{\text{side adjacent to } \angle\theta} = \frac{AC}{AB} = \frac{29}{21}$$

Example 22.3 : In Fig. 22.8, ΔABC is right-angled at B. If AB = 9cm, BC = 40cm and AC = 41cm, find the values cos C, cot C, tan A, and cosec A.

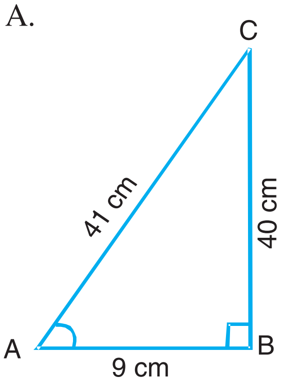


Fig. 22.8

Solution:

Now $$\cos C = \frac{\text{side adjacent to } \angle C}{\text{Hypotenuse}} = \frac{BC}{AC} = \frac{40}{41}$$

and $$\cot C = \frac{\text{side adjacent to } \angle C}{\text{side opposite to } \angle C} = \frac{BC}{AB} = \frac{40}{9}$$

With reference to ∠A, side adjacent to A is AB and side opposite to A is BC.

$$\therefore \quad \tan A = \frac{\text{side opposite to } \angle A}{\text{side adjacent to } \angle A} = \frac{BC}{AB} = \frac{40}{9}$$

and $$\operatorname{cosec} A = \frac{\text{Hypotenuse}}{\text{side opposite to } \angle A} = \frac{AC}{BC} = \frac{41}{40}$$

Example 22.4 : In Fig. 22.9, ΔABC is right angled at B, $\angle A = \angle C$, AC = $\sqrt{2}$ cm and AB = 1 cm. Find the values of sin C, cos C and tan C.

Solution: In ΔABC, $\angle A = \angle C$

$\therefore$ BC = AB = 1 cm (Given)

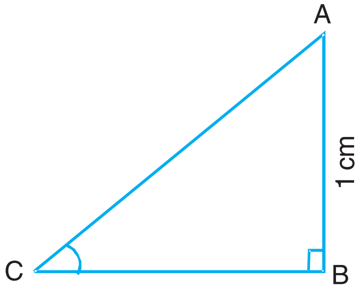


Fig. 22.9

$$\therefore \quad \sin C = \frac{\text{side opposite to } \angle C}{\text{Hypotenuse}} = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$$\cos C = \frac{\text{side adjacent to } \angle C}{\text{Hypotenuse}} = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$$

and $$\tan C = \frac{\text{side opposite to } \angle C}{\text{side adjacent to } \angle C} = \frac{AB}{BC} = \frac{1}{1} = 1$$

Remark: In the above example, we have $\angle A = \angle C$ and $\angle B = 90°$

$\therefore \ \angle A = \angle C = 45°$,

$\therefore$ We have $\sin 45° = \cos 45° = \dfrac{1}{\sqrt{2}}$

and $\tan 45^{\circ} = 1$

Example 22.5 : In Fig. 22.10. ΔABC is right-angled at C. If AB = c, AC = b and BC = a, which of the following is true?

(i) $\tan A = \dfrac{b}{c}$

(ii) $\tan A = \dfrac{c}{b}$

(iii) $\cot A = \dfrac{b}{a}$

(iv) $\cot A = \dfrac{a}{b}$

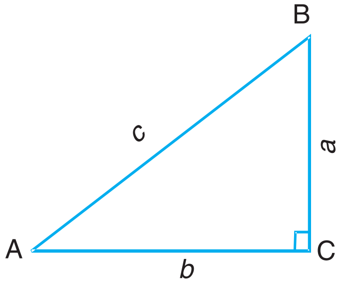


Fig. 22.10

Solution: Here $\tan A = \dfrac{\text{side opposite to } \angle A}{\text{side adjacent to } \angle A} = \dfrac{BC}{AC} = \dfrac{a}{b}$

and $\cot A = \frac{\text{side adjacent to } \angle A}{\text{side opposite to } \angle A} = \frac{b}{a}$

Hence the result (iii) i.e. $\cot A = \frac{b}{a}$ is true.

CHECK YOUR PROGRESS 22.1

1. In each of the following figures, ΔABC is a right triangle, right angled at B. Find all the trigonometric ratios of θ.

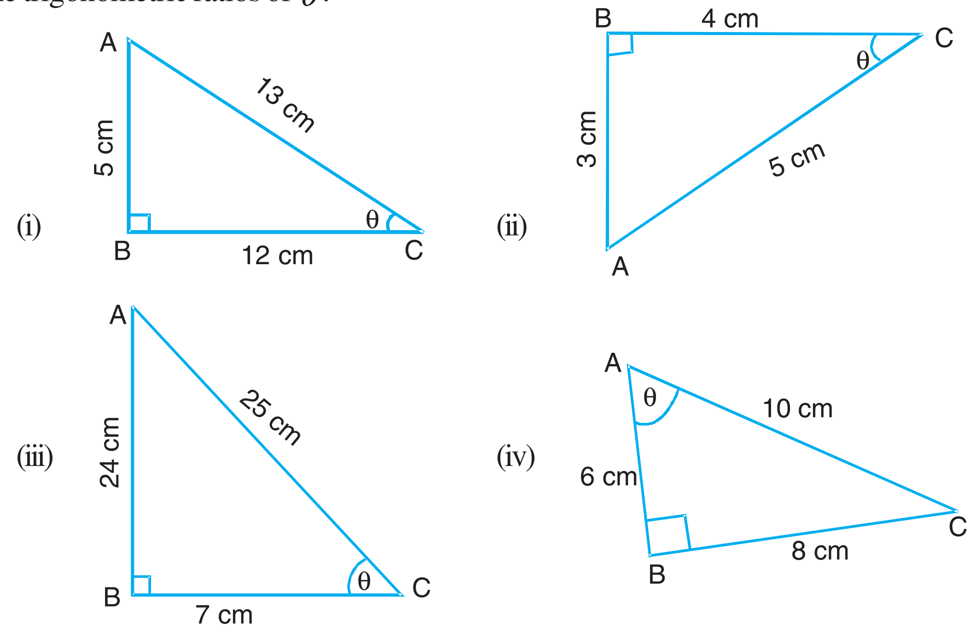


Fig. 22.11

2. In ΔABC, $\angle B = 90°$, BC = 5cm, AB = 4cm, and AC = $\sqrt{41}$ cm, find the value of sin A, cos A, and tan A.
3. In ΔABC right angled at B, if AB = 40 cm, BC = 9 cm and AC = 41 cm, find the values of sin C, cot C, cos, A and cot A.
4. In ΔABC, $\angle B = 90°$. If AB = BC = 2cm and AC = $2\sqrt{2}$ cm, find the value of sec C, cosec C, and cot C.
5. In Fig. 22.12, ΔABC is right angled at A. Which of the following is true?

(i) $\cot C = \frac{13}{12}$ (ii) $\cot C = \frac{12}{13}$

(iii) $\cot C = \frac{5}{12}$ (iv) $\cot C = \frac{12}{5}$

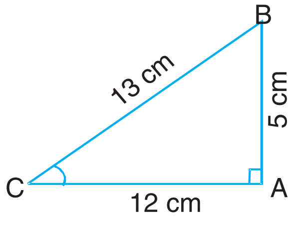


Fig. 22.12

6. In Fig. 22.13, AC = b, BC = a and AB = c. Which of the following is true?

(i) $\operatorname{cosec} A = \frac{a}{b}$ (ii) $\operatorname{cosec} A = \frac{c}{a}$

(iii) $\operatorname{cosec} A = \frac{c}{b}$ (iv) $\operatorname{cosec} A = \frac{b}{a}$.

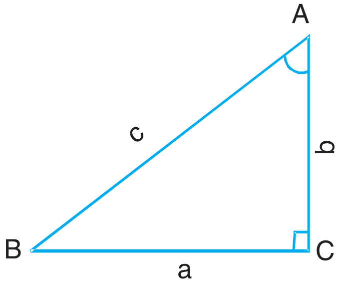


Fig. 22.13

22.2 GIVEN TWO SIDES OF A RIGHT-TRIANGLE, TO FIND TRIGONOMETRIC RATIO

When two sides of a right-triangle are given, its third side can be found out by using the Pythagoras theorem. Then we can find the trigonometric ratios of the given angle as learnt in the last section.

We take some examples to illustrate.

Example 22.6: In Fig. 22.14, ΔPQR is a right triangle, right angled at Q. If PQ = 5 cm and QR = 12 cm, find the values of sin R, cos R and tan R.

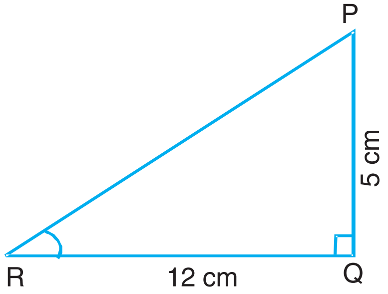


Fig. 22.14

Solution: We shall find the third side by using Pythagoras Theorem.

$$\because \Delta PQR \text{ is a right angled triangle at Q.}$$

$$\therefore \ PR = \sqrt{PQ^2 + QR^2} \qquad \text{(Pythagoras Theorem)}$$

$$= \sqrt{5^2 + 12^2} \text{ cm}$$

$$= \sqrt{25 + 144} \text{ cm}$$

$$= \sqrt{169} \text{ or } 13 \text{ cm}$$

We now use definition to evaluate trigonometric ratios:

$$\sin R = \frac{\text{side opposite to } \angle R}{\text{Hypotenuse}} = \frac{PQ}{PR} = \frac{5}{13}$$

$$\cos R = \frac{\text{side adjacent to } \angle R}{\text{Hypotenuse}} = \frac{QR}{PR} = \frac{12}{13}$$

and $$\tan R = \frac{\text{side opposite to } \angle R}{\text{side adjacent to } \angle R} = \frac{5}{12}$$

From the above example, we have the following:

Steps to find Trigonometric ratios when two sides of a right triangle are given.

Step1: Use Pythagoras Theorem to find the unknown (third) side of the triangle.

Step 2: Use definition of t-ratios and substitute the values of the sides.

Example 22.7 : In Fig. 22.15, ΔPQR is right-angled at Q, PR = 25cm, PQ = 7cm and $\angle PRQ = \theta$. Find the value of $\tan\theta$, $\text{cosec}\,\theta$ and $\sec\theta$.

Solution :

$\because$ ΔPQR is right-angled at Q

$$\therefore\ QR = \sqrt{PR^2 - PQ^2}$$

$$= \sqrt{25^2 - 7^2}\text{ cm}$$

$$= \sqrt{625 - 49}\text{ cm}$$

$$= \sqrt{576}\text{ cm}$$

$$= 24\text{ cm}$$

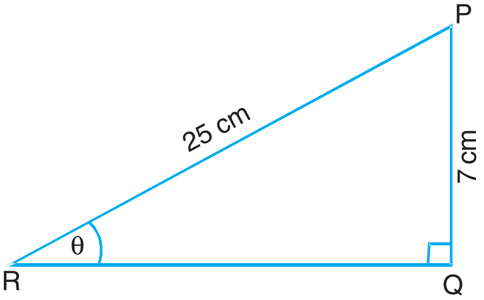


Fig. 22.15

$$\therefore \qquad \tan\theta = \frac{PQ}{QR} = \frac{7}{24}$$

$$\text{cosec}\,\theta = \frac{PR}{PQ} = \frac{25}{7}$$

$$\text{and} \qquad \sec\theta = \frac{PR}{QR} = \frac{25}{24}$$

Example 22.8 : In ΔABC, $\angle B = 90°$. If AB = 4 cm and BC = 3 cm, find the values of sin C, cos C, cot C, tan A, sec A and cosec A. Comment on the values of tan A and cot C. Also find the value of tan A – cot C.

Solution: By Pythagoras Theorem, in ΔABC,

$$AC = \sqrt{AB^2 + BC^2}$$

$$= \sqrt{4^2 + 3^2}\text{ cm}$$

$$= \sqrt{25}\text{ cm}$$

$$= 5\text{ cm}$$

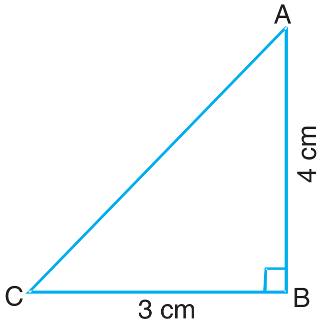


Fig. 22.16

$$\text{Now} \qquad \sin C = \frac{AB}{AC} = \frac{4}{5}$$

$$\cos C = \frac{BC}{AC} = \frac{3}{5}$$

$$\cot C = \frac{BC}{AB} = \frac{3}{4}$$

$$\tan A = \frac{BC}{AB} = \frac{3}{4}$$

$$\sec A = \frac{AC}{AB} = \frac{5}{4}$$

and $\qquad \operatorname{cosec} A = \frac{AC}{BC} = \frac{5}{3}$

The value of tan A and cot C are equal

$\therefore \qquad \tan A - \cot C = 0.$

Example 22.9: In Fig. 22.17, PQR is right triangle at R. If PQ = 13cm and QR = 5cm, which of the following is true?

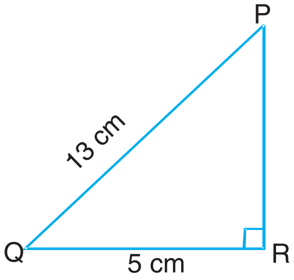


Fig. 22.17

(i) $\sin Q + \cos Q = \frac{17}{13}$ (ii) $\sin Q - \cos Q = \frac{17}{13}$

(iii) $\sin Q + \sec Q = \frac{17}{13}$ (iv) $\tan Q + \cot Q = \frac{17}{13}$

Solution: Here $PR = \sqrt{PQ^2 - QR^2} = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$ cm

$\therefore \qquad \sin Q = \frac{PR}{PQ} = \frac{12}{13}$ and $\cos Q = \frac{QR}{PQ} = \frac{5}{13}$

$\therefore \qquad \sin Q + \cos Q = \frac{12}{13} + \frac{5}{13} = \frac{17}{13}$

Hence statement (i) i.e. $\sin Q + \cos Q = \frac{17}{13}$ is true.

CHECK YOUR PROGRESS 22.2

1. In right ΔABC, right angled at B, AC = 10 cm, and AB = 6 cm. Find the values of sin C, cos C, and tan C.

2. In ΔABC, $\angle C = 90°$, BC = 24 cm and AC = 7 cm. Find the values of sin A, cosec A and cot A.

3. In ΔPQR, $\angle Q = 90°$, PR = $10\sqrt{2}$ cm and QR = 10cm. Find the values of sec P, cot P and cosec P.

4. In ΔPQR, $\angle Q = 90°$, PQ = $\sqrt{3}$ cm and QR = 1 cm. Find the values of tan R, cosec R, sin P and sec P.

5. In ΔABC, $\angle B = 90°$, AC = 25 cm, AB = 7 cm and $\angle ACB = \theta$. Find the values of cot θ, sin θ, sec θ and tan θ.

6. In right ΔPQR, right-angled at Q, PQ = 5 cm and PR = 7 cm. Find the values of sin P, cos P, sin R and cos R. Find the value of sin P – cos R.

7. ΔDEF is a right triangle at E in Fig. 22.18. If DE = 5 cm and EF = 12 cm, which of the following is true?

   (i) $\sin F = \frac{5}{12}$

   (ii) $\sin F = \frac{12}{5}$

   (iii) $\sin F = \frac{5}{13}$

   (iv) $\sin F = \frac{12}{13}$

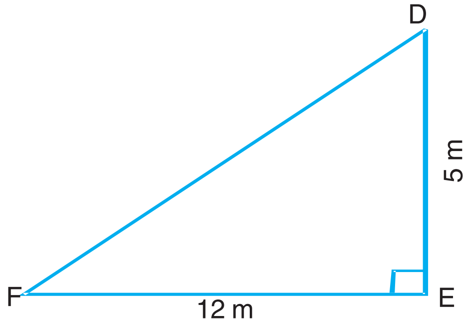


Fig. 22.18

22.3 GIVEN ONE TRIGONOMETRIC RATIO, TO FIND THE OTHERS

Sometimes we know one trigonometric ratio and we have to find the vaues of other t-ratios. This can be easily done by using the definition of t-ratios and the Pythagoras Theorem. Let us take $\sin\theta = \frac{12}{13}$. We now find the other t-ratios.

We draw a right-triangle ABC

Now $\sin\theta = \frac{12}{13}$ implies that sides AB and AC are in the ratio 12 : 13.

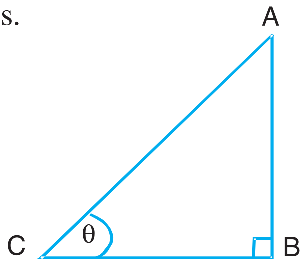


Fig. 22.19

Thus we suppose AB = 12 k and AC = 13 k.

$\therefore$ By Pythagoras Theorem,

$$\begin{aligned}BC &= \sqrt{AC^2 - AB^2}\\ &= \sqrt{(13k)^2 - (12k)^2}\\ &= \sqrt{169k^2 - 144k^2}\\ &= \sqrt{25k^2} = 5\text{ k}\end{aligned}$$

Now we can find all othe t-ratios.

$$\cos\theta = \frac{BC}{AC} = \frac{5k}{13k} = \frac{5}{13}$$

$$\tan\theta = \frac{AB}{BC} = \frac{12k}{5k} = \frac{12}{5}$$

$$\operatorname{cosec}\theta = \frac{AC}{AB} = \frac{13k}{12k} = \frac{13}{12}$$

$$\sec\theta = \frac{AC}{BC} = \frac{13k}{5k} = \frac{13}{5}$$

and $$\cot\theta = \frac{BC}{AB} = \frac{5k}{12k} = \frac{5}{12}$$

The method discussed above gives the following steps for the solution.

Steps to be followed for finding the t-ratios when one t-ratio is given.

1. Draw a right triangle ΔABC.
2. Write the given t-ratio in terms of the sides and let the constant of ratio be *k*.
3. Find the two sides in terms of *k*.
4. Use Pythagoras Theorem and find the third side.
5. Now find the remaining t-ratios by using the definition.

Let us consider some examples.

Example 22.10.: If $\cos\theta = \frac{7}{25}$, find the values of $\sin\theta$ and $\tan\theta$.

Solution : Draw a right-angled ΔABC in which $\angle B = 90°$ and $\angle C = \theta$.

We know that

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{BC}{AC} = \frac{7}{25}$$

Let BC = 7 k and AC = 25 k

Then by Pythagoras Theorem,

$$AB = \sqrt{AC^2 - BC^2}$$

$$= \sqrt{(25k)^2 - (7k)^2}$$

$$= \sqrt{625k^2 - 49k^2}$$

$$= \sqrt{576k^2} \text{ or } 24\text{ k}$$

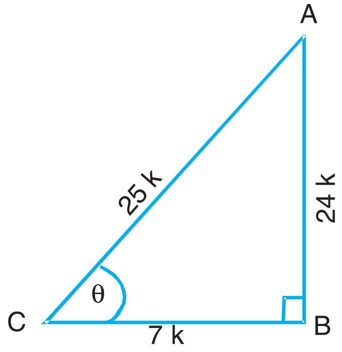


Fig. 22.20

$\therefore$ In ΔABC,

$$\sin\theta = \frac{AB}{AC} = \frac{24k}{25k} = \frac{24}{25}$$

and $$\tan\theta = \frac{AB}{BC} = \frac{24k}{7k} = \frac{24}{7}$$

Example 22.11.: If $\cot\theta = \frac{40}{9}$, find the value of $\frac{\cos\theta.\sin\theta}{\sec\theta}$.

Solution. Let ABC be a right triangle, in which $\angle B = 90^\circ$ and $\angle C = \theta$.

We know that

$$\cot\theta = \frac{BC}{AB} = \frac{40}{9}$$

Let BC $= 40k$ and AB $= 9k$

Then from right ΔABC,

$$AC = \sqrt{BC^2 + AB^2}$$

$$= \sqrt{(40k)^2 + (9k)^2}$$

$$= \sqrt{1600k^2 + 81k^2}$$

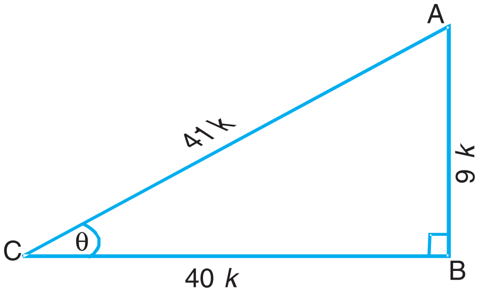


Fig. 22.21

$$= \sqrt{1681k^2} \text{ or } 41\text{ k}$$

Now $\sin\theta = \frac{AB}{AC} = \frac{9k}{41k} = \frac{9}{41}$

$$\cos\theta = \frac{BC}{AC} = \frac{40k}{41k} = \frac{40}{41}$$

and $\sec\theta = \frac{AC}{BC} = \frac{41k}{40k} = \frac{41}{40}$

$$\therefore \quad \frac{\cos\theta.\sin\theta}{\sec\theta} = \frac{\frac{9}{41}\times\frac{40}{41}}{\frac{41}{40}}$$

$$= \frac{9}{41}\times\frac{40}{41}\times\frac{40}{41}$$

$$= \frac{14400}{68921}$$

Example 22.12.: In PQR, $\angle Q = 90°$ and $\tan R = \frac{1}{\sqrt{3}}$. Then show that

$$\sin P \cos R + \cos P \sin R = 1$$

Solution: Let there be a right-triangle PQR, in which $\angle Q = 90°$ and $\tan R = \frac{1}{\sqrt{3}}$.

We know that

$$\tan R = \frac{PQ}{QR} = \frac{1}{\sqrt{3}}$$

Let $PQ = k$ and $QR = \sqrt{3}\,k$

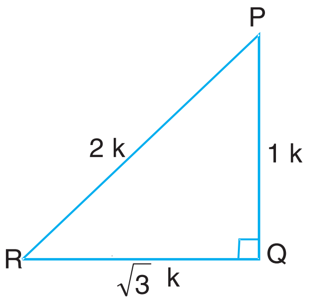


Fig. 22.22

Then, $PR = \sqrt{PQ^2 + QR^2}$

$$= \sqrt{k^2 + \left(\sqrt{3}k\right)^2}$$

$$= \sqrt{k^2 + 3k^2}$$

$$= \sqrt{4k^2} \text{ or } 2\text{ k}$$

$$\therefore \quad \sin P = \frac{\text{side opposite to } \angle P}{\text{Hypotenuse}} = \frac{QR}{PR} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2}$$

$$\cos P = \frac{\text{side adjacent to } \angle P}{\text{Hypotenuse}} = \frac{1k}{2k} = \frac{1}{2}$$

$$\sin R = \frac{\text{side opposite to } \angle R}{\text{Hypotenuse}} = \frac{PQ}{PR} = \frac{1k}{2k} = \frac{1}{2}$$

$$\text{and } \cos R = \frac{\text{side adjacent to } \angle R}{\text{Hypotenuse}} = \frac{QR}{PR} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2}$$

$$\therefore \qquad \sin P \cos R + \cos P \sin R = \frac{\sqrt{3}}{2} . \frac{\sqrt{3}}{2} + \frac{1}{2} . \frac{1}{2}$$

$$= \frac{3}{4} + \frac{1}{4} = \frac{4}{4}$$

$$= 1$$

Example 22.13.: In ΔABC, $\angle B$ is right-angle. If AB = c, BC = a and AC = b, which of the following is true?

(i) $\cos C + \sin A = \frac{2b}{a}$

(ii) $\cos C + \sin A = \frac{b}{a} + \frac{a}{b}$

(iii) $\cos C + \sin A = \frac{2a}{b}$

(iv) $\cos C + \sin A = \frac{a}{b} + \frac{c}{b}$

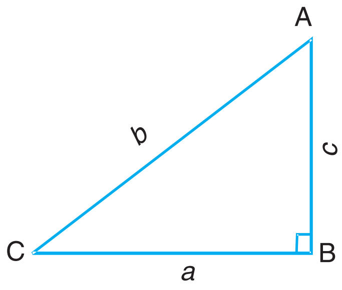


Fig. 22.23

Solution: Here $\cos C = \frac{BC}{AC} = \frac{a}{b}$

and $\quad \sin A = \frac{BC}{AC} = \frac{a}{b}$

$\therefore \quad \cos C + \sin A = \frac{a}{b} + \frac{a}{b} = \frac{2a}{b}$

$\therefore$ Statement (iii), i.e., $\cos C + \sin A = \frac{2a}{b}$ is true.

CHECK YOUR PROGRESS 22.3

1. If $\sin\theta = \frac{20}{29}$, find the values of $\cos\theta$ and $\tan\theta$.

2. If $\tan\theta = \frac{24}{7}$, find the values of $\sin\theta$ and $\cos\theta$.

3. If $\cos A = \frac{7}{25}$, find the values of sin A and tan A.

4. If $\cos\theta = \frac{m}{n}$, find the values of $\cot\theta$ and $\operatorname{cosec}\theta$.

5. If $\cos\theta = \frac{4}{5}$, evaluate $\frac{\cos\theta \,.\, \cot\theta}{1-\sec^2\theta}$.

6. If $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$, find the value of $\sin^2\theta\cos\theta + \tan^2\theta$.

7. If $\cot B = \frac{5}{4}$, then show that $\operatorname{cosec}^2 B = 1 + \cot^2 B$.

8. ΔABC is a right triangle with $\angle C = 90^o$. If $\tan A = \frac{3}{2}$, find the values of sin B and tan B.

9. If $\tan A = \frac{1}{\sqrt{3}}$ and $\tan B = \sqrt{3}$, then show that $\cos A \cos B - \sin A \sin B = 0$.

10. If $\cot A = \frac{12}{5}$, show that $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A$.

[Hint: Find the vlaues of tan A, sin A and sec A and substitute]

11. In Fig. 22.24, ΔABC is right-angled at vertex B. If AB = *c*, BC = *a* and CA = *b*, which of the following is true?

(i) $\sin A + \cos A = \frac{b+c}{a}$

(ii) $\sin A + \cos A = \frac{a+c}{b}$

(iii) $\sin A + \cos A = \frac{a+b}{c}$

(iv) $\sin A + \cos A = \frac{a+b+c}{b}$

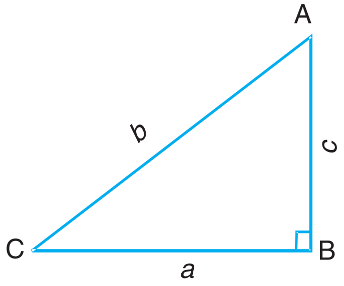


Fig. 22.24

22.4 RELATIONSHIPS BETWEEN TRIGONOMETRIC RATIOS

In a right triangle ABC, right angled at B, we have

$$\sin \theta = \frac{AB}{AC}$$

$$\cos \theta = \frac{BC}{AC}$$

and $$\tan \theta = \frac{AB}{BC}$$

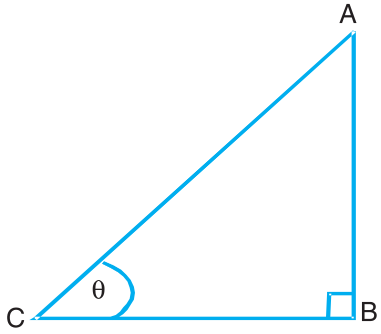


Fig. 22.25

Rewriting, $\tan \theta = \frac{AB}{BC} = \frac{AB}{AC} \div \frac{BC}{AC}$

$$= \frac{\frac{AB}{AC}}{\frac{BC}{AC}} = \frac{\sin \theta}{\cos \theta}$$

Thus, we see that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

We can verify this result by taking AB = 3 cm, BC = 4 cm and therefore AC = $\sqrt{AB^2 + BC^2} = \sqrt{3^2+5^2}$ or 5 cm

$\therefore\ \sin\theta = \dfrac{3}{5}, \cos\theta = \dfrac{4}{5}$ and $\tan\theta = \dfrac{3}{4}$

Now $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{3}{5}}{\frac{4}{5}} = \dfrac{3}{4} = \tan\theta.$

Thus, the result is verified.

Again $\sin\theta = \dfrac{AB}{AC}$ gives us

$$\frac{1}{\sin\theta} = \frac{1}{\frac{AB}{AC}} = \frac{AC}{AB} = \operatorname{cosec}\theta$$

Thus $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ or $\operatorname{cosec}\theta \,.\, \sin\theta = 1$

We say $\operatorname{cosec}\theta$ is the reciprocal of $\sin\theta$.

Again, $\cos\theta = \dfrac{BC}{AC}$ gives us

$$\frac{1}{\cos\theta} = \frac{1}{\frac{BC}{AC}} = \frac{AC}{BC} = \sec\theta$$

Thus $\sec\theta = \dfrac{1}{\cos\theta}$ or $\sec\theta \,.\, \cos\theta = 1$

We say that $\sec\theta$ is reciprocal of $\cos\theta$.

Finally, $\tan\theta = \dfrac{AB}{BC}$ gives us

$$\frac{1}{\tan\theta} = \frac{1}{\frac{AB}{BC}} = \frac{BC}{AB} = \cot\theta$$

Thus, $\cot\theta = \dfrac{1}{\tan\theta}$ or $\tan\theta \,.\, \cot\theta = 1$

Also $\cot\theta = \dfrac{1}{\sin\theta/\cos\theta} = \dfrac{\cos\theta}{\sin\theta}$

We say that cot θ is reciprocal of tan θ.

Thus, we have cosec θ, sec θ and cot θ are reciprocal of sin θ, cos θ and tan θ respectively.

We have, therefore, established the following results:

(i) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(ii) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

(iii) $\sec\theta = \dfrac{1}{\cos\theta}$

(iv) $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

Now we can make use of the above results in finding the values of different trigonometric ratios.

Example 22.14: If $\cos\theta = \dfrac{1}{2}$ and $\sin\theta = \dfrac{\sqrt{3}}{2}$, find the values of cosec θ, sec θ and tan θ.

Solution: We know that

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{1}{2}} = 2$$

and $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}$$

Example 22.15: For a right angled triangle ABC, right angled at C, $\tan A = 1$. Find the value of $\cos B$.

Solution: Let us construct a right angled ΔABC in which $\angle C = 90^o$.

We have $\tan A = 1$ (Given)

We know that

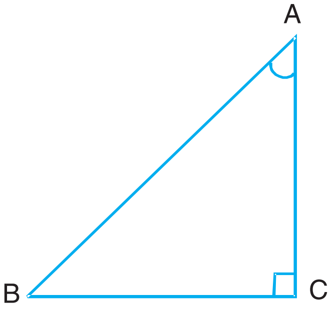


Fig. 22.26

$$\tan A = \frac{BC}{AC} = 1$$

$\therefore$ BC and AC are equal.

Let $BC = AC = k$

Then $AB = \sqrt{BC^2 + AC^2}$

$$= \sqrt{k^2 + k^2}$$

$$= \sqrt{2}k$$

Now $\cos B = \frac{BC}{AB} = \frac{k}{\sqrt{2}k}$

$$= \frac{1}{\sqrt{2}}$$

Hence $\cos B = \frac{1}{\sqrt{2}}$

CHECK YOUR PROGRESS 22.4

1. If $\sin\theta = \frac{1}{2}$ and $\cos\theta = \frac{\sqrt{3}}{2}$, find the values of $\cot\theta$ and $\sec\theta$.

2. If $\sin\theta = \frac{\sqrt{3}}{2}$ and $\tan\theta = \sqrt{3}$, find the value of $\cos^2\theta + \sin\theta\cot\theta$.

3. In a right angled ΔABC, right angled at C, cos A = $\frac{\sqrt{3}}{2}$. Find the value of sin A sin B + cos A cos B.

4. If cosec A = 2, find the value of sin A and tan A.

5. In a right angled ΔABC, right angled at B, tan A = $\sqrt{3}$, find the value of $\tan^2 B \sec^2 A - (\tan^2 A + \cot^2 B)$

22.5 IDENTITY

We have studied about equations in algebra in our earlier classes. Recall that when two expressions are connected by '=' (equal to) sign, we get an equation. In this section, we now introduce the concept of an identity. We get an identity when two expressions are connected by the equality sign. When we say that two expressions when connected by '=' give rise to an equation as well as identity, then what is the difference between the two.

The major difference between the two is that an equation involving a variable is true for some values only whereas the equation involving a variable is true for all values of the variable, is called an identity.

Thus $x^2 - 2x + 1 = 0$ is an equation as it is true for $x = 1$.

$x^2 - 5x + 6 = 0$ is an equation as it is true for $x = 2$ and $x = 3$.

If we consider $x^2 - 5x + 6 = (x - 2)(x - 3)$, it becomes an identity as it is true for $x = 2$, $x = 3$ and say $x = 0$, $x = 10$ etc. i.e. it is true for all values of x. In the next section, we shall consider some identities in trigonometry.

22.6 TRIGONOMETRIC IDENTITIES

We know that an angle is defined with the help of the rotation of a ray from initial to final position. You have learnt to define all trigonometric ratios of an angle. Let us recall them here.

Let XOX′ and YOY′ be the rectangular axes. Let A be any point on OX. Let the ray OA start rotating in the plane in an anti-clockwise direction about the point O till it reaches the final position OA′ after some interval of time. Let ∠A′OA = θ. Take any point P on the ray OA′. Draw PM ⊥ OX.

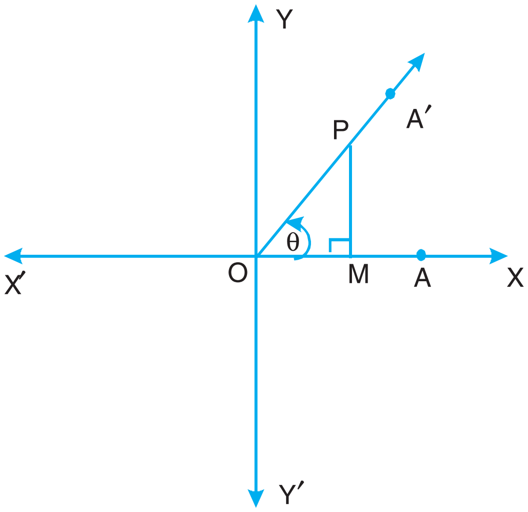


Fig. 22.27

In right angled ΔPMO,

$$\sin\theta = \frac{PM}{OP}$$

and $$\cos\theta = \frac{OM}{OP}$$

Squaring and adding, we get

$$\sin^2\theta + \cos^2\theta = \left(\frac{PM}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2$$

$$= \frac{PM^2 + OM^2}{OP^2} = \frac{OP^2}{OP^2}$$

$$= 1$$

Hence, $\sin^2\theta + \cos^2\theta = 1$ ...(1)

Also we know that

$$\sec\theta = \frac{OP}{OM}$$

and $$\tan\theta = \frac{PM}{OM}$$

Squaring and subtracting, we get

$$\sec^2\theta - \tan^2\theta = \left(\frac{OP}{OM}\right)^2 - \left(\frac{PM}{OM}\right)^2$$

$$= \frac{OP^2 - PM^2}{OM^2}$$

$$= \frac{OM^2}{OM^2} \text{ [By Pythagoras Theorm, } OP^2 - PM^2 = OM^2\text{]}$$

$$= 1$$

Hence, $\sec^2\theta - \tan^2\theta = 1$ ...(2)

Again, $\operatorname{cosec}\theta = \frac{OP}{PM}$

and $$\cot\theta = \frac{OM}{PM}$$

Squaring and subtracting, we get

$$\operatorname{cosec}^2\theta - \cot^2\theta = \left(\frac{OP}{PM}\right)^2 - \left(\frac{OM}{PM}\right)^2$$

$$= \frac{OP^2 - OM^2}{PM^2} = \frac{PM^2}{PM^2}$$

[By Pythagoras Theorm, $OP^2 - OM^2 = PM^2$]

$$= 1$$

Hence, $\operatorname{cosec}^2\theta - \cot^2\theta = 1$ ...(3)

Note: By using algebraic operations, we can write identities (1), (2) and (3) as

$\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

and $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \operatorname{cosec}^2\theta - 1$

respectively.

We shall solve a few examples, using the above identities.

Example 22.16: Prove that

$$\tan\theta + \cot\theta = \frac{1}{\sin\theta\cos\theta}$$

Solution: L.H.S. = $\tan\theta + \cot\theta$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta} \qquad (\because \sin^2\theta + \cos^2\theta = 1)$$

= R.H.S.

Hence, $\tan\theta + \cot\theta = \dfrac{1}{\sin\theta\cos\theta}$

Exampe 22.17: Prove that

$$\frac{\sin A}{1+\cos A} + \frac{1+\cos A}{\sin A} = 2\operatorname{cosec} A$$

Solution: $\text{L.H.S} = \dfrac{\sin A}{1+\cos A} + \dfrac{1+\cos A}{\sin A}$

$$= \frac{\sin^2 A + (1+\cos A)^2}{\sin A\,(1+\cos A)}$$

$$= \frac{\sin^2 A + 1 + \cos^2 A + 2\cos A}{\sin A\,(1+\cos A)}$$

$$= \frac{(\sin^2 A + \cos^2 A) + 1 + 2\cos A}{\sin A\,(1+\cos A)}$$

$$= \frac{1+1+2\cos A}{\sin A\,(1+\cos A)}$$

$$= \frac{2+2\cos A}{\sin A\,(1+\cos A)}$$

$$= \frac{2\,(1+\cos A)}{\sin A\,(1+\cos A)}$$

$$= \frac{2}{\sin A}$$

$$= 2\ \text{cosec}\ A$$

$$= \text{R.H.S.}$$

Hence, $\frac{\sin A}{1+\cos A} + \frac{1+\cos A}{\sin A} = 2\ \text{cosec}\ A$

Example 22.18: Prove that:

$$\frac{1-\sin A}{1+\sin A} = (\sec A - \tan A)^2$$

Solution: R.H.S. = $(\sec A - \tan A)^2$

$$= \left(\frac{1}{\cos A} - \frac{\sin A}{\cos A}\right)^2$$

$$= \left(\frac{1-\sin A}{\cos A}\right)^2$$

$$= \frac{(1-\sin A)^2}{\cos^2 A}$$

$$= \frac{(1-\sin A)^2}{1-\sin^2 A} \qquad \left(\because \cos^2 A = 1-\sin^2 A\right)$$

$$= \frac{(1-\sin A)^2}{(1-\sin A)(1+\sin A)}$$

$$= \frac{1-\sin A}{1+\sin A}$$

$$= \text{L.H.S.}$$

Hence, $\frac{1-\sin A}{1+\sin A} = (\sec A - \tan A)^2$

Alternative method

We can prove the identity by starting from L.H.S. in the following way:

$$\text{L.H.S.} = \frac{1-\sin A}{1+\sin A}$$

$$= \frac{1-\sin A}{1+\sin A} \times \frac{1-\sin A}{1-\sin A}$$

$$= \frac{(1-\sin A)^2}{1-\sin^2 A}$$

$$= \frac{(1-\sin A)^2}{\cos^2 A}$$

$$= \left(\frac{1-\sin A}{\cos A}\right)^2$$

$$= \left(\frac{1}{\cos A} - \frac{\sin A}{\cos A}\right)^2$$

$$= (\sec A - \tan A)^2$$

$$= \text{R.H.S.}$$

Remark: From the above examples, we get the following method for solving questions on Trigonometric identities.

Method to solve questions on Trigonometric identities

Step 1: Choose L.H.S. or R.H.S., whichever looks to be easy to simplify.

Step 2: Use different identities to simplify the L.H.S. (or R.H.S.) and arrive at the result on the other hand side.

Step 3: If you don't get the result on R.H.S. (or L.H.S.) arrive at an appropriate result and then simplify the other side to get the result already obtained.

Step 4: As both sides of the identity have been proved to be equal the identity is established.

We shall now, solve some more questions on Trigonometric identities.

Example 22.19: Prove that:

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{\cos\theta}{1+\sin\theta}$$

Solution: L.H.S. = $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$

$$= \frac{\sqrt{1-\sin\theta}}{\sqrt{1+\sin\theta}} \times \frac{\sqrt{1+\sin\theta}}{\sqrt{1+\sin\theta}}$$

$$= \frac{\sqrt{1-\sin^2\theta}}{(1+\sin\theta)}$$

$$= \frac{\sqrt{\cos^2\theta}}{1+\sin\theta} \qquad (\because 1-\sin^2\theta = \cos^2\theta)$$

$$= \frac{\cos\theta}{1+\sin\theta} = \text{R.H.S.}$$

Hence, $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{\cos\theta}{1+\sin\theta}$

Example 22.20: Prove that

$$\cos^4 A - \sin^4 A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$$

Solution: L.H.S. = $\cos^4 A - \sin^4 A$

$$= (\cos^2 A)^2 - (\sin^2 A)^2$$

$$= (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$$

$$= \cos^2 A - \sin^2 A \qquad (\because \cos^2 A + \sin^2 A = 1)$$

$$= \text{R.H.S.}$$

Again $\cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A \qquad (\because \cos^2 A = 1 - \sin^2 A)$

$$= 1 - 2\sin^2 A$$

$$= \text{R. H. S.}$$

Hence $\cos^4 A - \sin^4 A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$

Example 22.21: Prove that

$$\sec A\,(1 - \sin A)\,(\sec A + \tan A) = 1$$

Solution: L.H.S. $= \sec A\,(1 - \sin A)\,(\sec A + \tan A)$

$$= \frac{1}{\cos A}(1 - \sin A)\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)$$

$$= \frac{(1 - \sin A)(1 + \sin A)}{\cos^2 A}$$

$$= \frac{1 - \sin^2 A}{\cos^2 A}$$

$$= \frac{\cos^2 A}{\cos^2 A}$$

$$= 1 \ = \text{R.H.S.}$$

Hence, $\sec A\,(1 - \sin A)\,(\sec A + \tan A) = 1$

Example 22.22: Prove that

$$\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{1 + \sin\theta}{\cos\theta}$$

$$= \frac{\cos\theta}{1 - \sin\theta}$$

Solution: L.H.S. $= \dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$

$$= \frac{(\tan\theta + \sec\theta) - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1} \qquad (\because 1 = \sec^2\theta - \tan^2\theta)$$

$$= \frac{(\tan\theta + \sec\theta) - (\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}{\tan\theta - \sec\theta + 1}$$

$$= \frac{(\tan\theta + \sec\theta)[1 - (\sec\theta - \tan\theta)]}{\tan\theta - \sec\theta + 1}$$

$$= \frac{(\tan\theta + \sec\theta)(1 - \sec\theta + \tan\theta)}{\tan\theta - \sec\theta + 1}$$

$$= \tan\theta + \sec\theta$$

$$= \frac{1 + \sin\theta}{\cos\theta}$$

= R.H.S.

Again $\frac{1 + \sin\theta}{\cos\theta} = \frac{(1 + \sin\theta)(1 - \sin\theta)}{\cos\theta(1 - \sin\theta)}$

$$= \frac{1 - \sin^2\theta}{\cos\theta(1 - \sin\theta)}$$

$$= \frac{\cos^2\theta}{\cos\theta(1 - \sin\theta)}$$

$$= \frac{\cos\theta}{1 - \sin\theta}$$

= R.H.S.

Hence, $\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{1 + \sin\theta}{\cos\theta}$

$$= \frac{\cos\theta}{1 - \sin\theta}$$

Example 22.23: If $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$, then show that $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$.

Solution: We are given $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

or $\cos\theta = \sqrt{2}\sin\theta + \sin\theta$

or $\cos\theta = (\sqrt{2} + 1)\sin\theta$

or $$\frac{\cos\theta}{\sqrt{2}+1} = \sin\theta$$

or $$\sin\theta = \frac{\cos\theta}{\sqrt{2}+1} \times \frac{\left(\sqrt{2}-1\right)}{\left(\sqrt{2}-1\right)}$$

or $$\sin\theta = \frac{\sqrt{2}\cos\theta - \cos\theta}{2-1}$$

or $$\sin\theta + \cos\theta = \sqrt{2}\cos\theta$$

Hence, $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$.

Example 22.24: If $\tan^4\theta + \tan^2\theta = 1$, then show that

$$\cos^4\theta + \cos^2\theta = 1$$

Solution: We have $\tan^4\theta + \tan^2\theta = 1$

or $$\tan^2\theta\,(\tan^2\theta + 1) = 1$$

or $$1 + \tan^2\theta = \frac{1}{\tan^2\theta} = \cot^2\theta$$

or $$\sec^2\theta = \cot^2\theta \qquad (1 + \tan^2\theta = \sec^2\theta)$$

or $$\frac{1}{\cos^2\theta} = \frac{\cos^2\theta}{\sin^2\theta}$$

or $$\sin^2\theta = \cos^4\theta$$

or $$1 - \cos^2\theta = \cos^4\theta \qquad (\sin^2\theta = 1 - \cos^2\theta)$$

or $$\cos^4\theta + \cos^2\theta = 1$$

CHECK YOUR PROGRESS 22.5

Prove each of the following identities:

1. $(\text{cosec}^2\theta - 1)\sin^2\theta = \cos^2\theta$
2. $\sin^4 A + \sin^2 A\cos^2 A = \sin^2 A$
3. $\cos^2\theta\,(1 + \tan^2\theta) = 1$
4. $(1 + \tan^2\theta)\sin^2\theta = \tan^2\theta$

5. $\dfrac{\sin A}{1+\cos A}+\dfrac{\sin A}{1-\cos A}=2\operatorname{cosec}A$

6. $\sqrt{\dfrac{1+\cos A}{1-\cos A}}=\dfrac{1+\cos A}{\sin A}$

7. $\sqrt{\dfrac{\sec A-\tan A}{\sec A+\tan A}}=\dfrac{\cos A}{1+\sin A}$

8. $(\sin A-\cos A)^2+2\sin A\cos A=1$

9. $\cos^4\theta+\sin^4\theta-2\sin^2\theta\cos^2\theta=(2\cos^2\theta-1)^2$

10. $\dfrac{\sin A-\sin B}{\cos A+\cos B}+\dfrac{\cos A-\cos B}{\sin A+\sin B}=0$

11. $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cos\theta)=1$

12. $\sin A(1+\tan A)+\cos A(1+\cot A)=\sec A+\operatorname{cosec}A$

13. $\dfrac{1-\cos A}{1+\cos A}=(\operatorname{cosec}A-\cot A)^2$

14. $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}=1+\sec A\operatorname{cosec}A$

15. $\dfrac{\cot A+\operatorname{cosec}A-1}{\cot A-\operatorname{cosec}A+1}=\dfrac{1+\cos A}{\sin A}$

$=\dfrac{\sin A}{1-\cos A}$

16. If $\sin^2\theta+\sin\theta=1$, then show that

$\cos^2\theta+\cos^4\theta=1$

Select the correct alternative from the four given in each of the following questions (17 - 20):

17. $(\sin A+\cos A)^2-2\sin A\cos A$ is equal to

(i) 0 (ii) 2 (iii) 1 (iv) $\sin^2A-\cos^2A$

18. $\sin^4A-\cos^4A$ is equal to:

(i) 1 (ii) $\sin^2A-\cos^2A$ (iii) 0 (iv) $\tan^2A$

19. $\sin^2 A - \sec^2 A + \cos^2 A + \tan^2 A$ is equal to

(i) 0 (ii) 1 (iii) $\sin^2 A$ (iv) $\cos^2 A$

20. (sec A – tan A) (sec A + tan A) – (cosec A – cot A) (cosec A + cot A) is equal to

(i) 2 (ii) 1 (iii) 0 (iv) $\frac{1}{2}$

22.7 TRIGONOMETRIC RATIOS FOR COMPLEMENTARY ANGLES

In geometry, we have studied about complementary and supplementary angles. Recall that two angles are complementary if their sum is 90°. If the sum of two angles A and B is 90°, then ∠A and ∠B are complementary angles and each of them is complement of the other. Thus, angles of 20° and 70° are complementary and 20° is complement of 70° and vice versa.

Let XOX′ and YOY′ be a rectangular system of coordinates. Let A be any point on OX. Let ray OA be rotated in an anti clockwise direction and trace an angle θ from its initial position. Let ∠POM = θ. Draw PM ⊥ OX. Then ΔPMO is a right angled triangle.

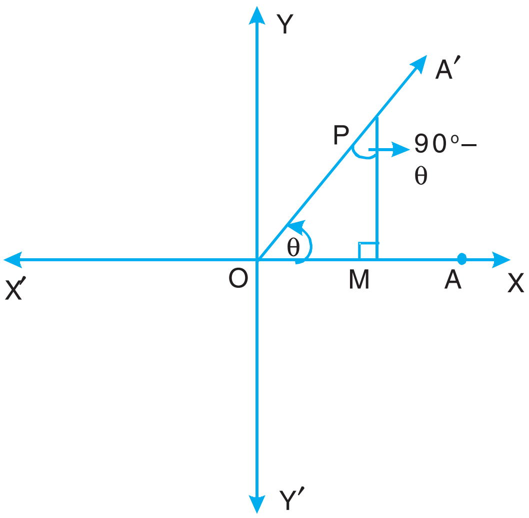


Fig. 22.28

$$\text{Also,} \quad \angle POM + \angle OPM + \angle PMO = 180^\circ$$

$$\text{or} \quad \angle POM + \angle OPM + 90^\circ = 180^\circ$$

$$\text{or} \quad \angle POM + \angle OPM = 90^\circ$$

$$\therefore \quad \angle OPM = 90^\circ - \angle POM = 90^\circ - \theta$$

Thus ∠OPM and ∠POM are complementary angles. Now in right angled triangle PMO,

$$\sin\theta = \frac{PM}{OP}, \cos\theta = \frac{OM}{OP} \text{ and } \tan\theta = \frac{PM}{OM}$$

$$\operatorname{cosec}\theta = \frac{OP}{PM}, \sec\theta = \frac{OP}{OM} \text{ and } \cot\theta = \frac{OM}{PM}$$

For reference angle (90°–θ), we have in right ∠d ΔOPM,

$$\sin(90^\circ - \theta) = \frac{OM}{OP} = \cos\theta$$

$$\cos(90^\circ - \theta) = \frac{PM}{OP} = \sin\theta$$

$$\tan(90^\circ - \theta) = \frac{OM}{PM} = \cot\theta$$

$$\cot(90^\circ - \theta) = \frac{PM}{OM} = \tan\theta$$

$$\text{cosec}(90^\circ - \theta) = \frac{OP}{OM} = \sec\theta$$

and $$\sec(90^\circ - \theta) = \frac{OP}{PM} = \text{cosec}\,\theta$$

The above six results are known as trigonometric ratios of complementary angles. For example,

$\sin(90^\circ - 20^\circ) = \cos 20^\circ$ i.e. $\sin 70^\circ = \cos 20^\circ$

$\tan(90^\circ - 40^\circ) = \cot 40^\circ$ i.e. $\tan 50^\circ = \cot 40^\circ$ and so on.

Let us take some examples to illustrate the use of above results.

Example 22.25: Prove that $\tan 13^\circ = \cot 77^\circ$

Solution: R.H.S. $= \cot 77^\circ$

$= \cot(90^\circ - 13^\circ)$

$= \tan 13^\circ$ ....[$\because \cot(90^\circ - \theta) = \tan\theta$]

$=$ L.H.S.

Thus, $\tan 13^\circ = \cot 77^\circ$

Example 22.26: Evaluate $\sin^2 40^\circ - \cos^2 50^\circ$

Solution: $\cos 50^\circ = \cos(90^\circ - 40^\circ)$

$= \sin 40^\circ$ ....[$\because \cos(90^\circ - \theta) = \tan\theta$]

$\therefore$ $\sin^2 40^\circ - \cos^2 50^\circ = \sin^2 40^\circ - \sin^2 40^\circ = 0$

Example 22.27: Evaluate : $\dfrac{\cos 41^\circ}{\sin 49^\circ} + \dfrac{\sec 37^\circ}{\text{cosec}\, 53^\circ}$

Solution: $\sin 49^\circ = \sin(90^\circ - 41^\circ) = \cos 41^\circ$ ...[$\because \sin(90^\circ - \theta) = \cos\theta$]

and $\operatorname{cosec} 53^\circ = \operatorname{cosec}(90^\circ - 37^\circ) = \sec 37^\circ$ ...[$\because \operatorname{cosec}(90^\circ - \theta) = \sec\theta$]

$$\therefore \quad \frac{\cos 41^\circ}{\sin 49^\circ} + \frac{\sec 37^\circ}{\operatorname{cosec} 53^\circ} = \frac{\cos\ 41^\circ}{\cos\ 41^\circ} + \frac{\sec\ 37^\circ}{\sec\ 37^\circ}$$

$$= 1+1 = 2$$

Example 22.28: Show that

$3 \sin 17^\circ \sec 73^\circ + 2 \tan 20^\circ \tan 70^\circ = 5$

Solution: $3 \sin 17^\circ \sec 73^\circ + 2 \tan 20^\circ \tan 70^\circ$

$= 3 \sin 17^\circ \sec(90^\circ - 17^\circ) + 2 \tan 20^\circ \tan(90^\circ - 20^\circ)$

$= 3 \sin 17^\circ \operatorname{cosec} 17^\circ + 2 \tan 20^\circ \cot 20^\circ$

...[$\because \sec(90^\circ - \theta) = \operatorname{cosec}\theta$ and $\tan(90^\circ - \theta) = \cot\theta$]

$$= 3\sin 17^\circ.\frac{1}{\sin 17^\circ} + 2\tan 20^\circ.\frac{1}{\tan 20^\circ}$$

$= 3 + 2 = 5$

Example 22.29: Show that $\tan 7^\circ \tan 23^\circ \tan 67^\circ \tan 83^\circ = 1$

Solution: $\tan 67^\circ = \tan(90^\circ - 23^\circ) = \cot 23^\circ$

and $\tan 83^\circ = \tan(90^\circ - 7^\circ) = \cot 7^\circ$

Now. L.H.S. = $\tan 7^\circ \tan 23^\circ \tan 67^\circ \tan 83^\circ$

$= \tan 7^\circ \tan 23^\circ \cot 23^\circ \cot 7^\circ$

$= (\tan 7^\circ \cot 7^\circ)(\tan 23^\circ \cot 23^\circ)$

$= 1\ .1 = 1$

= R.H.S.

Hence, $\tan 7^\circ \tan 23^\circ \tan 67^\circ \tan 83^\circ = 1$

Example 22.30: If $\tan A = \cot B$, prove that $A + B = 90^\circ$.

Solution: We are given

$\tan A = \cot B$

or $\tan A = \tan(90^\circ - B)$ ... [$\because \cot\theta = \tan(90^\circ - \theta)$]

$\therefore$ $A = 90^\circ - B$

or $A + B = 90^\circ$

Example 22.31: For a ΔABC, show that $\sin\left(\frac{B+C}{2}\right)=\cos\left(\frac{A}{2}\right)$, where A, B and C are interior angles of ΔABC.

Solution: We know that sum of angles of triangle is 180°.

$$\therefore \quad A + B + C = 180^{o}$$

$$\text{or} \quad B + C = 180^{o} - A$$

$$\text{or} \quad \frac{B+C}{2} = 90^{o} - \frac{A}{2}$$

$$\therefore \quad \sin\left(\frac{B+C}{2}\right) = \sin\left(90^{o} - \frac{A}{2}\right)$$

$$\text{or} \quad \sin\left(\frac{B+C}{2}\right) = \cos\left(\frac{A}{2}\right)$$

Example 22.32: Prove that $\frac{\cos\theta}{\sin(90^{o}-\theta)}+\frac{\sin\theta}{\cos(90^{o}-\theta)}=2$.

Solution: L.H.S. $= \frac{\cos\theta}{\sin(90^{o}-\theta)}+\frac{\sin\theta}{\cos(90^{o}-\theta)}$

$$= \frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\sin\theta} \quad \text{... [}\because \sin(90^{o}-\theta)=\cos\theta \text{ and } \cos(90^{o}-\theta)=\sin\theta\text{]}$$

$$= 1 + 1 = 2$$

$$= \text{R.H.S.}$$

Hence, $\frac{\cos\theta}{\sin(90^{o}-\theta)}+\frac{\sin\theta}{\cos(90^{o}-\theta)}=2$

Example 22.33: Show that $\frac{\sin(90^{o}-\theta)}{\text{cosec}(90^{o}-\theta)}+\frac{\cos(90^{o}-\theta)}{\sec(90^{o}-\theta)}=1$

Solution: L.H.S. $= \frac{\sin(90^{o}-\theta)}{\text{cosec}(90^{o}-\theta)}+\frac{\cos(90^{o}-\theta)}{\sec(90^{o}-\theta)}$

$$= \frac{\cos\theta}{\sec\theta}+\frac{\sin\theta}{\text{cosec}\,\theta} \quad \text{...[}\because \sin(90^{o}-\theta)=\cos\theta,\ \cos(90^{o}-\theta)=\sin\theta,$$

$$\text{cosec}(90^{o}-\theta)=\sec\theta \text{ and } \sec(90^{o}-\theta)=\text{cosec}\,\theta\text{]}$$

$$= \frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\sin\theta} = \cos^2\theta + \sin^2\theta \ = 1$$

= R.H.S.

Hence, $\dfrac{\sin(90^\circ - \theta)}{\mathrm{cosec}(90^\circ - \theta)} + \dfrac{\cos(90^\circ - \theta)}{\sec(90^\circ - \theta)} = 1$

Example 22.34: Simplify:

$$\frac{\cos(90^\circ - \theta)\sec(90^\circ - \theta)\tan\theta}{\mathrm{cosec}(90^\circ - \theta)\sin(90^\circ - \theta)\cot(90^\circ - \theta)} + \frac{\tan(90^\circ - \theta)}{\cot\theta}$$

Solution: The given expression

$$= \frac{\cos(90^\circ - \theta)\sec(90^\circ - \theta)\tan\theta}{\mathrm{cosec}(90^\circ - \theta)\sin(90^\circ - \theta)\cot(90^\circ - \theta)} + \frac{\tan(90^\circ - \theta)}{\cot\theta}$$

$$= \frac{\sin\theta\mathrm{in}\theta.\mathrm{co}\ \theta.\tan\theta}{\sec\theta\mathrm{ec}\theta.\mathrm{cotan}\,\theta} + \frac{\cot\theta}{\cot\theta} \quad ...[\because \sin\theta \,.\, \cos\theta = 1 \text{ and } \sec\theta \,.\, \cos\theta = 1]$$

$= 1 + 1$

$= 2$

Example 22.35: Express tan 68° + sec 68° in terms of angles between 0° and 45°.

Solution: We know that

$\tan(90^\circ - \theta) = \cot\theta$

and $\sec(90^\circ - \theta) = \mathrm{cosec}\,\theta$

$\therefore$ $\tan 68^\circ = \tan(90^\circ - 22^\circ) = \cot 22^\circ$

and $\sec 68^\circ = \sec(90^\circ - 22^\circ) = \mathrm{cosec}\, 22^\circ$

Hence $\tan 68^\circ + \sec 68^\circ = \cot 22^\circ + \mathrm{cosec}\, 22^\circ$.

Remark: While using notion of complementary angles, usually we change that angle which is > 45° to its complement.

Example 22.36: If tan 2A = cot (A – 18°) where 2A is an acute angle, find the value of A.

Solution: We are given tan 2A = cot (A – 18°)

or $\cot(90^\circ - 2A) = \cot(A - 18^\circ)$ ...[$\because$ $\cot(90^\circ - 2A = \tan 2A$]

$\therefore \quad 90^o - 2A = A - 18^o$

or $\quad 3A = 90^o + 18^o$

or $\quad 3A = 108^o$

or $\quad A = 36^o$

CHECK YOUR PROGRESS 22.6

1. Show that:

   (i) $\cos 55^o = \sin 35^o$

   (ii) $\sin^2 11^o - \cos^2 79^o = 0$

   (iii) $\cos^2 51^o - \sin^2 39^o = 0$

2. Evaluate each of the following:

   (i) $\dfrac{3\sin 19^o}{\cos 71^o}$ (ii) $\dfrac{\tan 65^o}{2\cot 25^o}$ (iii) $\dfrac{\cos 89^o}{3\sin 1^o}$

   (iv) $\cos 48^o - \sin 42^o$ (v) $\dfrac{3\sin 5^o}{\cos 85^o} + \dfrac{2\tan 33^o}{\cot 57^o}$

   (vi) $\dfrac{\cot 54^o}{\tan 36^o} + \dfrac{\tan 20^o}{\cot 70^o} - 2$

   (vii) $\sec 41^o \sin 49^o + \cos 49^o \operatorname{cosec} 41^o$

   (viii) $\dfrac{\cos 75^o}{\sin 15^o} + \dfrac{\sin 12^o}{\cos 78^o} - \dfrac{\cos 18^o}{\sin 72^o}$

3. Evaluate each of the following:

   (i) $\left(\dfrac{\sin 47^o}{\cos 43^o}\right)^2 + \left(\dfrac{\cos 43^o}{\sin 47^o}\right)^2$

   (ii) $\dfrac{\cos^2 20^o + \cos^2 70^o}{3\left(\sin^2 59^o + \sin^2 31^o\right)}$

4. Prove that:

   (i) $\sin\theta \cos(90^o - \theta) + \cos\theta \sin(90^o - \theta) = 1$

(ii) $\cos\theta \cos(90^o - \theta) - \sin\theta \sin(90^o - \theta) = 0$

(iii) $\dfrac{\cos(90^o - \theta)}{1+\sin(90^o - \theta)} + \dfrac{1+\sin(90^o - \theta)}{\cos(90^o - \theta)} = 2\text{cosec}\theta$

(iv) $\sin(90^o - \theta).\cos(90^o - \theta) = \dfrac{\tan(90^o - \theta)}{1+\tan^2(90^o - \theta)}$

(v) $\tan 45^o \tan 13^o \tan 77^o \tan 85^o = 1$

(vi) $2 \tan 15^o \tan 25^o \tan 65^o \tan 75^o = 2$

(vii) $\sin 20^o \sin 70^o - \cos 20^o \cos 70^o = 0$

5. Show that $\sin(50^o + \theta) - \cos(40^o - \theta) = 0$

6. If $\sin A = \cos B$ where A and B are acute angles, prove that $A + B = 90^o$.

7. In a ΔABC, prove that

(i) $\tan\left(\dfrac{B+C}{2}\right) = \cot\left(\dfrac{A}{2}\right)$

(ii) $\cos\left(\dfrac{A+B}{2}\right) = \sin\left(\dfrac{C}{2}\right)$

8. Express $\tan 59^o + \text{cosec } 85^o$ in terms of trigonometric ratios of angles between 0^o and 45^o.

9. Express $\sec 46^o - \cos 87^o$ in terms of trigonometric ratios of angles between 0^o and 45^o.

10. Express $\sec^2 62^o + \sec^2 69^o$ in terms of trigonometric ratios of angles between 0^o and 45^o.

Select the correct alternative for each of the following questions (11-12):

11. The value of $\dfrac{\sin 40^o}{2\cos 50^o} - \dfrac{2\sec 41^o}{3\text{cosec}49^o}$ is

(i) -1 (ii) $\dfrac{1}{6}$ (iii) $-\dfrac{1}{6}$ (iv) 1

12. If $\sin(\theta + 36^o) = \cos\theta$, where $\theta + 36^o$ is an acute angle, then θ is

(i) 54^o (ii) 18^o (iii) 21^o (iv) 27^o

LET US SUM UP

- In a right angled triangle, we define trignometric ratios as under:

$$\sin\theta = \frac{\text{side opposite to angle }\theta}{\text{Hypotenuse}} = \frac{AB}{AC}$$

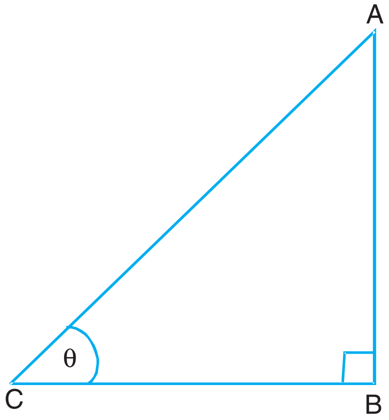
$$\cos\theta = \frac{\text{side adjacent to angle }\theta}{\text{Hypotenuse}} = \frac{BC}{AC}$$

$$\tan\theta = \frac{\text{side opposite to angle }\theta}{\text{side adjacent to angle }\theta} = \frac{AB}{BC}$$

$$\cot\theta = \frac{\text{side adjacent to angle }\theta}{\text{side opposite to angle }\theta} = \frac{BC}{AB}$$

$$\sec\theta = \frac{\text{Hypotenuse}}{\text{side adjacent to angle }\theta} = \frac{AC}{BC}$$

$$\text{cosec}\,\theta = \frac{\text{Hypotenuse}}{\text{side opposite to angle }\theta} = \frac{AC}{AB}$$



- The following relationships exist between different trigonometric ratios:

(i) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (ii) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(iii) $\sec\theta = \dfrac{1}{\cos\theta}$ (iv) $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$

(v) $\cot\theta = \dfrac{1}{\tan\theta}$

- The trigonometric identities are:

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $\sec^2\theta - \tan^2\theta = 1$

(iii) $\text{cosec}^2\theta - \cot^2\theta = 1$

- Two angles, whose sum is 90°, are called complementary angles.

- $\sin(90^\circ - A) = \cos A$, $\cos(90^\circ - A) = \sin A$ and $\tan(90^\circ - A) = \cot A$.
- $\text{cosec}(90^\circ - A) = \sec A$, $\sec(90^\circ - A) = \text{cosec} A$ and $\cot(90^\circ - A) = \tan A$

Supportive website:

- http://www.wikipedia.org
- http://mathworld:wolfram.com

TERMINAL EXERCISE

1. If $\sin A = \frac{4}{5}$, find the values of cos A and tan A.

2. If $\tan A = \frac{20}{21}$, find the values of cosec A and sec A.

3. If $\cot\theta = \frac{3}{4}$, find the value of $\sin\theta + \cos\theta$.

4. If $\sec\theta = \frac{m}{n}$, find the values of $\sin\theta$ and $\tan\theta$.

5. If $\cos\theta = \frac{3}{5}$, find the value of

$$\frac{\sin\theta\tan\theta - 1}{2\tan^2\theta}$$

6. If $\sec\theta = \frac{5}{4}$, find the value of $\frac{\tan\theta}{1+\tan\theta}$

7. If $\tan A = 1$ and $\tan B = \sqrt{3}$, find the value of cos A cos B – sin A sin B.

Prove each of the following identities (8 –20):

8. $(\sec\theta + \tan\theta)(1 - \sin\theta) = \cos\theta$.

9. $\frac{\cot\theta}{1-\tan\theta} = \frac{\text{cosec}\theta}{\sec\theta}$

10. $\frac{1-\cos\theta}{1+\cos\theta} = (\text{cosec}\theta - \cot\theta)^2$

11. $\dfrac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta}=\dfrac{\sec\theta+1}{\sec\theta-1}$

12. $\dfrac{\tan A+\cot B}{\cot A+\tan B}=\tan A\cot B$

13. $\sqrt{\dfrac{1+\cos A}{1-\cos A}}=\operatorname{cosec}A+\cot A$

14. $\sqrt{\dfrac{\operatorname{cosec}A+1}{\operatorname{cosec}A-1}}=\dfrac{\cos A}{1-\sin A}$

15. $\sin^3 A-\cos^3 A=(\sin A-\cos A)\,(1+\sin A\cos A)$

16. $\dfrac{\cos A}{1-\tan A}+\dfrac{\sin A}{1-\cot A}=\cos A+\sin A$

17. $\sqrt{\dfrac{\sec A-1}{\sec A+1}}+\sqrt{\dfrac{\sec A+1}{\sec A-1}}=2\operatorname{cosec}A$

18. $(\operatorname{cosec}A-\sin A)(\sec A-\cos A)=\dfrac{1}{\tan A+\cot A}$

19. $(1+\cot\theta-\operatorname{cosec}\theta)\,(1+\tan\theta+\sec\theta)=2$

20. $2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)+1=0$

21. If $\sec\theta+\tan\theta=p$, show that $\sin\theta=\dfrac{p^2-1}{p^2+1}$

22. Prove that $\dfrac{\cos(90^\circ-A)}{1+\sin(90^\circ-A)}+\dfrac{1+\sin(90^\circ-A)}{\cos(90^\circ-A)}=2\sec(90^\circ-A)$

23. Prove that $\dfrac{\sin(90^\circ-A).\cos(90^\circ-A)}{\tan A}=\sin^2(90^\circ-A)$

24. If $\tan\theta=\dfrac{3}{4}$ and $\theta+\alpha=90^\circ$, find the value of $\cot\alpha$.

25. If $\cos(2\theta+54^\circ)=\sin\theta$ and $(2\theta+54^\circ)$ is an acute angle, find the value of θ.

26. If $\sec Q=\operatorname{cosec}P$ and P and Q are acute angles, show that $P+Q=90^\circ$.

ANSWERS TO CHECK YOUR PROGRESS

22.1

1. (i) $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\tan\theta = \frac{5}{12}$

$\operatorname{cosec}\theta = \frac{13}{5}$, $\sec\theta = \frac{13}{12}$ and $\cot\theta = \frac{12}{5}$

(ii) $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$

$\operatorname{cosec}\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$ and $\cot\theta = \frac{4}{3}$

(iii) $\sin\theta = \frac{24}{25}$, $\cos\theta = \frac{7}{25}$, $\tan\theta = \frac{24}{7}$

$\operatorname{cosec}\theta = \frac{25}{24}$, $\sec\theta = \frac{25}{7}$ and $\cot\theta = \frac{7}{24}$

(iv) $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$

$\operatorname{cosec}\theta = \frac{5}{4}$, $\sec\theta = \frac{5}{3}$ and $\cot\theta = \frac{3}{4}$

2. $\sin A = \frac{5}{\sqrt{41}}$, $\cos A = \frac{4}{\sqrt{41}}$ and $\tan A = \frac{5}{4}$

3. $\sin C = \frac{40}{41}$, $\cot C = \frac{9}{40}$, $\cos A = \frac{40}{41}$ and $\cot A = \frac{40}{9}$

4. $\sec C = \sqrt{2}$, $\operatorname{cosec} C = \sqrt{2}$ and $\cot C = 1$

5. (iv)

6. (ii)

22.2

1. $\sin C = \frac{3}{5}$, $\cos C = \frac{4}{5}$ and $\tan C = \frac{3}{4}$

2. $\sin A = \frac{24}{25}$, $\operatorname{cosec} A = \frac{25}{24}$ and $\cot A = \frac{7}{24}$

3. $\sec P = \sqrt{2}$, $\cot P = 1$, and $\operatorname{cosec} P = \sqrt{2}$

4. $\tan R = \sqrt{3}$, $\operatorname{cosec} R = \frac{2}{\sqrt{3}}$, $\sin P = \frac{1}{2}$ and $\sec P = \frac{2}{\sqrt{3}}$

5. $\cot \theta = \frac{24}{7}$, $\sin \theta = \frac{7}{25}$, $\sec \theta = \frac{25}{24}$, and $\tan \theta = \frac{7}{24}$

6. $\sin P = \frac{2\sqrt{6}}{7}$, $\cos P = \frac{5}{7}$, $\sin R = \frac{5}{7}$ and $\cos R = \frac{2\sqrt{6}}{7}$, $\sin P - \cos R = 0$

7. (iii)

22.3

1. $\cos \theta = \frac{21}{29}$ and $\tan \theta = \frac{20}{21}$

2. $\sin \theta = \frac{24}{25}$ and $\cos \theta = \frac{7}{25}$

3. $\sin A = \frac{24}{25}$ and $\tan A = \frac{24}{7}$

4. $\cot \theta = \frac{m}{\sqrt{n^2 - m^2}}$ and $\operatorname{cosec} \theta = \frac{n}{\sqrt{n^2 - m^2}}$

5. $-\frac{256}{135}$

6. $\frac{27}{8}$

7. $\sin B = \frac{2}{\sqrt{13}}$ and $\tan B = \frac{2}{3}$

11. (ii)

22.4

1. $\cot \theta = \sqrt{3}$ and $\sec \theta = \frac{2}{\sqrt{3}}$

2. $\frac{3}{4}$

3. $\frac{\sqrt{3}}{2}$

4. $\sin A = \frac{1}{2}$ and $\tan A = \frac{1}{\sqrt{3}}$

5. $-\frac{14}{3}$

22.5

17. (iii)

18. (ii)

19. (i)

20. (iii)

22.6

1. (i) 3 (ii) $\frac{1}{2}$ (iii) $\frac{1}{3}$ (iv) 0

   (v) 5 (vi) 0 (vii) 2 (viii) 1

3. (i) 2 (ii) $\frac{1}{3}$

8. $\cot 31^\circ + \sec 5^\circ$

9. $\operatorname{cosec} 44^\circ - \sin 3^\circ$

10. $\operatorname{cosec}^2 28^\circ + \operatorname{cosec}^2 21^\circ$

11. (ii)

12. (iv)

ANSWERS TO TERMINAL EXERCISE

1. $\cos A = \frac{3}{5}$ and $\tan A = \frac{4}{3}$

2. $\text{cosec A} = \frac{29}{20}$ and $\sec A = \frac{29}{21}$

3. $\frac{7}{5}$

4. $\sin\theta = \frac{\sqrt{m^2 - n^2}}{m}$ and $\tan\theta = \frac{\sqrt{m^2 - n^2}}{n}$

5. $\frac{3}{160}$

6. $\frac{3}{7}$

7. $\frac{1-\sqrt{3}}{2\sqrt{2}}$

24. $\frac{3}{4}$

25. 12°

23

TRIGONOMETRIC RATIOS OF SOME SPECIAL ANGLES

In the last lesson, we have defined trigonometric ratios for acute angles in a right triangle and also developed some relationship between them. In this lesson we shall find the values of trigonometric ratios of angles of 30°, 45° and 60° by using our knowledge of geometry. We shall also write the values of trigonometric ratios of 0° and 90° and we shall observe that some trigonometric ratios of 0° and 90° are not defined. We shall also use the knowledge of trigonometry to solve simple problems on heights and distances from day to day life.

OBJECTIVES

After studying this lesson, you will be able to

- *find the values of trigonometric ratios of angles of 30°, 45° and 60°;*
- *write the values of trigonometric ratios of 0° and 90°;*
- *tell, which trigonometric ratios of 0° and 90°are not defined;*
- *solve daily life problems of heights and distances;*

EXPECTED BACKGROUND KNOWLEDGE

- Pythagoras Theorem i.e. in a right angled triangle ABC, right angled at B,

  $AC^2 = AB^2 + BC^2$.

- In a right triangle ABC, right angled at B,

$$\sin C = \frac{\text{side opposite to } \angle C}{\text{Hypotenuse}}, \quad \operatorname{cosec} C = \frac{\text{Hypotenuse}}{\text{side opposite to } \angle C}$$

$$\cos C = \frac{\text{side adjacent to } \angle C}{\text{Hypotenuse}}, \quad \sec C = \frac{\text{Hypotenuse}}{\text{side adjacent to } \angle C}$$

$$\tan C = \frac{\text{side opposite to } \angle C}{\text{side adjacent to } \angle C} \text{ and } \cot C = \frac{\text{side adjacent to } \angle C}{\text{side opposite to } \angle C}$$

$$\text{cosec } C = \frac{1}{\sin C}, \ \sec C = \frac{1}{\cos C} \text{ and } \cot C = \frac{1}{\tan C}$$

- $\sin(90^\circ - \theta) = \cos\theta$, $\cos(90^\circ - \theta) = \sin\theta$

  $\tan(90^\circ - \theta) = \cot\theta$, $\cot(90^\circ - \theta) = \tan\theta$
- $\sec(90^\circ - \theta) = \text{cosec}\,\theta$ and $\text{cosec}(90^\circ - \theta) = \sec\theta$

23.1 TRIGONOMETRIC RATIOS FOR AN ANGLE OF 45$^\text{O}$

Let a ray OA start from OX and rotate in the anticlock wise direction and make an angle of 45° with the x-axis as shown in Fig. 23.1.

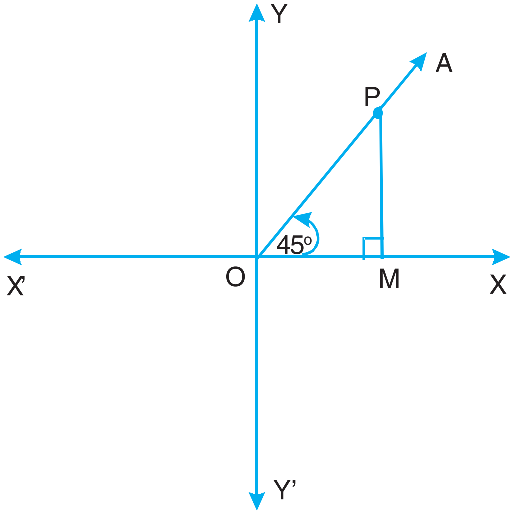


Fig. 23.1

Take any point P on OA. Draw PM ⊥ OX.

Now in right ΔPMO,

$$\angle POM + \angle OPM + \angle PMO = 180^\circ$$

or $\quad 45^\circ + \angle OPM + 90^\circ = 180^\circ$

or $\quad \angle OPM = 180^\circ - 90^\circ - 45^\circ = 45^\circ$

∴ In $\quad \Delta PMO, \angle OPM = \angle POM = 45^\circ$

$$\therefore \quad OM = PM$$

Let OM = a units, then PM = a units.

In right triangle PMO,

$$OP^2 = OM^2 + PM^2 \text{ (Pythagoras Theorem)}$$

$$= a^2 + a^2$$

$$= 2\,a^2$$

$\therefore$ OP = $\sqrt{2}\,a$ units

Now $\quad \sin 45^\circ = \frac{PM}{OP} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$

$$\cos 45^\circ = \frac{OM}{OP} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$$

$$\tan 45^{o} = \frac{PM}{OM} = \frac{a}{a} = 1$$

$$\text{cosec } 45^{o} = \frac{1}{\sin 45^{o}} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$$

$$\sec 45^{o} = \frac{1}{\cos 45^{o}} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$$

and $$\cot 45^{o} = \frac{1}{\tan 45^{o}} = \frac{1}{1} = 1$$

23.2 TRIGONOMETRIC RATIOS FOR AN ANGLE OF 30^{O}

Let a ray OA start from OX and rotate in the anti clockwise direction and make an angle of 30^{o} with x-axis as shown in Fig. 23.2.

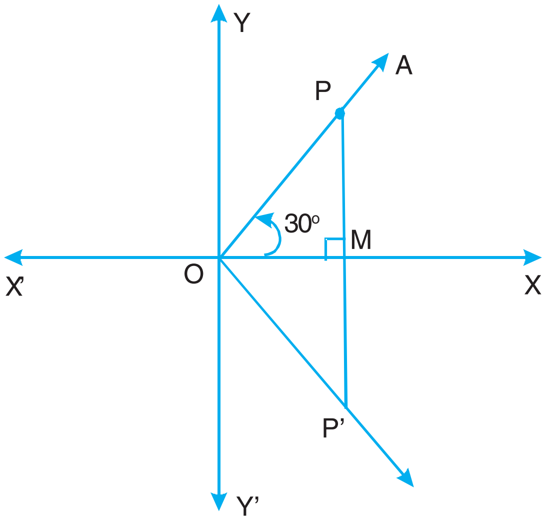


Fig. 23.2

Take any point P on OA.

Draw PM ⊥ OX and produce

PM to P′ such that PM = P′M. Join OP′

Now in ΔPMO and ΔP′MO,

OM = OM ...(Common)

$\angle PMO = \angle P'MO$ ...(Each = 90^{o})

and PM = P′M ...(Construction)

$\therefore$ $\Delta PMO \cong \Delta P'MO$

$\therefore$ $\angle OPM = \angle OP'M = 60^{o}$

$\therefore$ OPP′ is an equilateral triangle

$\therefore$ OP = OP′

Let PM = a units

PP′ = PM + MP′

= $(a + a)$ units ...($\because$ MP′ = MP)

= $2a$ units

$\therefore$ OP = OP′ = PP′ = $2a$ units

Now in right triangle PMO,

$OP^2 = PM^2 + OM^2$ ...(Pythagoras Theorem) or $(2a)^2 = a^2 + OM^2$

$\therefore$ $OM^2 = 3a^2$

or $OM = \sqrt{3}\ a$ units

$$\therefore \quad \sin 30^o = \frac{PM}{OP} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^o = \frac{OM}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^o = \frac{PM}{OM} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$$

$$\text{cosec } 30^o = \frac{1}{\sin 30^o} = \frac{1}{1/2} = 2$$

$$\sec 30^o = \frac{1}{\cos 30^o} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$$

and $$\cot 30^o = \frac{1}{\tan 30^o} = \frac{1}{1/\sqrt{3}} = \sqrt{3}$$

23.3 TRIGONOMETRIC RATIOS FOR AN ANGLE OF 60^o

Let a ray OA start from OX and rotate in anticlock wise direction and make an angle of 60° with x-axis.

Take any point P on OA.

Draw PM ⊥ OX.

Produce OM to M′ such that

OM = MM′. Join PM′.

Let OM = a units

In ΔPMO and ΔPMM′,

PM = PM ...(Common)

∠PMO = ∠PMM′ ...(Each = 90°)

OM = MM′ ...(Construction)

$\therefore$ ΔPMO ≅ ΔPMM′

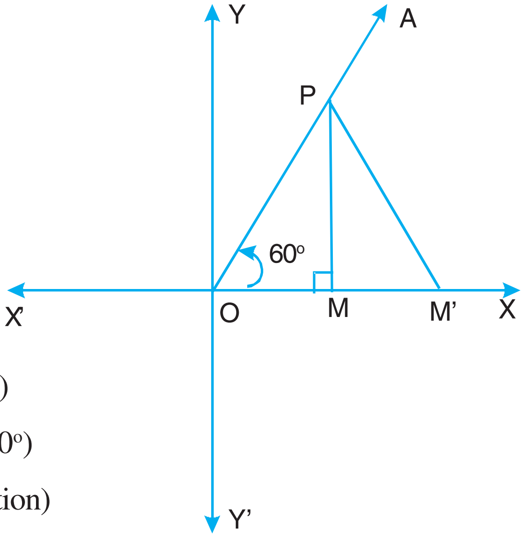


Fig. 23.3

$\therefore \quad \angle POM = \angle PM'M = 60^o$

$\therefore \quad \Delta POM'$ is an equilateral triangle.

$\therefore \quad OP = PM' = OM' = 2a$ units

In right ΔPMO,

$OP^2 = PM^2 + OM^2$ ...(Pythagorus Theorem)

$\therefore \quad (2a)^2 = PM^2 + a^2$

or $\quad PM^2 = 3a^2$

$\therefore \quad PM = \sqrt{3}\, a$ units

$$\therefore \quad \sin 60^o = \frac{PM}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^o = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60^o = \frac{PM}{OM} = \frac{\sqrt{3}a}{a} = \sqrt{3}$$

$$\text{cosec } 60^o = \frac{1}{\sin 60^o} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

$$\sec 60^o = \frac{1}{\cos 60^o} = \frac{1}{1/2} = 2$$

and $$\cot 60^o = \frac{1}{\tan 60^0} = \frac{1}{\sqrt{3}}$$

23.4 TRIGONOMETRIC RATIOS FOR ANGLES OF 0^O AND 90^O

In Section 23.1, 23.2 and 23.3, we have defined trigonometric ratios for angles of 45^o, 30^o and 60^o. For angles of 0^o and 90^o, we shall assume the following results and we shall not be discussing the logical proofs of these.

(i) $\sin 0^o = 0$ and therefore cosec 0^o is not defined

(ii) $\cos 0^o = 1$ and therefore $\sec 0^o = 1$

(iii) tan $0^\circ = 0$ therefore cot 0° is not defined.

(iv) sin $90^\circ = 1$ and therefore cosec $90^\circ = 1$

(v) cos $90^\circ = 0$ and therefore sec 90° is not defined.

(vi) cot $90^\circ = 0$ and therefore tan 90° is not defined.

The values of trignometric ratios for 0°, 30°, 45°, 60° and 90° can be put in a tabular form which makes their use simple. The following table also works as an aid to memory.

θ Trig. ratio	0°	30°	45°	60°	90°
sin θ	$\sqrt{\frac{0}{4}}=0$	$\sqrt{\frac{1}{4}}=\frac{1}{2}$	$\sqrt{\frac{2}{4}}=\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$	$\sqrt{\frac{4}{4}}=1$
cos θ	$\sqrt{\frac{4}{4}}=1$	$\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$	$\sqrt{\frac{2}{4}}=\frac{1}{\sqrt{2}}$	$\sqrt{\frac{1}{4}}=\frac{1}{2}$	$\sqrt{\frac{0}{4}}=0$
tan θ	$\sqrt{\frac{0}{4-0}}=0$	$\sqrt{\frac{1}{4-1}}=\frac{1}{\sqrt{3}}$	$\sqrt{\frac{2}{4-2}}=1$	$\sqrt{\frac{3}{4-3}}=\sqrt{3}$	Not defined
cot θ	Not defined	$\sqrt{\frac{3}{4-3}}=\sqrt{3}$	$\sqrt{\frac{2}{4-2}}=1$	$\sqrt{\frac{1}{4-1}}=\frac{1}{\sqrt{3}}$	$\sqrt{\frac{0}{4-0}}=0$
cosec θ	Not defined	$\sqrt{\frac{4}{1}}=2$	$\sqrt{\frac{4}{2}}=\sqrt{2}$	$\sqrt{\frac{4}{3}}=\frac{2}{\sqrt{3}}$	$\sqrt{\frac{4}{4}}=1$
sec θ	$\sqrt{\frac{4}{4}}=1$	$\sqrt{\frac{4}{3}}=\frac{2}{\sqrt{3}}$	$\sqrt{\frac{4}{2}}=\sqrt{2}$	$\sqrt{\frac{4}{1}}=2$	Not defined

Let us, now take some examples to illustrate the use of these trigonometric ratios.

Example 23.1: Find the value of $\tan^2 60^\circ - \sin^2 30^\circ$.

Solution: We know that $\tan 60^\circ = \sqrt{3}$ and $\sin 30^\circ = \frac{1}{2}$

$$\therefore \qquad \tan^2 60^\circ - \sin^2 30^\circ = \left(\sqrt{3}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$= 3 - \frac{1}{4} = \frac{11}{4}$$

Example 23.2: Find the value of

$$\cot^2 30^o \sec^2 45^o + \operatorname{cosec}^2 45^o \cos 60^o$$

Solution: We know that

$$\cot 30^o = \sqrt{3}, \sec 45^o = \sqrt{2}, \operatorname{cosec} 45^o = \sqrt{2} \text{ and } \cos 60^o = \frac{1}{2}$$

$\therefore$ $\cot^2 30^o \sec^2 45^o + \operatorname{cosec}^2 45^o \cos 60^o$

$$= \left(\sqrt{3}\right)^2 \left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 . \frac{1}{2}$$

$$= 3 \times 2 + 2 \times \frac{1}{2}$$

$$= 6 + 1$$

$$= 7$$

Example 23.3: Evaluate : $2(\cos^2 45^o + \tan^2 60^o) - 6(\sin^2 45^o - \tan^2 30^o)$

Solution: $2(\cos^2 45^o + \tan^2 60^o) - 6(\sin^2 45^o - \tan^2 30^o)$

$$= 2\left[\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\sqrt{3}\right)^2\right] - 6\left[\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right]$$

$$= 2\left(\frac{1}{2} + 3\right) - 6\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= 1 + 6 - 3 + 2$$

$$= 6$$

Example 23.4: Verify that

$$\frac{\tan 45^o}{\operatorname{cosec} 30^o} + \frac{\sec 60^o}{\cot 45^o} - \frac{5 \sin 90^o}{2 \cos 0^o} = 0$$

Solution: L.H.S. $= \dfrac{\tan 45^o}{\operatorname{cosec} 30^o} + \dfrac{\sec 60^o}{\cot 45^o} - \dfrac{5 \sin 90^o}{2 \cos 0^o}$

$$= \frac{1}{2} + \frac{2}{1} - \frac{5 \times 1}{2 \times 1}$$

$$= \frac{1}{2} + 2 - \frac{5}{2} = 0 = \text{R.H.S.}$$

Hence, $\dfrac{\tan 45^\circ}{\operatorname{cosec} 30^\circ} + \dfrac{\sec 60^\circ}{\cot 45^\circ} - \dfrac{5 \sin 90^\circ}{2 \cos 0^\circ} = 0$

Example 23.5: Show that

$$\frac{4}{3}\cot^2 30^\circ + 3\sin^2 60^\circ - 2\operatorname{cosec}^2 60^\circ - \frac{3}{4}\tan^2 30^\circ = \frac{10}{3}$$

Solution: L.H.S. $= \dfrac{4}{3}\cot^2 30^\circ + 3\sin^2 60^\circ - 2\operatorname{cosec}^2 60^\circ - \dfrac{3}{4}\tan^2 30^\circ$

$$= \frac{4}{3}\times\left(\sqrt{3}\right)^2 + 3\left(\frac{\sqrt{3}}{2}\right)^2 - 2\left(\frac{2}{\sqrt{3}}\right)^2 - \frac{3}{4}\left(\frac{1}{\sqrt{3}}\right)^2$$

$$= \frac{4}{3}\times 3 + 3\times\frac{3}{4} - 2\times\frac{4}{3} - \frac{3}{4}\times\frac{1}{3}$$

$$= 4 + \frac{9}{4} - \frac{8}{3} - \frac{1}{4}$$

$$= \frac{48 + 27 - 32 - 3}{12}$$

$$= \frac{40}{12} = \frac{10}{3}$$

$$= \text{R.H.S.}$$

Hence, $\dfrac{4}{3}\cot^2 30^\circ + 3\sin^2 60^\circ - 2\operatorname{cosec}^2 60^\circ - \dfrac{3}{4}\tan^2 30^\circ = \dfrac{10}{3}$

Example 23.6 : Verify that

$$\frac{4\cot^2 60^\circ + \sec^2 30^\circ - 2\sin^2 45^\circ}{\cos^2 30^\circ + \cos^2 45^\circ} = \frac{4}{3}$$

Solution: L.H.S. $= \dfrac{4\cot^2 60^\circ + \sec^2 30^\circ - 2\sin^2 45^\circ}{\cos^2 30^\circ + \cos^2 45^\circ}$

$$= \frac{4.\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2 - 2.\left(\frac{1}{\sqrt{2}}\right)^2}{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{4\times\frac{1}{3}+\frac{4}{3}-2\times\frac{1}{2}}{\frac{3}{4}+\frac{1}{2}}$$

$$= \frac{\frac{8}{3}-1}{\frac{5}{4}} = \frac{\frac{5}{3}}{\frac{5}{4}}$$

$$= \frac{5}{3}\times\frac{4}{5} = \frac{4}{3}$$

= R.H.S.

Hence, $\dfrac{4\cot^2 60^o + \sec^2 30^o - 2\sin^2 45^o}{\cos^2 30^o + \cos^2 45^o} = \dfrac{4}{3}$

Example 23.7: If $\theta = 30^o$, verfity that

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Solution: For $\theta = 30^o$

L.H.S. = $\tan 2\theta$

$= \tan(2\times 30^o)$

$= \tan 60^o$

$= \sqrt{3}$

and R.H.S. = $\dfrac{2\tan\theta}{1-\tan^2\theta}$

$$= \frac{2\tan 30^o}{1-\tan^2 30^o}$$

$$= \frac{2.\left(\frac{1}{\sqrt{3}}\right)}{1-\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{\frac{2}{\sqrt{3}}}{1-\frac{1}{3}} = \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}}$$

$$= \frac{2}{\sqrt{3}} \times \frac{3}{2} = \sqrt{3}$$

$\therefore$ L.H.S. = R.H.S.

Hence, $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Example 23.8: Let $A = 30^o$. Verify that

$$\sin 3A = 3\sin A - 4\sin^3 A$$

Solution: For $A = 30^o$,

$$\begin{aligned} \text{L.H.S.} &= \sin 3A \\ &= \sin(3 \times 30^o) \\ &= \sin 90^o \\ &= 1 \end{aligned}$$

and $\quad$ $\text{R.H.S.} = 3\sin A - 4\sin^3 A$

$$\begin{aligned} &= 3\sin 30^o - 4\sin^3 30^o \\ &= 3 \times \frac{1}{2} - 4 \times \left(\frac{1}{2}\right)^3 \\ &= \frac{3}{2} - \frac{4}{8} \\ &= \frac{3}{2} - \frac{1}{2} \\ &= 1 \end{aligned}$$

$\therefore$ $\quad$ L.H.S. = R.H.S.

Hence, $\sin 3A = 3\sin A - 4\sin^3 A$

Example 23.9: Using the formula $\sin(A-B) = \sin A\cos B - \cos A\sin B$, find the value of $\sin 15^o$.

Solution: $\sin(A - B) = \sin A \cos B - \cos A \sin B$ ...(i)

Let $A = 45^o$ and $B = 30^o$

$\therefore$ From (i),

$$\sin(45^o - 30^o) = \sin 45^o \cos 30^o - \cos 45^o \sin 30^o$$

or $$\sin 15^o = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Hence, $\sin 15^o = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

***Remark:** In the above examples we can also take $A = 60^o$ and $B = 45^o$.*

Example 23.10: If $\sin(A + B) = 1$ and $\cos(A - B) = 1$, $0^o < A + B \le 90^o$, $A \ge B$, find A and B.

Solution: $\because$ $\sin(A + B) = 1 = \sin 90^o$

$\therefore$ $A + B = 90^o$ ...(i)

Again $\cos(A - B) = 1 = \cos 0^o$

$\therefore$ $A - B = 0^o$ ...(ii)

Adding (i) and (ii), we get

$2A = 90^o$ or $A = 45^o$

From (ii), we get

$B = A = 45^o$

Hence, $A = 45^o$ and $B = 45^o$

Example 23.11: If $\cos(20^o + x) = \sin 30^o$, find x.

Solution: $\cos(20^o + x) = \sin 30^o = \dfrac{1}{2} = \cos 60^o$ $\quad ...\left(\because \cos 60^o = \dfrac{1}{2}\right)$

$\therefore$ $20^o + x = 60^o$

or $x = 60^o - 20^o = 40^o$

Hence, $x = 40^o$

Example 23.12: In ΔABC, right angled at B, if BC = 5 cm, $\angle BAC = 30^o$, find the length of the sides AB and AC.

Solution: We are given $\angle BAC = 30^o$ i.e., $\angle A = 30^o$

and BC = 5 cm

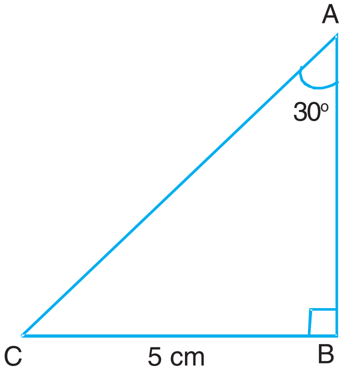


Fig. 23.4

Now $\sin A = \frac{BC}{AC}$

or $\sin 30^o = \frac{5}{AC}$

or $\frac{1}{2} = \frac{5}{AC}$

$\therefore$ AC = 2 × 5 or 10 cm

By Pythagoras Theorem,

$$AB = \sqrt{AC^2 - BC^2}$$

$$= \sqrt{(10)^2 - 5^2} \text{ cm}$$

$$= \sqrt{75} \text{ cm}$$

$$= 5\sqrt{3} \text{ cm}$$

Hence AC = 10 cm and AB = $5\sqrt{3}$ cm.

Example 23.13: In ΔABC, right angled at C, AC = 4 cm and AB = 8 cm. Find $\angle A$ and $\angle B$.

Solution: We are given, AC = 4 cm and AB = 8 cm

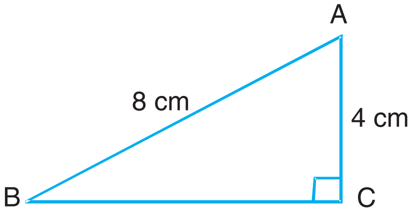


Fig. 23.5

Now $\sin B = \frac{AC}{AB}$

$= \frac{4}{8}$ or $\frac{1}{2}$

$\therefore B = 30^o$ ...$\left[\because \sin 30^o = \frac{1}{2}\right]$

Now $\angle A = 90^o - \angle B$ ....$\left[\because \angle A + \angle B = 90^o\right]$

$= 90^o - 30^o$

$= 60^o$

Hence, $\angle A = 60^o$ and $\angle B = 30^o$

Example 23.14: ΔABC is right angled at B. If $\angle A = \angle C$, find the value of

(i) sin A cos C + cos A sin C

(ii) sin A sin B + cos A cos B

Solution: We are given that in ΔABC,

$\angle B = 90^o$

$\therefore \quad \angle A + \angle C = 180^o - 90^o \qquad ...(\because \angle A + \angle B + \angle C = 180^o)$

$= 90^o$

Also it is given that $\angle A = \angle C$

$\therefore \quad \angle A = \angle C = 45^o$

(i) sin A cos C + cos A sin C

$$= \sin 45^o \cos 45^o + \cos 45^o \sin 45^o$$

$$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

(ii) sin A sin B + cos A cos B

$$= \sin 45^o \sin 90^o + \cos 45^o \cos 90^o$$

$$= \frac{1}{\sqrt{2}} \times 1 + \frac{1}{\sqrt{2}} \times 0$$

$$= \frac{1}{\sqrt{2}}$$

Example 23.15: Find the value of x if $\tan 2x - \sqrt{3} = 0$.

Solution: We are given

$\tan 2x - \sqrt{3} = 0$

or $\quad \tan 2x = \sqrt{3} = \tan 60^o$

$\therefore\ 2x = 60^\circ$

or $x = 30^\circ$

Hence value of x is 30°.

CHECK YOUR PROGRESS 23.1

1. Evaluate each of the following:

   (i) $\sin^2 60^\circ + \cos^2 45^\circ$

   (ii) $2\sin^2 30^\circ - 2\cos^2 45^\circ + \tan^2 60^\circ$

   (iii) $4\sin^2 60^\circ + 3\tan^2 30^\circ - 8\sin^2 45^\circ \cos 45^\circ$

   (iv) $4(\sin^4 30^\circ + \cos^4 60^\circ) - 3(\cos^2 45^\circ - 2\sin^2 45^\circ)$

   (v) $\dfrac{\tan 45^\circ}{\operatorname{cosec} 30^\circ} + \dfrac{\sec 60^\circ}{\cot 45^\circ} - \dfrac{5\sin 90^\circ}{2\cos 0^\circ}$

   (vi) $\dfrac{5\cos^2 60^\circ + 4\sec^2 30^\circ - \tan^2 45^\circ}{\sin^2 30^\circ + \cos^2 30^\circ}$

2. Verify each of the following:

   (i) $\operatorname{cosec}^3 30^\circ \times \cos 60^\circ \times \tan^3 45^\circ \times \sin^2 90^\circ \times \sec^2 45^\circ \times \cot 30^\circ = 8\sqrt{3}$

   (ii) $\tan^2 30^\circ + \dfrac{1}{2}\sin^2 45^\circ + \dfrac{1}{3}\cos^2 30^\circ + \cot^2 60^\circ = \dfrac{7}{6}$

   (iii) $\cos^2 60^\circ - \sin^2 60^\circ = -\cos 60^\circ$

   (iv) $4(\sin^4 30^\circ + \cos^4 60^\circ) - 3(\cos^2 45^\circ - \sin^2 90^\circ) = 2$

   (v) $\dfrac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ} = \tan 30^\circ$

3. If $\angle A = 30^\circ$, verify each of the following:

   (i) $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$

   (ii) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

   (iii) $\cos 3A = 4\cos^3 A - 3\cos A$

4. If $A = 60^\circ$ and $B = 30^\circ$, verify each of the following:

   (i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

   (ii) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

5. Taking $2A = 60^\circ$, find $\sin 30^\circ$ and $\cos 30^\circ$, using $\cos 2A = 2\cos^2 A - 1$.

6. Using the formula $\cos(A + B) = \cos A \cos B - \sin A \sin B$, evaluate $\cos 75^\circ$.

7. If $\sin(A - B) = \dfrac{1}{2}$, $\cos(A + B) = \dfrac{1}{2}$, $0^\circ < A + B < 90^\circ$, $A > B$, find A and B.

8. If $\sin(A + 2B) = \dfrac{\sqrt{3}}{2}$ and $\cos(A + 4B) = 0$, find A and B.

9. In ΔPQR right angled at Q, $PQ = 5$ cm and $\angle R = 30^\circ$, find QR and PR.

10. In ΔABC, $\angle B = 90^\circ$, $AB = 6$ cm and $AC = 12$ cm. Find $\angle A$ and $\angle C$.

11. In ΔABC, $\angle B = 90^\circ$. If $A = 30^\circ$, find the value of $\sin A \cos B + \cos A \sin B$.

12. If $\cos(40^\circ + 2x) = \sin 30^\circ$, find x.

Choose the correct alternative for each of the following (13-15):

13. The value of $\sec 30^\circ$ is

    (A) 2 (B) $\dfrac{\sqrt{3}}{2}$ (C) $\dfrac{2}{\sqrt{3}}$ (D) $\sqrt{2}$

14. If $\sin 2A = 2 \sin A$, then A is

    (A) 30° (B) 0° (C) 60° (D) 90°

15. $\dfrac{2\tan 60^o}{1 + \tan^2 60^o}$ is equal to

    (A) $\sin 60^\circ$ (B) $\sin 30^\circ$ (C) $\cos 60^\circ$ (D) $\tan 60^\circ$

23.5 APPLICATION OF TRIGONOMETRY

We have so far learnt to define trigonometric ratios of an angle. Also, we have learnt to determine the values of trigonometric ratios for the angles of 30°, 45° and 60°. We also know those trigonometric ratios for angles of 0° and 90° which are well defined. In this section, we will learn how trigonometry can be used to determine the distance between the objects or the distance between the objects or the heights of objects by taking examples from day to day life. We shall first define some terms which will be required in the study of heights and distances.

23.5.1 Angle of Elevation

When the observer is looking at an object (P) which is at a greater height than the observer (A), he has to lift his eyes to see the object and an angle of elevation is formed between the line of sight joining the observer's eye to the object and the borizontal line. In Fig. 23.6, A is the observer, P is the object, AP is the line of sight and AB is the horizontal line, then $\angle\theta$ is the angle of elevation.

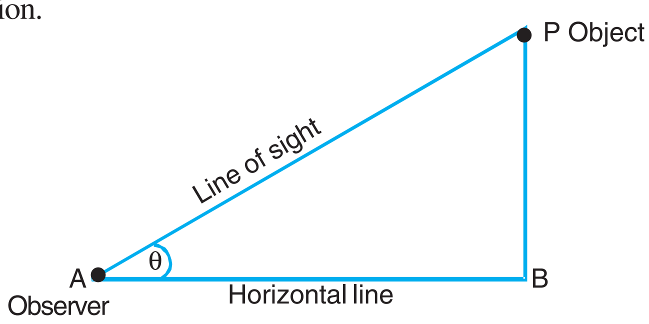


Fig. 23.6

23.5.2 Angle of Depression

When the observer (A) (at a greater height), is looking at an object (at a lesser height), the angle formed between the line of sight and the horizontal line is called an angle of depression. In Fig. 23.7, AP is the line of sight and AK is the horizontal line. Here α is the angle of depression.

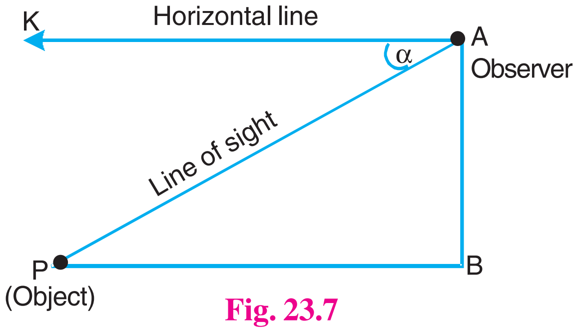


Fig. 23.7

Example 23.16: A ladder leaning against a window of a house makes an angle of 60° with the ground. If the length of the ladder is 6 m, find the distance of the foot of the ladder from the wall.

Solution: Let AC be a ladder leaning against the wall, AB making an angle of 60° with the level ground BC.

Here AC = 6 m ...(Given)

Now in right angled ΔABC,

$$\frac{BC}{AC} = \cos 60^{\circ}$$

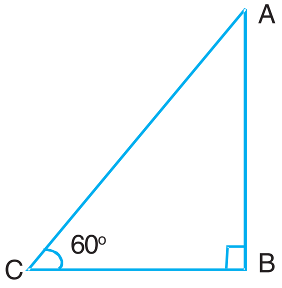


Fig. 23.8

or $\frac{BC}{6} = \frac{1}{2}$

or $BC = \frac{1}{2} \times 6$ or 3 m

Hence, the foot of the ladder is 3 m away from the wall.

Example 23.17: The shadow of a vertical pole is $\frac{1}{\sqrt{3}}$ of its height. Show that the sun's elevation is 60°.

Solution: Let AB be vertical pole of height h units and BC be its shadow.

Then $BC = h \times \frac{1}{\sqrt{3}}$ units

Let θ be the sun's elevation.

Then in right ΔABC,

$$\tan\theta = \frac{AB}{BC} = \frac{h}{h/\sqrt{3}} = \sqrt{3}$$

or $\tan\theta = \tan 60^\circ$

$\therefore$ $\theta = 60^\circ$

Hence, the sun's elevation is 60°.

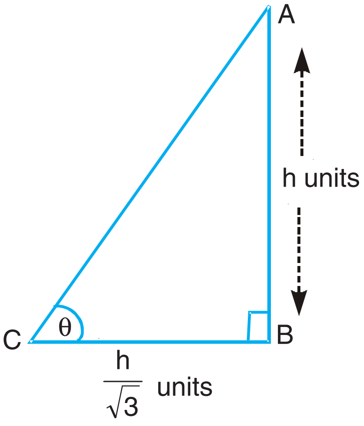


Fig. 23.9

Example 23.18: A tower stands vertically on the ground. The angle of elevation from a point on the ground, which is 30 m away from the foot of the tower is 30°. Find the height of the tower. (Take $\sqrt{3} = 1.73$)

Solution: Let AB be the tower h metres high.

Let C be a point on the ground, 30 m away from B, the foot of the tower

$\therefore$ $BC = 30$ m

Then by question, $\angle ACB = 30^\circ$

Now in right ΔABC,

$$\frac{AB}{BC} = \tan 30^\circ$$

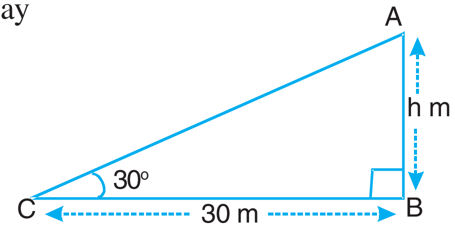


Fig. 23.10

or $$\frac{h}{30} = \frac{1}{\sqrt{3}}$$

∴ $$h = \frac{30}{\sqrt{3}} \text{ m}$$

$$= \frac{30}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \text{ m}$$

$$= 10\sqrt{3} \text{ m}$$

$$= 10 \times 1.73 \text{ m}$$

$$= 17.3 \text{ m}$$

Hence, height of the tower is 17.3 m.

Example 23.19: A balloon is connected to a meterological ground station by a cable of length 100 m inclined at 60° to the horizontal. Find the height of the balloon from the ground assuming that there is no slack in the cable.

Solution: Let A be the position of the balloon, attached to the cable AC of length 100 m. AC makes an angle of 60° with the level ground BC.

Let AB, the height of the balloon be h metres

Now in right ΔABC,

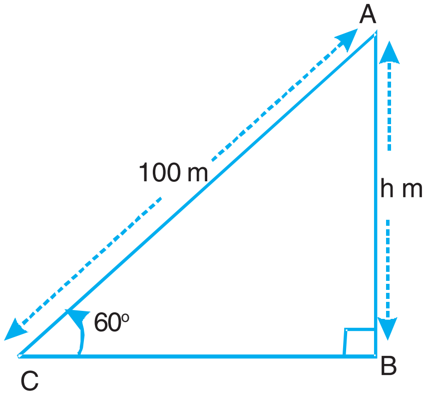


Fig. 23.11

$$\frac{AB}{AC} = \sin 60^o$$

or $$\frac{h}{100} = \frac{\sqrt{3}}{2}$$

or $$h = 50\sqrt{3}$$

$$= 50 \times 1.732$$

$$= 86.6$$

Hence, the balloon is at a height of 86.6 metres.

Example 23.20: The upper part of a tree is broken by the strong wind. The top of the tree makes an angle of 30° with the horizontal ground. The distance between the base of the tree and the point where it touches the ground is 10 m. Find the height of the tree.

Solution: Let AB be the tree, which was broken at C, by the wind and the top A of the tree touches the ground at D, making an angle of 30° with BD and BD= 10 m.

Let BC = x metres

Now in right ΔCBD,

$$\frac{BC}{BD} = \tan 30^\circ$$

or $$\frac{x}{10} = \frac{1}{\sqrt{3}}$$

or $$x = \frac{10}{\sqrt{3}} \text{ m} \qquad \text{...(i)}$$

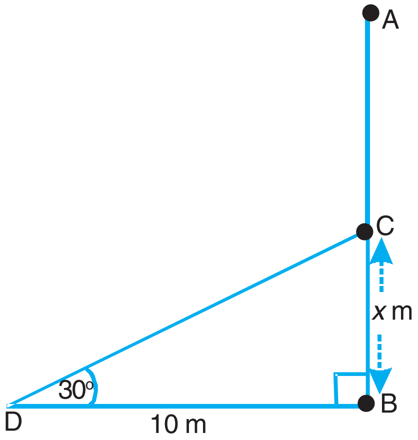


Fig. 23.12

Again in right ΔCBD,

$$\frac{BC}{DC} = \sin 30^\circ$$

or $$\frac{x}{DC} = \frac{1}{2}$$

or $$DC = 2x$$

$$= \frac{20}{\sqrt{3}} \text{ m} \qquad \text{...[By (i)]}$$

$$\therefore \quad AC = DC = \frac{20}{\sqrt{3}} \qquad \text{...(ii)}$$

Now height of the tree $= BC + AC$

$$= \left(\frac{10}{\sqrt{3}} + \frac{20}{\sqrt{3}}\right)$$

$$= \frac{30}{\sqrt{3}} \text{ or } 10\sqrt{3} \text{ m}$$

$$= 17.32 \text{ m}$$

Hence height of the tree = 17.32 m

Example 23.21: The shadown of a tower, when the angle of elevation of the sun is 45° is found to be 10 metres longer than when it was 60°. Find the height of the tower.

Solution: Let AB be the tower h metres high and C and D be the two points where the angles of elevation are 45° and 60° respectively.

Then CD = 10 m, $\angle ACB = 45^\circ$ and $\angle ADB = 60^\circ$

Let BD be x metres.

Then BC = BD + CD = (x + 10) m

Now in rt. $\angle$d ΔABC,

$$\frac{AB}{BC} = \tan 45^\circ$$

or $$\frac{h}{x+10} = 1$$

∴ $$x = (h - 10) \text{ m} \qquad ...(i)$$

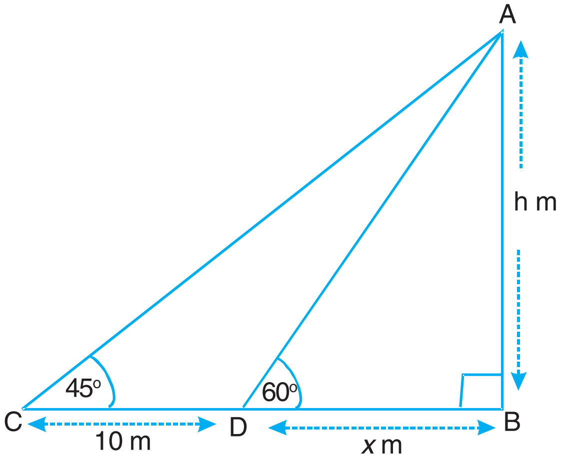


Fig. 23.13

Again in rt $\angle$d ΔABD,

$$\frac{AB}{BD} = \tan 60^\circ$$

or $$\frac{h}{x} = \sqrt{3}$$

or $$h = \sqrt{3}\,x \qquad ...(ii)$$

From (i) and (ii), we get

$$h = \sqrt{3}\,(h - 10)$$

or $$h = \sqrt{3}\,h - 10\sqrt{3}$$

or $$(\sqrt{3} - 1)h = 10\sqrt{3}$$

∴ $$h = \frac{10\sqrt{3}}{\sqrt{3}-1}$$

$$= \frac{10\sqrt{3}}{\sqrt{3}-1} \times \frac{(\sqrt{3}+1)}{(\sqrt{3}+1)} = \frac{10\sqrt{3}(\sqrt{3}+1)}{2}$$

$$= 5\sqrt{3}(\sqrt{3} + 1) = 15 + 5 \times 1.732 = 15 + 8.66 = 23.66$$

Hence, height of the tower is 23.66 m.

Example 23.22: An aeroplane when 3000 m high passes vertically above another aeroplane at an instant when the angles of elevation of the two aeroplanes from the same point on the ground are 60° and 45° respectively. Find the vertical distance between the two planes.

Solution: Let O be the point of observation.

Let P and Q be the two planes

Then AP = 3000 m and $\angle AOQ = 45^\circ$

and $\angle AOP = 60^\circ$

In rt. $\angle$d ΔQAO,

$$\frac{AQ}{AO} = \tan 45^\circ = 1$$

or AQ = AO ...(i)

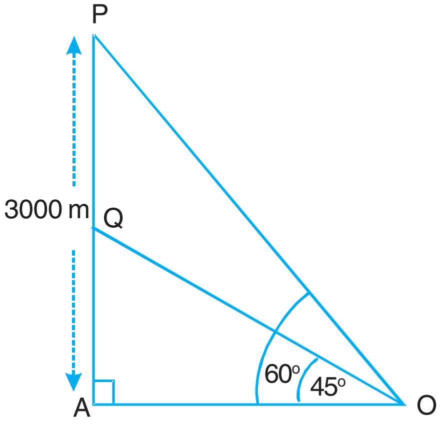


Fig. 23.14

Again in rt. $\angle$d ΔPAO,

$$\frac{PA}{AO} = \tan 60^\circ = \sqrt{3}$$

$$\therefore \quad \frac{3000}{AO} = \sqrt{3} \text{ or } AO = \frac{3000}{\sqrt{3}} \quad ...(ii)$$

From (i) and (ii), we get

$$AQ = \frac{3000}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = 1000\sqrt{3} = 1732 \text{ m}$$

$\therefore$ PQ = AP – AQ = (3000 – 1732) m = 1268 m

Hence, the required distance is 1268 m.

Example 23.23: The angle of elevation of the top of a building from the foot of a tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 50 m high, find the height of the building.

Solution: Let PQ be the tower 50 m high and AB be the building x m high.

Then $\angle AQB = 30^\circ$ and $\angle PBQ = 60^\circ$

In rt. $\angle$d ΔABQ, $\frac{x}{BQ} = \tan 30^\circ$ ...(i)

and in rt. $\angle$d ΔPQB, $\frac{PQ}{BQ} = \tan 60^\circ$

or $$\frac{50}{BQ} = \tan 60^\circ \quad ...(ii)$$

Dividing (i) by (ii), we get,

$$\frac{x}{50} = \frac{\tan 30^\circ}{\tan 60^\circ} = \frac{1}{3}$$

or $$x = \frac{50}{3} = 16.67$$

Hence, height of the building is 16.67 m.

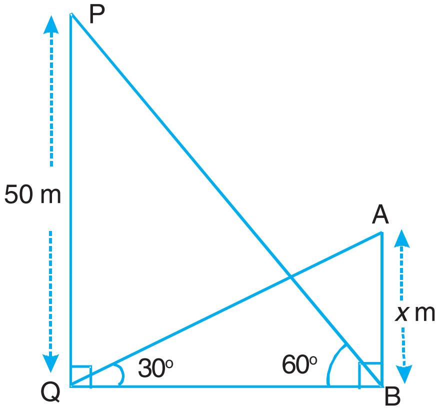


Fig. 23.15

Example 23.24: A person standing on the bank of a river observes that the angle of elevation of the top of a tree standing on the opposite bank is 60°. When he moves 40 metres away from the bank, he finds the angle be 30°. Find the height of the tree and the width of the river.

Solution: Let AB be a tree of height *h* metres.

Let BC = *x* metres represents the width of the river.

Let C and D be the two points where the tree subtends angles of 60° and 30° respectively

In right ΔABC,

$$\frac{AB}{BC} = \tan 60^\circ$$

or $$\frac{h}{x} = \sqrt{3}$$

or $$h = \sqrt{3}\,x \qquad ...(i)$$

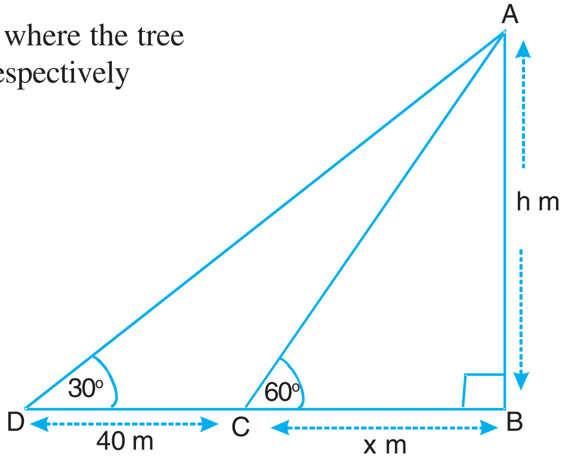


Fig. 23.16

Again in right ΔABD,

$$\frac{AB}{BD} = \tan 30^\circ$$

or $$\frac{h}{x+40} = \frac{1}{\sqrt{3}} \qquad ...(ii)$$

From (i) and (ii), we get,

$$\frac{\sqrt{3}x}{x+40} = \frac{1}{\sqrt{3}}$$

or $3x = x + 40$

or $2x = 40$

$\therefore$ $x = 20$

$\therefore$ From (i), we get

$$h = \sqrt{3} \times 20 = 20 \times 1.732$$

$$= 34.64$$

Hence, width of the river is 20 m and height of the tree is 34.64 metres.

Example 23.25: Standing on the top of a tower 100 m high, Swati observes two cars on the opposite sides of the tower. If their angles of depression are 45° and 60°, find the distance between the two cars.

Solution: Let PM be the tower 100 m high. Let A and B be the positions of the two cars. Let the angle of depression of car at A be 60° and of the car at B be 45° as shown in Fig. 23.17.

Now $\angle QPA = 60^\circ = \angle PAB$

and $\angle RPB = 45^\circ = \angle PBA$

In right ΔPMB,

$$\frac{PM}{MB} = \tan 45^\circ$$

or $$\frac{100}{MB} = 1$$

or $MB = 100$ m ...(i)

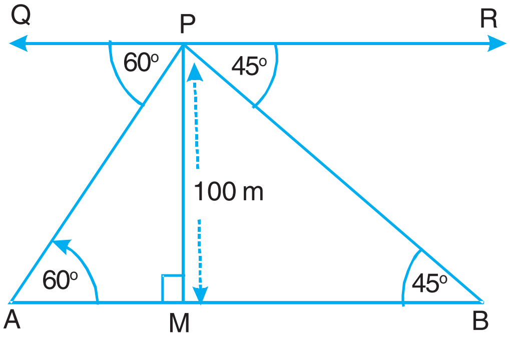


Fig. 23.17

Also in right ΔPMA,

$$\frac{PM}{MA} = \tan 60^\circ$$

or $$\frac{100}{MA} = \sqrt{3}$$

$\therefore$ $$MA = \frac{100}{\sqrt{3}}$$

$$= \frac{100\sqrt{3}}{3}$$

$$= \frac{100 \times 1.732}{3}$$

$$= 57.74$$

$\therefore$ MA = 57.74 m ...(ii)

Hence, the distance between the two cars

$$= MA + MB$$

$$= (57.74 + 100) \text{ m} \qquad \text{[By (i) and (ii)]}$$

$$= 157.74 \text{ m}$$

Example 23.26: Two pillars of equal heights are on either side of a road, which is 100 m wide. At a point on the road between the pillars, the angles of elevation of the top of the pillars are 60° and 30° respectively. Find the position of the point between the pillars and the height of each pillar.

Solution: Let AB and CD be two pillars each of height h metres. Let O be a point on the road. Let BO = x metres, then

$$OD = (100 - x) \text{ m}$$

By question, $\angle AOB = 60^\circ$ and $\angle COD = 30^\circ$

In right ΔABO,

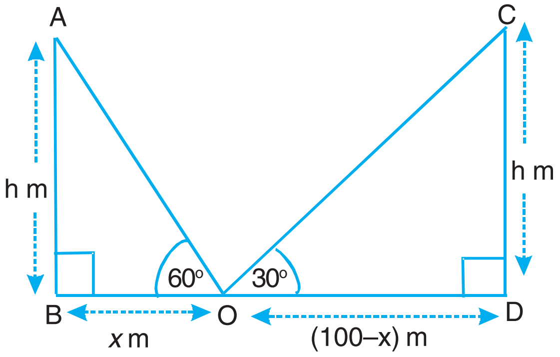


Fig. 23.18

$$\frac{AB}{BO} = \tan 60^\circ$$

or $$\frac{h}{x} = \sqrt{3}$$

or $$h = \sqrt{3}\ x \qquad \text{...(i)}$$

In right ΔCDO,

$$\frac{CD}{OD} = \tan 30^\circ$$

or $$\frac{h}{100 - x} = \frac{1}{\sqrt{3}} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\frac{\sqrt{3}x}{100 - x} = \frac{1}{\sqrt{3}}$$

or $3x = 100 - x$

or $4x = 100$

∴ $x = 25$

∴ From (i), we get $h = \sqrt{3} \times 25 = 1.732 \times 25$ or 43.3

∴ The required point from one pillar is 25 metres and 75 m from the other.

Height of each pillar = 43.3 m.

Example 23.27: The angle of elevation of an aeroplane from a point on the ground is 45°. After a flight of 15 seconds, the elevation changes to 30°. If the aeroplane is flying at a constant height of 3000 metres, find the speed of the plane.

Solution: Let A and B be two positions of the plane and let O be the point of observation. Let OCD be the horizontal line.

Then $\angle AOC = 45^o$ and $\angle BOD = 30^o$

By question, AC = BD = 3000 m

In rt ∠d ΔACO,

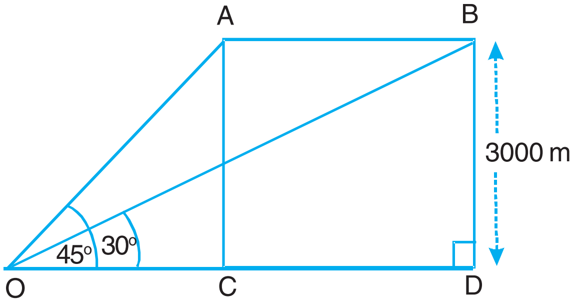


Fig. 23.19

$$\frac{AC}{OC} = \tan 45^o$$

or $$\frac{3000}{OC} = 1$$

or $OC = 3000$ m ...(i)

In rt ∠d ΔBDO,

$$\frac{BD}{OD} = \tan 30^o$$

or $$\frac{3000}{OC + CD} = \frac{1}{\sqrt{3}}$$

or $3000\sqrt{3} = 3000 + CD$ ...[By (i)]

or $CD = 3000(\sqrt{3} - 1)$

$= 3000 \times 0.732$

$= 2196$

∴ Distance covered by the aeroplane in 15 seconds = AB = CD = 2196 m

∴ Speed of the plane $= \left(\frac{2196}{15} \times \frac{60 \times 60}{1000}\right)$ km/h

$= 527.04$ km/h

Example 23.28: The angles of elevation of the top of a tower from two points P and Q at distanes of *a* and *b* respectively from the base and in the same straight line with it are complementary. Prove that the height of the tower is $\sqrt{ab}$.

Solution: Let AB be the tower of height *h*, P and Q be the given points such taht PB = *a* and QB = *b*.

Let $\angle APB = \alpha$ and $\angle AQB = 90^o - \alpha$

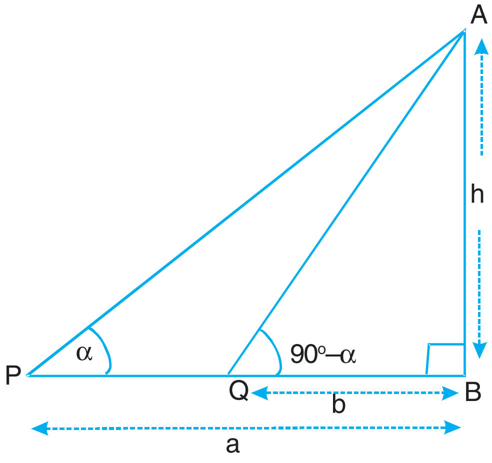


Fig. 23.20

Now in rt ∠d ΔABQ,

$$\frac{AB}{QB} = \tan(90^o - \alpha)$$

or $\frac{h}{b} = \cot\alpha$ ...(i)

and in rt ∠d ΔABP,

$$\frac{AB}{PB} = \tan\alpha$$

or $\frac{h}{a} = \tan\alpha$ ...(ii)

Multiplying (i) and (ii), we get

$$\frac{h}{b} \times \frac{h}{a} = \cot\alpha \, . \tan\alpha = 1$$

or $h^2 = ab$

or $h = \sqrt{ab}$

Hence, height of the tower is $\sqrt{ab}$.

CHECK YOUR PROGRESS 23.2

1. A ladder leaning against a vertical wall makes an angle of 60° with the ground. The foot of the ladder is at a distance of 3 m from the wall. Find the length of the ladder.

2. At a point 50 m away from the base of a tower, an observer measures the angle of elevation of the top of the tower to be 60^o. Find the height of the tower.
3. The angle of elevation of the top of the tower is 30^o, from a point 150 m away from its foot. Find the height of the tower.
4. The string of a kite is 100 m long. It makes an angle of 60^o with the horizontal ground. Find the height of the kite, assuming that there is no slack in the string.
5. A kite is flying at a height of 100 m from the level ground. If the string makes an angle of 60^o with a point on the ground, find the length of the string assuming that there is no slack in the string.
6. Find the angle of elevation of the top of a tower which is $100\sqrt{3}$ m high, from a point at a distance of 100 m from the foot of the tower on a horizontal plane.
7. A tree 12 m high is broken by the wind in such a way that its tip touches the ground and makes an angle of 60^o with the ground. At what height from the ground, the tree is broken by the wind?
8. A tree is broken by the storm in such way that its tip touches the ground at a horizontal distance of 10 m from the tree and makes an angle of 45^o with the ground. Find the height of the tree.
9. The angle of elevation of a tower at a point is 45^o. After going 40 m towards the foot of the tower, the angle of elevation becomes 60^o. Find the height of the tower.
10. Two men are on either side of a cliff which is 80 m high. They observe the angles of elevation of the top of the cliff to be 30^o and 60^o respectively. Find the distance between the two men.
11. From the top of a building 60 m high, the angles of depression of the top and bottom of a tower are observed to be 45^o and 60^o respectively. Find the height of the tower and its distance from the building.
12. A ladder of length 4 m makes an angle of 30^o with the level ground while leaning against a window of a room. The foot of the ladder is kept fixed on the same point of the level ground. It is made to lean against a window of another room on its opposite side, making an angle of 60^o with the level ground. Find the distance between these rooms.
13. At a point on the ground distant 15 m from its foot, the angle of elevation of the top of the first storey is 30^o. How high the second storey will be, if the angle of elevation of the top of the second storey at the same point is 45^o?
14. An aeroplane flying horizontal 1 km above the ground is observed at an elevation of 60^o. After 10 seconds its elevation is observed to be 30^o. Find the speed of the aeroplane.

15. The angle of elevation of the top of a building from the foot of a tower is 30^o and the angle of elevation of the top of the tower from the foot of the building is 60^o. If the tower is 50 m high, find the height of the building.

LET US SUM UP

- Table of values of Trigonometric Ratios

θ Trig. ratio	0^o	30^o	45^o	60^o	90^o
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cot θ	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
cosec θ	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined

Supportive website:

- http://www.wikipedia.org
- http://mathworld:wolfram.com

TERMINAL EXERCISE

1. Find the value of each of the following:

   (i) $4\cos^2 60^o + 4\sin^2 45^o - \sin^2 30^o$

   (ii) $\sin^2 45^o - \tan^2 45^o + 3(\sin^2 90^o + \tan^2 30^o)$

(iii) $\dfrac{5\sin^2 30^\circ + \cos^2 45^\circ - 4\tan^2 30^\circ}{2\sin^2 30^\circ \cos^2 30^\circ + \tan 45^\circ}$

(iv) $\dfrac{\cot 45^\circ}{\sec 30^\circ + \operatorname{cosec} 30^\circ}$

2. Prove each of the following:

   (i) $2\cot^2 30^\circ - 2\cos^2 60^\circ - \dfrac{3}{4}\sin^2 45^\circ - 4\sec^2 30^\circ = -\dfrac{5}{24}$

   (ii) $2\sin^2 30^\circ + 2\tan^2 60^\circ - 5\cos^2 45^\circ = 4$

   (iii) $\cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

   (iv) $\dfrac{\cot 30^\circ \cot 60^\circ - 1}{\cot 30^\circ + \cot 60^\circ} = \cot 90^\circ$

3. If $\theta = 30^\circ$, verify that

   (i) $\sin 2\theta = 2\sin\theta\cos\theta$

   (ii) $\cos 2\theta = 1 - 2\sin^2\theta$

   (iii) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

4. If $A = 60^\circ$ and $B = 30^\circ$, verify that

   (i) $\sin(A+B) \neq \sin A + \sin B$

   (ii) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

   (iii) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

   (iv) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

   (v) $\tan A = \dfrac{\sqrt{1-\cos^2 A}}{\cos A}$

5. Using the formula $\cos(A-B) = \cos A\cos B + \sin A\sin B$, find the value of $\cos 15^\circ$.

6. If $\sin(A+B) = 1$ and $\cos(A-B) = \dfrac{\sqrt{3}}{2}$, $0^\circ < A+B \leq 90^\circ$, $A > B$, find A and B.

7. An observer standing 40 m from a building observes that the angle of elevation of the top and bottom of a flagstaff, which is surmounted on the building are 60° and 45° respectively. Find the height of the tower and the length of the flagstaff.

8. From the top of a hill, the angles of depression of the consecutive kilometre stones due east are found to be 60° and 30°. Find the height of the hill.

9. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 45°. Find the height of the tower.

10. A man on the top of a tower on the sea shore finds that a boat coming towards him takes 10 minutes for the angle of depression to change from 30° to 60°. How soon will the boat reach the sea shore?

11. Two boats approach a light-house from opposite directions. The angle of elevation of the top of the lighthouse from the boats are 30° and 45°. If the distance between these boats be 100 m, find the height of the lighthouse.

12. The shadow of a tower standing on a level ground is found to be $45\sqrt{3}$ m longer when the sun's altitude is 30° than when it was 60°. Find the height of the tower.

13. The horizontal distance between two towers is 80 m. The angle of depression of the top of the first tower when seen from the top of the second tower is 30°. If the height of the second tower is 160 m, find the height of the first tower.

14. From a window, 10 m high above the ground, of a house in a street, the angles of elevation and depression of the top and the foot of another house on opposite side of the street are 60° and 45° respectively. Find the height of the opposite house (Take $\sqrt{3}$ =1.73)

15. A statue 1.6 m tall stands on the top of a pedestal from a point on the gound, the angle of elevation of the top of the statue is 60° and from the same point, the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal.

ANSWERS TO CHECK YOUR PROGRESS

23.1

1. (i) $\frac{5}{4}$ (ii) $\frac{5}{2}$ (iii) 0 (iv) 2 (v) 0 (vi) $\frac{67}{12}$

5. $\sin 30^\circ = \frac{1}{2}$, $\cos 30^\circ = \frac{\sqrt{3}}{2}$

6. $\frac{\sqrt{3}-1}{2\sqrt{2}}$

7. $A = 45^\circ$ and $B = 15^\circ$

8. A = 30° and B = 15°

9. QR = $5\sqrt{3}$ and PR = 10 cm

10. $\angle A = 60^\circ$ and $\angle C = 30^\circ$

11. $\frac{\sqrt{3}}{2}$

12. $x = 10^\circ$

13. C

14. B

15. A

23.2

1. 6 m
2. 86.6 m
3. 86.6 m
4. 86.6 m
5. 115.46 m
6. 60°
7. 5.57 m
8. 24.14 m
9. 94.64 m
10. 184.75 m
11. 25.35 m
12. 5.46 m
13. 6.34 m
14. 415.66 km/h
15. 16.67 m

ANSWERS TO TERMINAL EXERCISE

1. (i) $\frac{11}{4}$ (ii) $\frac{7}{2}$ (iii) $\frac{40}{121}$ (iv) $\frac{\sqrt{3}}{2\left(\sqrt{3}+1\right)}$

5. $\frac{\sqrt{3}+1}{2\sqrt{2}}$

6. A = 60° and B = 30°

7. 40m , 29.28 m

8. 433 m

9. 19.124 m

10. 5 minutes

11. 36.6 m

12. 67.5 m

13. 113.8 m

14. 27.3 m

15. 2.18656 m

Secondary Course Mathematics

Practice Work-Trignometry

Maximum Marks: 25 **Time : 45 Minutes**

Instructions:

1. Answer all the questions on a separate sheet of paper.
2. Give the following informations on your answer sheet

   Name

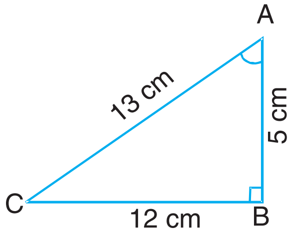
   Enrolment number

   Subject

   Topic of practice work

   Address
3. Get your practice work checked by the subject teacher at your study centre so that you get positive feedback about your performance.

1. In the adjoining figure, the value of sin A is 1



(A) $\frac{5}{13}$

(B) $\frac{12}{13}$

(C) $\frac{5}{12}$

(D) $\frac{13}{12}$

2. If 4 cot A = 3, then value of $\frac{\sin A - \cos A}{\sin A + \cos A}$ is 1

(A) $\frac{1}{7}$ (B) $\frac{6}{7}$

(C) $\frac{5}{6}$ (D) $\frac{3}{4}$

3. The value of sec 30^o is 1

(A) 2 (B) $\frac{\sqrt{3}}{2}$

(C) $\frac{2}{\sqrt{3}}$ (D) $\sqrt{2}$

4. In ΔABC, right angled at B, if AB = 6 cm and AC = 12 cm, then $\angle A$ is 1

(A) 60^o

(B) 30^o

(C) 45^o

(D) 15^o

5. The value of 1

$$\frac{\sin 36^o}{2\cos 54^o} - \frac{2\sec 41^o}{3\,\text{cosec}49^o} \text{ is}$$

(A) – 1

(B) $\frac{1}{6}$

(C) $-\frac{1}{6}$

(D) 1

6. If $\sin A = \frac{1}{2}$, show that 2

$3\cos A - 4\cos^3 A = 0$

7. Using the formula $\sin(A - B) = \sin A \cos B - \cos A \sin B$, find the value of sin $15^{o.}$ 2

8. Find the value of

$\tan 15^o \tan 25^o \tan 60^o \tan 65^o \tan 75^o$ 2

9. Show that $\sqrt{\dfrac{1+\sin A}{1-\sin A}} = \sec A + \tan A$ 2

10. If $\sin^2\theta + \sin\theta = 1$, then show that 2

$\cos^2\theta + \cos^4\theta = 1$

11. Prove that $\dfrac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1} = \dfrac{1+\cos A}{\sin A}$ 4

12. An observer standing 40 m from a building notices that the angles of elevation of the top and the bottom of a flagstaff surmounted on the building are 60° and 45° respectively. Find the height of the building and the flag staff. 6

MODULE 6

Statistics

The modern society is essentially data oriented. It is difficult to imagine any facet of our life untouched in newspapers, advertisements, magazines, periodicals and other forms of publicity over radio, television etc. These data may relate to cost of living, moritality rate, literacy rate, cricket averages, rainfall of different cities, temperatures of different towns, expenditures in varioius sectors of a five year plan and so on. It is, therefore, essential to know how to extract 'meaningful' information from such data. This extraction of useful or meaningful information is studied in the branch of mathematics called ***statistics.***

In the lesson on "Data and their Representations" the learner will be introduced to different types of data, collection of data, presentation of data in the form of frequency distributions, cumulative frequency tables, graphical representaitons of data in the form of bar charts (graphs), histograms and frequency polygons.

Sometimes, we are required to describe the data arithmetically, like describing mean age of a class of studens, mean height of a group of students, median score or model shoe size of a group. Thus, we need to find certain measures which summarise the main features of the data. In lesson on "measures of Central Tendency", the learner will be introduced to some measures of central tendency i.e., mean, median, mode of ungrouped data and mean of grouped data.

In the lesson on "Introduction to Probablity", the learner will get acquainted with the concept of theoretical probability as a measure of uncertainity, through games of chance like tossing a coin, throwing a die etc.

24

DATA AND THEIR REPRESENTATIONS

Statistics is a special and an important branch of mathematics which deals mainly with data and their representations. In this lesson, we shall make a beginning of this study of this branch of mathematics with collection, classification, presentation and analysis of data. We shall study how to classify the given data into ungrouped as well as grouped frequency distributions. We shall also learn about cumulative frequency of a class and cumulative frequency table.

Further we shall learn graphical representation of data in the form of bar charts, histograms and frequency polygons.

OBJECTIVES

After studying this lesson, you will be able to

- *know meaning of 'statistics' in singular and plural form;*
- *differentiate between primary and secondary data;*
- *understand the meaning of a class, class mark, class limits, discrete and continuous data, frequency of a class, class size or class width through examples;*
- *condense and represent data into a frequency table;*
- *form a cumulative frequency table of a frequency distribution;*
- *draw a bar chart or bar graph of a frequency distribution;*
- *draw a bar chart or bar graph for the given data;*
- *draw a histogram and frequency polygon for a given continuous data;*
- *read and interpret given bar graphs, histograms.*

EXPECTED BACKGROUND KNOWLEDGE

- Writing of numbers in increasing/decreasing order.

- Finding average of two numbers.
- Plotting of points in a plane with respect to two perpendicular axes
- Idea of ratio and proportion.

24.1 STATISTICS AND STATISTICAL DATA

In our day to day life, we come across statements such as:

1. This year the results of the school will be better.
2. The price of petrol/diesel may go up next month.
3. There is likelihood of heavy rains in the evening.
4. The patient may recover soon from illness, etc.

Concentrate on the above statements:

- The first statement can be from a teacher or the head of an institution. It shows that he/she has observed the performance of the present batch of students in comparison with the earlier ones.
- The second statement may be from a person who has seen the trend of increasing of oil prices from a newspaper.
- The third statement can be from a person who has been observing the weather reports in meteorological department. If so, then one can expect that it is based on some sound observations and analysis of the weather reports.
- The last statement can be from a doctor which is based on his/her observations and analysis.

The reliability of the statements such as given above, depends upon the individual's capacity for observation and analysis based on some numerical data. **Statistics is the science which deals with the collection, organisation, analysis and interpretation of the numerical data.**

Collection and analysis of numerical data is essential in studying many problems such as the problem of economic development of the country, educational development, the problem of health and population, the problem of agricultural development etc.

The word 'statistics' has different meanings in different contexts. Obseve the following sentences:

1. May I have the latest copy of "Educational Statistics of India".
2. I like to study statistics. It is an interesting subject.

In the first sentence, statistics is used in a **plural** sense, meaning numerical data. These may include a number of schools/colleges/institutions in India, literacy rates of states etc.

In the second sentence, the word 'statistics' is used as a **singular** noun, meaning the subject which deals with classification, tabulation/organisation, analysis of data as well as drawing of meaningful conclusions from the data.

24.2 COLLECTION OF DATA

In any field of investigation, the first step is to collect the data. It is these data that will be analysed by the investigator or the statistician to draw inferences. It is, therefore, of utmost importance that these data be reliable and relevant and collected according to a plan or design which must be laid out in advance.

Data are said to be **primary** if the investigator himself is responsible for the collection of data. Some examples of primary data are: voters' lists, data collected in census-questionnaire etc.

It is not always possible for an investigator to collect data due to lack of time and resources. In that case, he/her may use data collected by other governmental or private agency in the form of published reports. They are called **secondary data**. Data may be primary for one individual or agency but it becomes secondary for other using the same data.

Since these data are collected for a purpose other than that of the original investigators, the user may lose some details or the data may not be all that relevant to his/her study. Therefore, such data must be used with great care.

CHECK YOUR PROGRESS 24.1

1. Fill in the blanks with suitable word(s) so that the following sentences give the proper meaning:

   (a) Statistics, in singular sense, means the subject which deals with _______, _____, analysis of data as well as drawing of meaningful _______ from the data.

   (b) Statistics is used, in a plural sense, meaning _______________.

   (c) The data are said to be __________ if the investigator himself is responsible for its collection.

   (d) Data taken from governmental or private agencies in the form of published reports are called __________ data.

   (e) Statistics is the science which deals with collection, organisation, analysis and interpretation of the ____________.

2. Javed wanted to know the size of shoes worn by the maximum number of persons in a locality. So, he goes to each and every house and notes down the information on a sheet. The data so collected is an example of ____________ data.

3. To find the number of absentees in each day of each class from I to XII, you collect the information from the school records. The data so collected is an example of _______ data.

24.3 PRESENTATION OF DATA

When the work of collection of data is over, the next step to the investigator is to find ways to condense and organise them in order to study their salient features. Such an arrangement of data is called **presentation of data.**

Suppose there are 20 students in a class. The marks obtained by the students in a mathematics test (out of 100) are as follows:

45, 56, 61, 56, 31, 33, 70, 61, 76, 56,

36, 59, 64, 56, 88, 28, 56, 70, 64, 74

The data in this form is called **raw data**. Each entry such as 45, 56 etc. is called a **value** or **observation.** By looking at it in this form, can you find the highest and the lowest marks? What more information do you get?

Let us arrange these numbers in ascending order:

28, 31, 33, 36, 45, 56, 56, 56, 56, 56,

59, 61, 61, 64, 64, 70, 70, 74, 76, 88 ...(1)

Now you can get the following information:

(a) Highest marks obtained : 88

(b) Lowest marks obtained : 28

(c) Number of students who got 56 marks: 5

(d) Number of students who got marks more than 60 : 9

The data arranged in the form (1) above, are called **arrayed data**.

Presentation of data in this form is time cousuming, when the number of observations is large. To make the data more informative we can present these in a tabular form as shown below:

Marks in Mathematics of 20 students

Marks	Number of Students
28	1
31	1
33	1
36	1
45	1
56	5
59	1
61	2
64	2
70	2
74	1
76	1
88	1
Total	20

This presentation of the data in the form of a table is an improvement over the arrangement of numbers (marks) in an array, as it presents a clear idea of the data. From the table, we can easily see that 1 student has secured 28 marks, 5 students have secured 56 marks, 2 students have secured 70 marks, and so on. Number 1, 1, 1, 1, 1, 5, 2, ...are called respective **frequencies** of the observations (also called variate or variable) 28, 31, 33, 36, 45, 56, 70, ...

Such a table is claled a **frequency distribution table** for **ungrouped** data or simply **ungrouped frequency table**.

Note: When the number of observations is large, it may not be convenient to find the frequencies by simple counting. In such cases, we make use of bars (1), called **tally marks)** which are quite helpful in finding the frequencies.

In order to get a further condensed form of the data (when the number of observation is large), we classify the data into **classes** or **groups** or class intervals as below:

Step 1: We determine the **range** of the raw data i.e. the differenece between the maximum and minimum observations (values) occurring in the data. In the above example range is 88 – 28 = 60.

Step 2: We decide upon the number of classes or groups into which the raw data are to be grouped. There is no hard and fast rule for determining the number of classes, but generally there should not be less than 5 and not more than 15.

Step 3: We divide the range (it is 60 here) by the desired number of classes to determine the approximate **size** (or width) of a **class-interval**.In the above example, suppose we decide to have 9 classes. Than the size of each class is $\frac{60}{9} \approx 7$.

Step 4: Next, we set up the **class limits** using the size of the interval determined in Step 3. We make sure that we have a class to include the minimum as well as a class to include the maximum value occurring in the data. The classes should be non-overlapping, no gaps between the classes, and classes should be of the same size.

Step 5: We take each item (observation) from the data, one at a time, and put a tally mark (|) against the class to which it belongs. For the sake of convenience, we record the tally marks in bunches of five, the fifth one crossing the other four diagonally as 𝍸.

Step 6: By counting tally marks in each class, we get the frequency of that class. (obviously, the total of all frequencies should be equal to the total number of observations in the data)

Step 7: The frequency table should be given a proper title so as to convey exactly what the table is about.

Using the above steps, we obtain the following table for the marks obtained by 20 students.

Frequency Table of the marks obtained by 20 students in a mathematics test

Class Interval (Marks out of 100)	Tally Marks	Frequency
28-34	𝍷𝍷𝍷	3
35-41	𝍷	1
42-48	𝍷	1
49-55	–	0
56-62	𝍸 𝍷𝍷𝍷	8
63-69	𝍷𝍷	2
70-76	𝍷𝍷𝍷𝍷	4
77-83	–	0
84-90	𝍷	1
Total		**20**

The above table is called a **frequency distribution table** for grouped data or briefly, a **grouped frequency table**. The data in the above form are called **grouped data**.

In the above table, the class 28-34 includes the observations 28, 29, 30, 31, 32, 33 and 34; class 35-41 includes 35, 36, 37, 38, 39, 40 and 41 and so on. So, there is no overlapping.

For the class 28-34, 28 is called the **lower class limit** and 34, the **upper class limit**, and so on.

From this type of presentation, we can draw better conclusions about the data. Some of these are.

(i) The number of students getting marks from 28 to 34 is 3.

(ii) No students has got marks in the class 49-55, i.e., no students has got marks 49, 50, 51, 52, 53, 54 and 55.

(iii) Maximum number of students have got marks from 56 to 62 etc.

We can also group the same 20 observations into 9 groups 28-35, 35-42, 42-49, 49-56, 56-63, 63-70, 70-77, 77-84, 84-91 as shown in the following table.

It appears from classes 28-35 and 35-42, etc. that the observation 35 may belong to both those classes. But as you know, no observation could belong simultaneously to two classes. To avoid this, we adopt the convention that the common observation 35 belongs to the higher class, i.e. 35-42 (and **not** to 28-35). Similarly 42 belogs to 42-49 and so on. Thus, class 28-35 contains all observations which are greater than or equal to 28 but less than 35, etc.

Frequency Table of the marks obtained by 20 students in a mathematics test

Class Interval (Marks out of 100)	Tally Marks	Frequency
28-35	III	3
35-42	I	1
42-49	I	1
49-56	–	0
56-63	~~IIII~~ III	8
63-70	II	2
70-77	IIII	4
77-84	–	0
84-91	I	1
Total		**20**

Why do we prepare frequency distribution as given in the above table, it will be clear to you from the next example.

Now let us consider the following frequency distribution table which gives the weight of 50 students of a class:

Weight (in kg)	Number of Students
31-35	10
36-40	7
41-45	15
45-50	4
51-55	2
56-60	3
61-65	4
66-70	3
71-75	2
Total	**50**

Suppose two students of weights 35.5 kg and 50.54 kg are admitted in this class. In which class (interval) will we include them? Can we include 35.5 in class 31-35? In class 36-40?

No! The class 31-35 includes numbers upto 35 and the class 36-40, includes numbers from 36 onwards. So, there are gaps in between the upper and lower limits of two consecutive classes. To overcome this difficulty, we divide the intervals in such a way that the upper and lower limits of consecutive classes are the same. For this, we find the difference between the upper limit of a class and the lower limit of its succeeding class. We than add half of this difference to each of the upper limits and subtract the same from each of the lower limits. For example

Consider the classes 31-35 and 36-40

The lower limit of 36-40 is 36

The upper limit of 31-35 is 35

The difference $= 36 - 35 = 1$

So, half the difference $= \frac{1}{2} = 0.5$

So, the new class interval formed from 31-35 is $(31 - 0.5) - (35 + 0.5)$, i.e., $30.5 - 35.5$. Similarly, class 36-40 will be $(36 - 0.5) - (40 + 0.5)$, i.e., $35.5 - 40.5$ and so on.

This way, the new classes will be

30.5-35.5, 35.5-40.5, 40.5-45.5, 45.5-50.5, 50.5-55.5, 55.5-60.5, 60.5-65.5, 65.5-70.5 and 70.5-75.5. These are now continuous classes.

Note that the width of the class is again the same, i.e., 5. These changed limits are called

true class limits. Thus, for the class 30.5-35.5, 30.5 is the **true lower class limit** and 35.5 is the **true upper class limit**.

Can we now include the weight of the new students? In which classes?

Obviously, 35.5 will be included in the class 35.5-40.5 and 50.54 in the class 50.5-55.5 (Can you explain why?).

So, the new frequency distribution will be as follows:

Weight (in kg)	Number of Students
30.5-35.5	10
35.5-40.5	8
40.5-45.5	15
45.5-50.5	4
50.5-55.5	3
55.5-60.5	3
60.5-65.5	4
65.5-70.5	3
70.5-75.5	2
Total	**52**

35.5 included in the class

50.54 included in the class

Note: Here, in the above case, we could have also taken the classes as 30-35, 35-40, 40-45, ..., 65-70 and 70-75.

Example 24.1: Construct a frequency table for the following data which give the daily wages (in rupees) of 32 persons. Use class intervals of size 10.

110	184	129	141	105	134	136	176	155
145	150	160	160	152	201	159	203	146
177	139	105	140	190	158	203	108	129
118	112	169	140	185				

Solution: Range of data = 205 - 105 = 98

It is convenient, therefore, to have 10 classes each of size 10.

Frequency distribution table of the above data is given below:

Frequency table showing the daily wages of 32 persons

Daily wages (in Rs.)	Tally Marks	Number of persons or frequency
105-115	~~IIII~~	5
115-125	I	1
125-135	III	3
135-145	~~IIII~~	5
145-155	IIII	4
155-165	~~IIII~~	5
165-175	I	1
175-185	III	3
185-195	II	2
195-205	III	3
Total		**32**

Example 24.2: The heights of 30 students, (in centimetres) have been found to be as follows:

161	151	153	165	167	154
162	163	170	165	157	156
153	160	160	170	161	167
154	151	152	156	157	160
161	160	163	167	168	158

(i) Represent the data by a grouped frequency distribution table, taking the classes as 161-165, 166-170, etc.

(ii) What can you conclude about their heights from the table?

Solution:

(i) **Frequency distribution table showing heights of 30 students**

Height (in cm)	Tally Marks	Frequency
151-155	IIII II	7
156-160	IIII IIII	9
161-165	IIII III	8
166-170	IIII I	6
Total		**30**

(ii) One conclusion that we can draw from the above table is that more than 50% of the students (i.e., 16) are shorter than 160 cm.

CHECK YOUR PROGRESS 24.2

1. Give an example of a raw data and an arrayed data.

2. Heights (in cm) of 30 girls in Class IX are given below:

140	140	160	139	153	146	151	150	150	154
148	158	151	160	150	149	148	140	148	153
140	139	150	152	149	142	152	140	146	148

Determine the range of the data.

3. Differentiate between a primary data and secondary data.

4. 30 students of Class IX appeared for mathematics olympiad. The marks obtained by them are given as follows:

46	31	74	68	42	54	14	93	72	53
59	38	16	88	27	44	63	43	81	64
77	62	53	40	71	60	8	68	50	58

Construct a grouped frequency distribution of the data using the classes 0-9, 10-19 etc. Also, find the number of students who secured marks more than 49.

5. Construct a frequency table with class intervals of equal sizes using 250-270 (270 not included) as one of the class interval for the following data:

268	230	368	248	242	310	272	342
310	300	300	320	315	304	402	316
406	292	355	248	210	240	330	316
406	215	262	238				

6. Following is the frequency distribution of ages (in years) of 40 teachers in a school:

Age (in years)	Number of teachers
25-31	12
31-37	15
37-43	7
43-49	5
49-55	1
Total	40

(i) What is the class size?

(ii) What is the upper class limit of class 37-43?

(iii) What is the lower class limit of class 49-55?

24.4 CUMULATIVE FREQUENCY TABLE

Consider the frequency distribution table:

Weight (in kg)	Number of Students
30-35	10
35-40	7
40-45	15
45-50	4
50-55	2
55-60	3
60-65	4
65-70	3
70-75	2
Total	**50**

Now try to answer the following questions:

(i) How many students have their weights less than 35 kg?

(ii) How many students have their weights less than 50 kg?

(iii) How many students have their weights less than 60 kg?

(iv) How many students have their weights less than 70 kg?

Let us put the answers in the following way:

Number of students with weight:

Less than 35 kg : 10
Less than 40 kg : (10) + 7 = 17
Less than 45 kg : (10 + 7) + 15 =32
Less than 50 kg : (10 + 7 + 15) + 4 = 36
Less than 55 kg : (10 + 7 + 15 + 4) + 2 = 38
Less than 60 kg : (10 + 7 + 15 + 4 + 2) + 3 = 41
Less than 65 kg : (10 + 7 + 15 + 4 + 2 + 3) + 4 = 45
Less than 70 kg : (10 + 7 + 15 + 4 + 2 + 3 + 4) + 3 = 48
Less than 75 kg : (10 + 7 + 15 + 4 + 2 + 3 + 4 + 3) + 2 = 50

From the above, it is easy to see that answers to questions (i), (ii), (iii) and (iv) are 10, 36, 41 and 48 respectively.

The frequencies 10, 17, 32, 36, 38, 41, 48, 50 are called the **cumulative frequencies** of the respective classes. Obviously, the cumulative frequency of the last class, i.e., 70-75 is 50 which is the total number of observations (Here it is total number of students).

In the table under consideration, if we insert a column showing the cumulative frequency of each class, we get what we call **cumulative frequency distribution** or simply **cumulative frequency table** of the data.

Cumulative Frequency Distribution Table

Weight (in kg)	Number of students (frequency)	Cumulative frequency
0-35	10	10
35-40	7	17
40-45	15	32
45-50	4	36
50-55	2	38
55-60	3	41
60-65	4	45
65-70	3	48
70-75	2	50
Total	**50**	

Example 24.3: The following table gives the distribution of employees residig in a locality into different income groups

Income (per week) (in ₹)	Number of Employees
0-1000	12
1000-2000	35
2000-3000	75
3000-4000	225
4000-5000	295
5000-6000	163
6000-7000	140
7000-8000	55
Total	**1000**

Form a cumulative frequency table for the data above and answer the question given below.

How many employees earn less than

(i) ₹ 2000? (ii) ₹ 5000? (iii) ₹ 8000 (per week)?

Solution: Cumulative frequency table of the given distribution is given below:

Cumulative Frequency Table

Income (per week) (in ₹)	Number of Employees (frequency)	Cumulative frequency
0-1000	12	12
1000-2000	35	47
2000-3000	75	122
3000-4000	225	347
4000-5000	295	642
5000-6000	163	805
6000-7000	140	945
7000-8000	55	1000
Total	**1000**	

From the above table, we see that:

(i) Number of employees earning less than ₹ 2000 = 47

(ii) Number of employees earning less than ₹ 5000 = 642

(iii) Number of employees earning less than ₹ 8000 = 1000

CHECK YOUR PROGRESS 24.3

1. Construct a cumulative frequency distribution for each of the following distributions:

(i)

Classes	Frequency
1-5	4
6-10	6
11-15	10
16-20	13
21-25	6
26-30	2

(ii)

Classes	Frequency
0-10	3
10-20	10
20-30	24
30-40	32
40-50	9
50-60	7

2. Construct a cumulative frequency distribution from the following data:

Heights (in cm)	110-120	120-130	130-140	140-150	150-160	Total
Number of students	14	30	60	42	14	160

How many students have their heights less than 150 cm?

24.5 GRAPHICAL REPRESENTATION OF DATA

24.5.1 Bar Charts (Graphs)

Earlier, we have discussed presentation of data by tables. There is another way to present the data called **graphical representation** which is more convenient for the purpose of comparison among the individual items. Bar chart (graph) is one of the graphical representation of numerical data. For example Fig 24.1 represents the data given in the table regarding blood groups.

Blood groups of 35 students in a class

Blood Group	Number of students
A	13
B	9
AB	6
O	7
Total	**35**

We can represent this data by Fig. 24.1

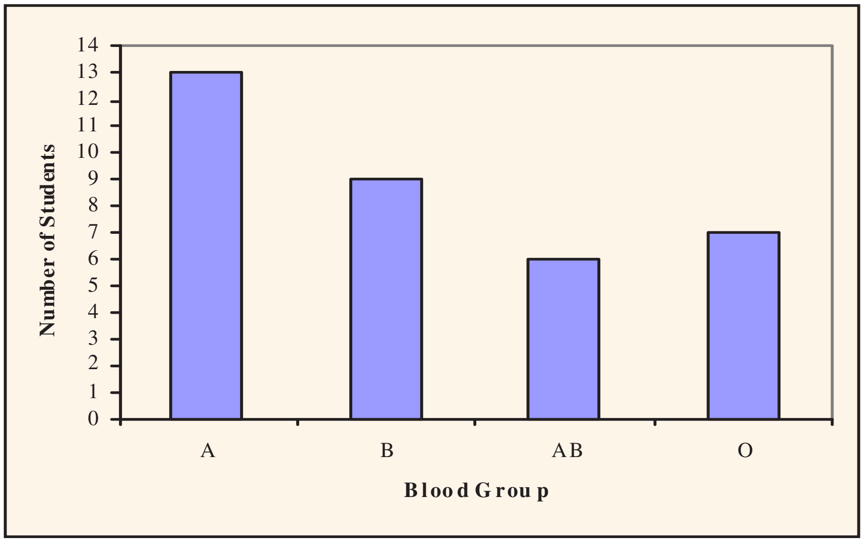


Fig. 24.1

This is called a **bar chart** or **bar graph.**

Bars (rectangles) of unifoirm width are drawn with equal spaces in between them, on the horizontal axis-called x-axis. The heights of the rectangles are shown along the vertical axis-called y-axis and are proportional to their respective frequencies (number of students).

The width of the rectangle has no special meaning except to make it pictorially more attractive. If you are given the bar chart as Fig. 24.1 what can you conclude from it?

You can conclude that

(i) The number of students in the class having blood group A is the maximum.

(ii) The number of students in the class having blood group AB is the minimum.

Bar graphs are used by economists, businessmen, medical journals, government departments for representing data.

Another form of the bar graph shown in Fig. 24.2, is the following where blood groups of the students are represented along y-axis and their frequencies along x-axis.

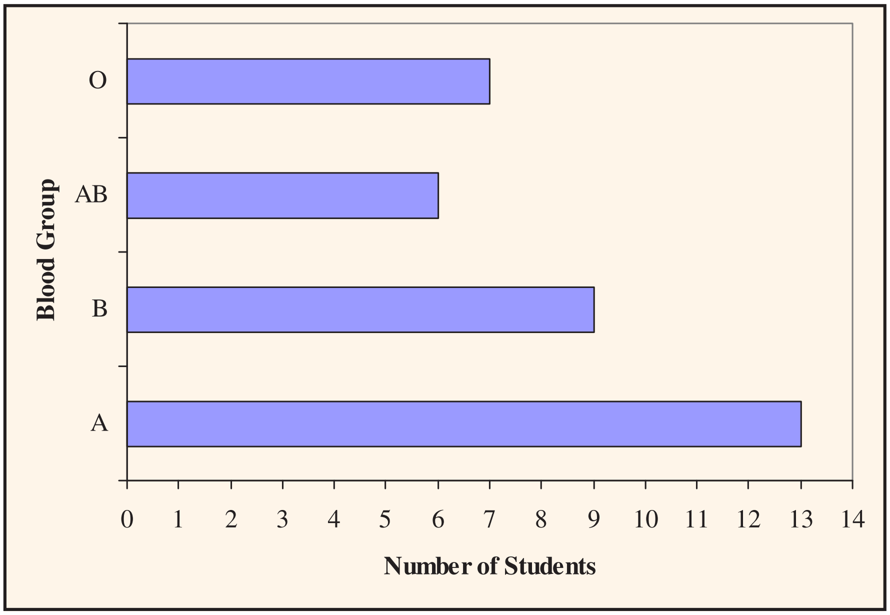


Fig. 24.2

There is not much difference between the bar graphs in Fig. 24.1 and Fig. 24.2 except that it depends upon the person's liking to represent data with vertical bars or with horizontal bars. Generally vertical bar graphs are preferred.

Example 24.4: Given below (Fig. 24.3) is the bar graph of the number of students in Class IX during academic years 2001-02 to 2005-06. Read the bar graph and answer the following questions:

(i) What is the information given by the bar graph?

(ii) In which year is the number of students in the class, 250?

(iii) State whether true or false:

The enrolment during 2002-03 is twice that of 2001-02.

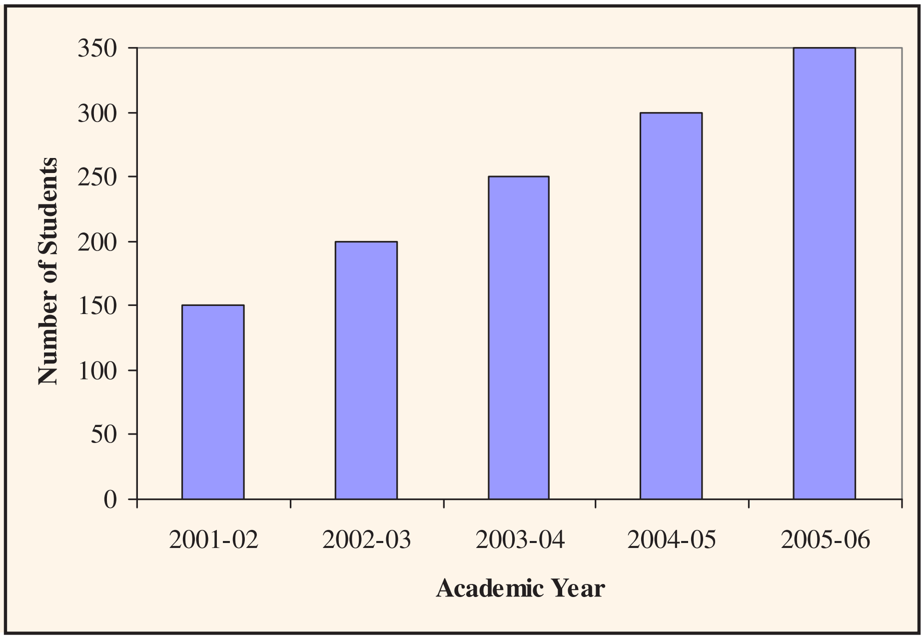


Fig. 24.3

Solution:

(i) The bar graph represents the number of students in class IX of a school during academic year 2001-02 to 2005-06.

(ii) In 2003-04, the number of students in the class was 250.

(iii) Enrolment in 2002-03 = 200

Enrolment in 2001-02 = 150

$$\frac{200}{150} = \frac{4}{3} = 1\frac{1}{3} < 2$$

Therefore, the given statement is false.

Example 24.5: The bar graph given in Fig. 24.4 represents the circulation of newspapers in six languages in a town (the figures are in hundreds). Read the bar graph and answer the following questions:

(i) Find the total number of newspapers read in Hindi, English and Punjabi.

(ii) Find the excess of the number of newspapers read in Hindi over those of Urdu, Marathi and Tamil together.

(iii) In which language is the number of newspapers read the least?

(iv) Write, in increasing order, the number of newspapers read in different languages.

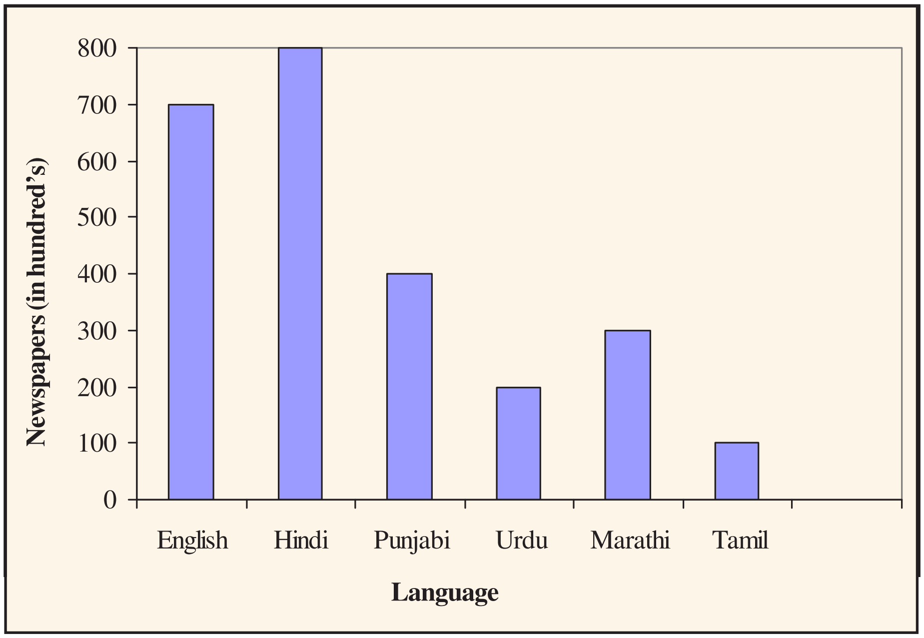


Fig. 24.4

Solution:

(i) Number of newspapers (in hundreds) read in Hindi, English and Punjabi = 800 + 700 + 400 = 1900

(ii) Number of newspapers (in hundreds) read in Hindi = 800

Number of newspapers (in hundreds) in Urdu,
Marathi and Tamil = 200 + 300 +100 = 600

So, difference (in hundreds) = 800 – 600) = 200

(iii) In Tamil, the number of newspapers read is the least.

(iv) Tamil, Urdu, Marathi, Punjabi, English, Hindi

Construction of Bar Graphs

We now explain the construction of bar graphs through examples:

Example 24.6: The following data give the amount of loans (in crores of rupees) given by a bank during the years 2000 to 2004:

Year	Loan (in crores of rupees)
2000	25
2001	30
2002	40
2003	55
2004	60

Construction a bar graph representing the above information.

Solution:

Step 1: Take a graph paper and draw two perpendicular lines and call them horizontal and vertical axes (Fig. 24.5)

Step 2: Along the horizontal axis, represent the information 'years' and along the vertical axis, represent the corresponding 'loans (in crores of rupees)'.

Step 3: Along the horizontal axis, choose a uniform (equal) width of bars and a uniform gap between them, according to the space available.

Step 4: Choose a suitable scale along the vertical axis in view of the data given to us.

Let us choose the scale:

1 unit of graph paper = 10 crore of rupees for the present data.

Step 5: Calculate the heights of the bars for different years as given below:

$$2000 : \frac{1}{10} \times 25 = 2.5 \text{ units}$$

$$2001 : \frac{1}{10} \times 30 = 3 \text{ units}$$

$$2002 : \frac{1}{10} \times 40 = 4 \text{ units}$$

$$2003 : \frac{1}{10} \times 55 = 5.5 \text{ units}$$

$$2004 : \frac{1}{10} \times 60 = 6 \text{ units}$$

Step 6: Draw five bars of equal width and heights obtained in Step 5 above, the corresponding years marked on the horizontal axis, with equal spacing between them as shown in Fig. 24.5.

Bar graph of loans (in crores of rupees) given by a bank during the years 2000 to 2004

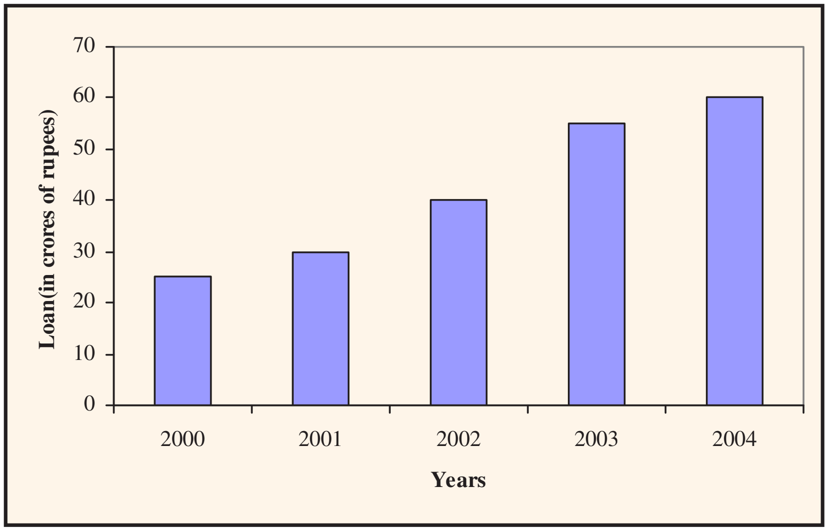


Fig. 24.5

Thus, Fig. 24.5 gives the required bar graph.

Example 24.7: The data below shows the number of students present in different classes on a particular day.

Class	VI	VII	VIII	IX	X
Number of students present	40	45	35	40	50

Represent the above data by a bar graph.

Solution: The bar graph for the above data is shown in Fig. 24.6.

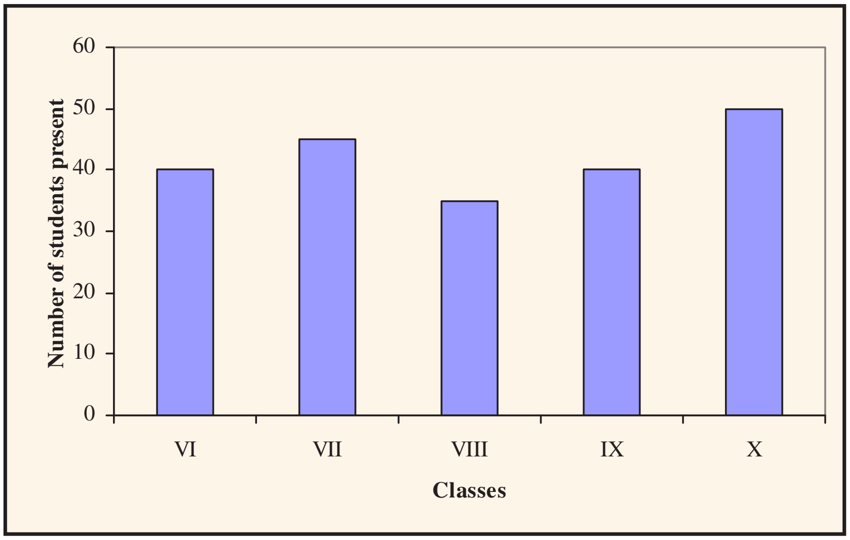


Fig. 24.6

Example 24.8: A survey of 200 students of a school was done to find which activity they prefer to do in their free time and the information thus collected is recorded in the following table:

Preferred activity	Number of students
Playing	60
Reading story books	45
Watching TV	40
Listening to music	25
Painting	30

Draw a bar graph for this data.

Solution: The bar graph representing the above data is shown in Fig. 24.7 below:

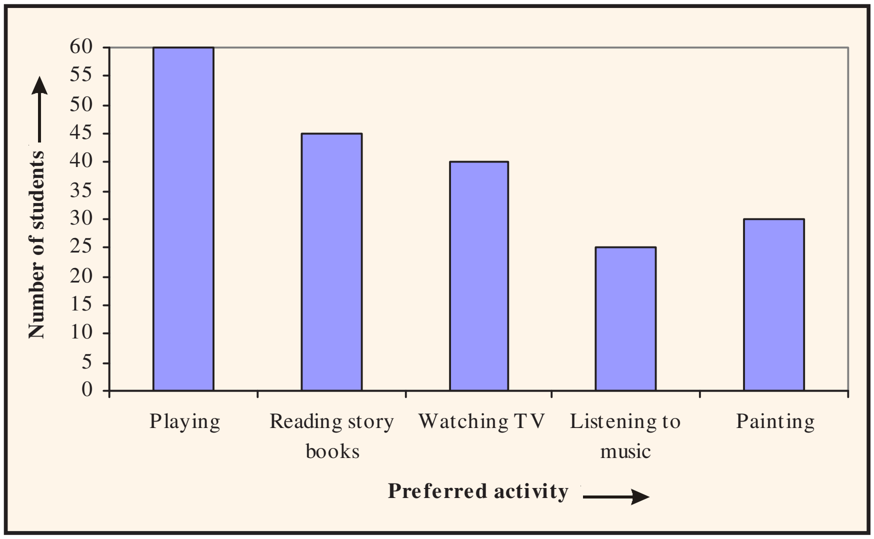


Fig. 24.7

CHECK YOUR PROGRESS 24.4

1. Fill in the blanks:

   (i) A bar graph is a graphical representation of numerical data using ________ of equal width.

   (ii) In a bar graph, bars are drawn with __________ spaces in between them.

   (iii) In a bar graph, heights of rectangles are __________ to their respective frequencies.

2. The following bar graph shows how the members of the staff of a school come to school.

Mode of transport of school staff

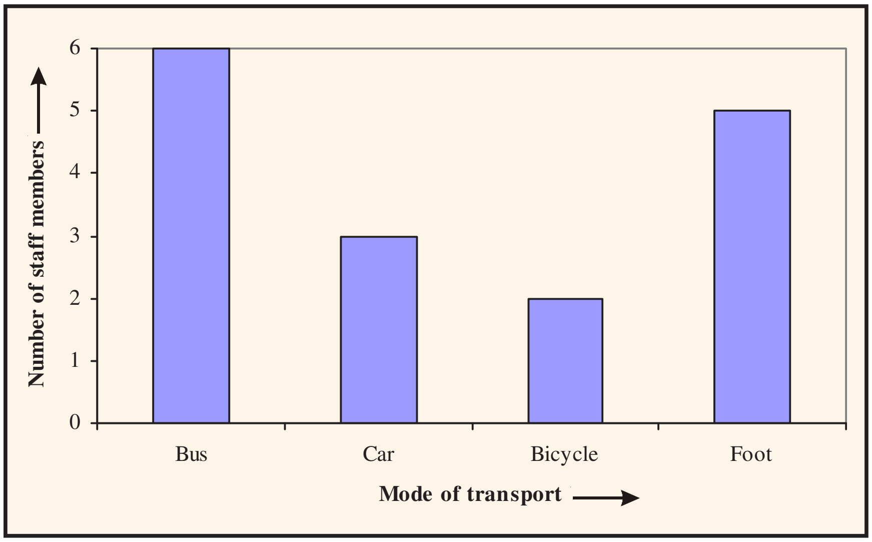


Fig. 24.8

Study the bar graph and answer the following questions:

(i) How many members of staff come to school on bicycle?

(ii) How many member of staff come to school by bus?

(iii) What is the most common mode of transfport of the members of staff?

3. The bar graph given below shows the number of players in each team of 4 given games:

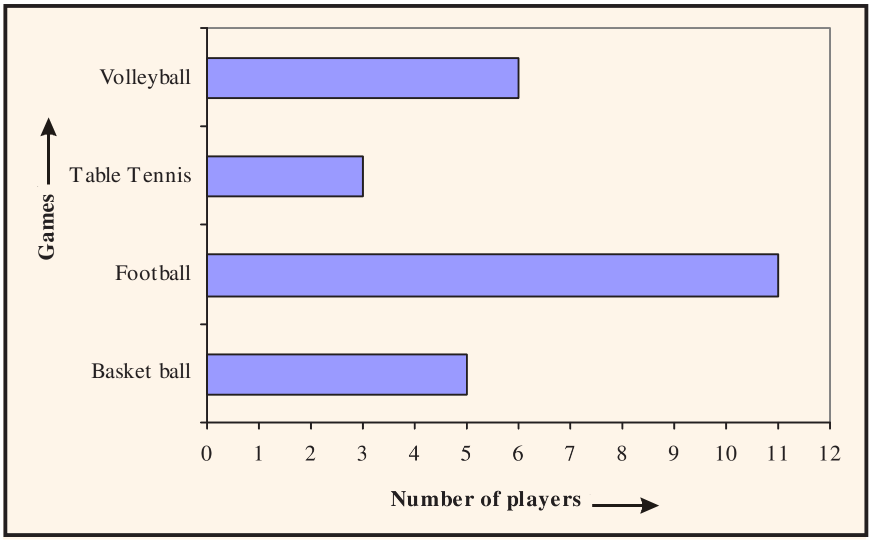


Fig. 24.9

Read the bar graph and answer the following questions:

(i) How many players play in the volley ball team?

(ii) Which game is played by the maximum number of players?

(iii) Which game is played by only 3 players?

4. The following bar graph shows the number of trees planted by an agency in different years:

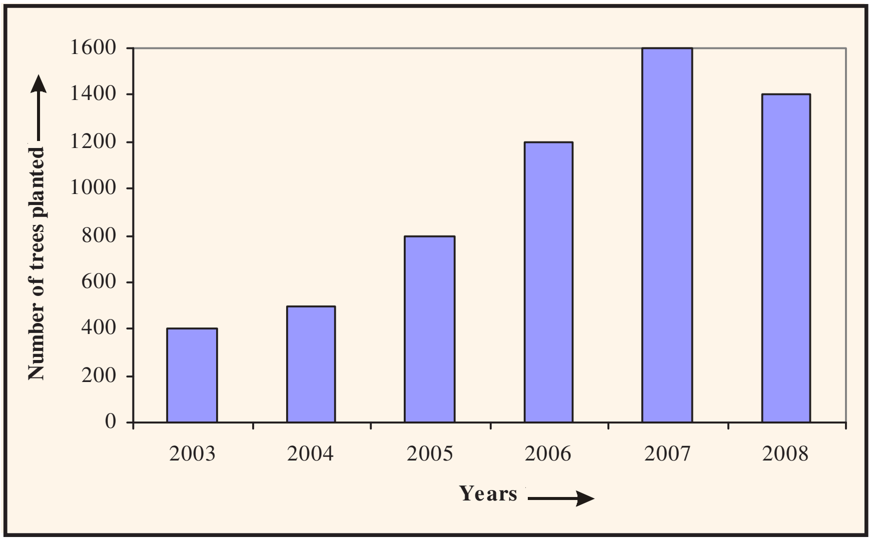


Fig. 24. 10

Study the above bar graph and answer the following questions:

(i) What is the total number of trees planted by the agency from 2003 to 2008?

(ii) In which year is the number of trees planted the maximum?

(iii) In which year is the number of trees planted the minimum?

(iv) In which year, the number of trees planted is less than the number of trees planted in the year preceding it?

5. The expenditure of a company under different heads (in lakh of rupees) for a year is given below:

Head	Expenditure (in lakhs of rupees)
Salary of employees	200
Travelling allowances	100
Electricity and water	50
Rent	125
Others	150

Construct a bar chart to represent this data.

24.5.2 Histograms and Frequency Polygons

Earlier, we have learnt to represent a given information by means of a bar graph. Now, we will learn how to represent a continuous grouped frequency distribution graphically. A continuous grouped frequency distribution can be represented graphically by a **histogram**.

A histogram is a vertical bar graph with no space between the bars.

(i) The classes of the grouped data are taken along the horizontal axis and

(ii) the respective class frequencies on the vertical axis, using a suitable scale on each axis.

(iii) For each class a rectangle is constructed with base as the width of the class and height determined from the class frequencies. The areas of rectangles are proportional to the frequencies of their respective classes.

Let us illustrate this with the help of examples.

Example 24.9: The following is the frequency distribution of marks obtained by 20 students in a class test.

Marks obtained	20-30	30-40	40-50	50-60	60-70	70-80
Number of students	1	3	1	6	4	5

Draw a histogram for the above data.

Solution: We go through the following steps for drawing a histogram.

Step 1: On a graph paper, draw two perpendicular lines and call them as horizontal and vertical axes.

Step 2: Along the horizontal axis, we take classes (marks) 20-30, 30-40, ... (Here each is of equal width 10)

Step 3: Choose a suitable scale on the vertical axis to represent the frequencies (number of students) of classes.

Step 4: Draw the rectangles as shown in Fig. 24.11.

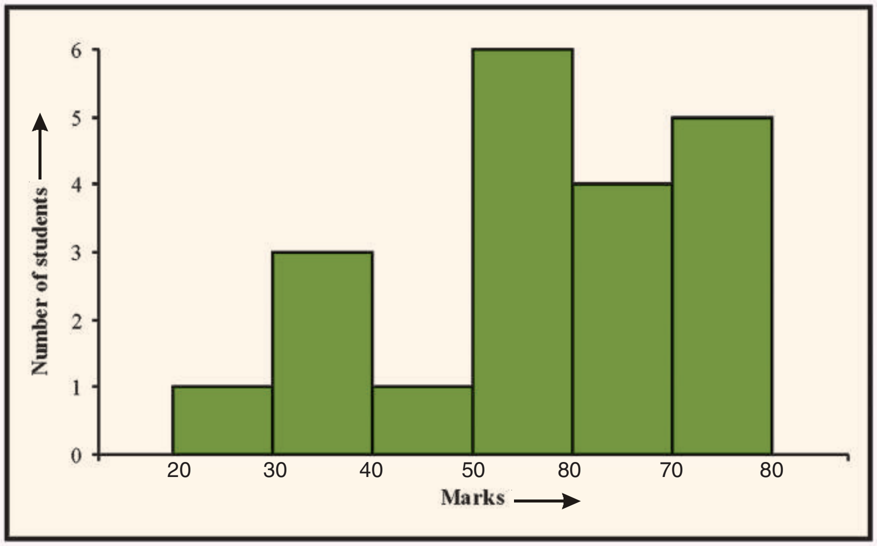


Fig. 24.11

Fig. 24.11 shows the histogram for the frequency distribution of marks obtained by 20 students in a class test.

Example 24.10: Draw a histogram for the following data:

Height (in cm)	125-130	130-135	135-140	140-145	145-150	150-155	155-160
Number of students	1	2	3	5	4	3	2

Solution: Following the steps as suggested in the above example, the histogram representing the given data is given below:

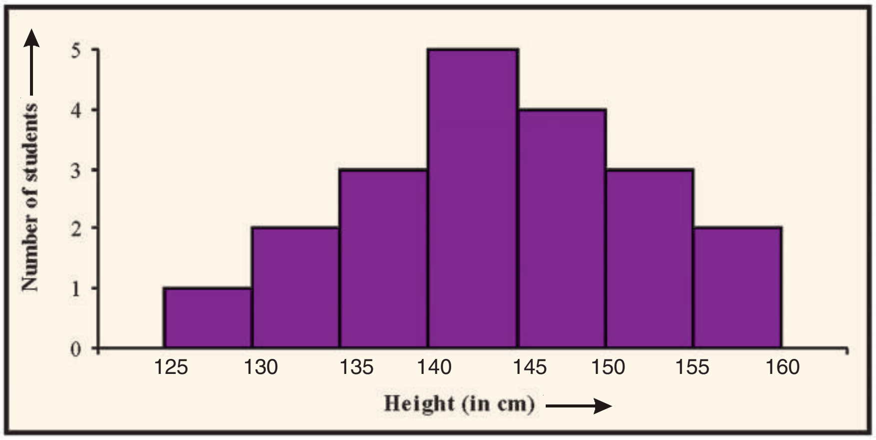


Fig. 24.12

Frequency Polygon

There is yet another way of representing a grouped frequency distribution graphically. This is called **frequency polygen.** To see what we mean, consider the histogram in Fig. 24.13.

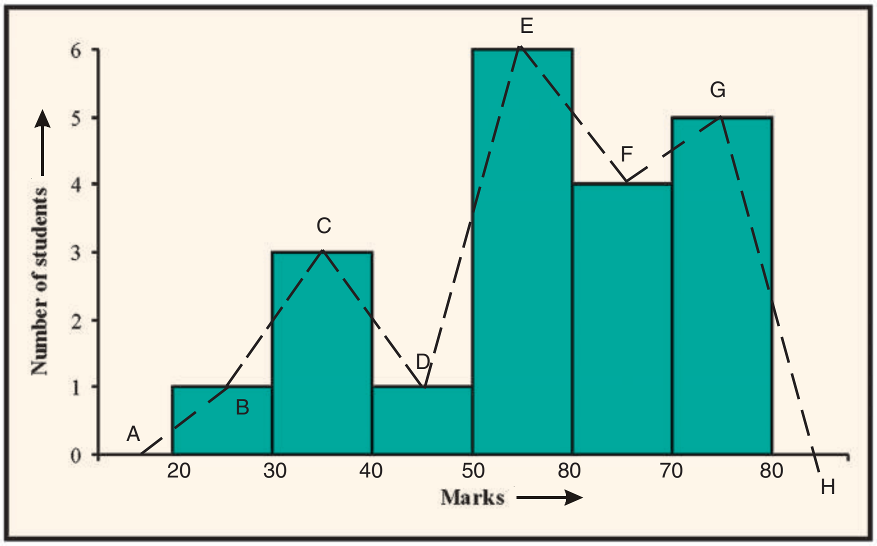


Fig. 24.13

Let B, C, D, E, F and G be the mid points of the tops of the adjacent rectangles (Fig. 24.13). Join B to C, C to D, D to E, E to F and F to G by means of line segments (dotted).

To complete the polygon, join B to A (the mid point of class 10-20) and join G to H (the mid point of the class 80-90).

Thus, A B C D E F G H is the **frequency polygon** representing the data given in Example 24.9

Note: Although, there exists no class preceding the lowest class and no class succeeding the highest class, we add the two classes each with zero frequency so that we can make the area of the frequency polygon the same as the area of the histogram.

Example 24.11: Draw a frequency polygon for the data in Example 24.12.

Solution: Histogram representing the given data is shown in Fig. 24.12. For frequency polygon, we follow the procedure as given above. The frequency polygen ABCDEFGHI representing the given data is given below:

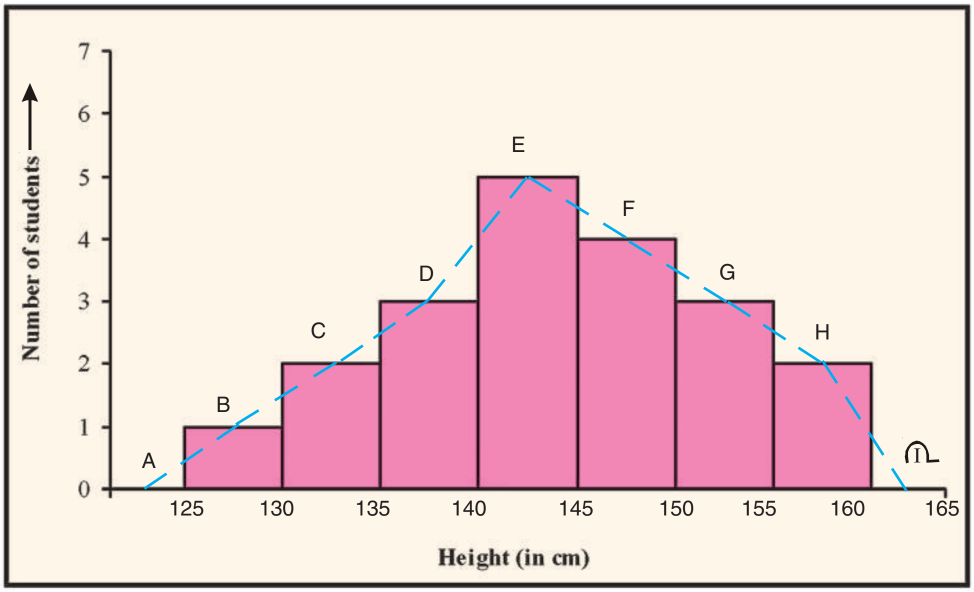


Fig. 24.14 →

Example 24.12: Marks (out of 50) obtained by 30 students of Class IX in a mathematics test are given in the following table:

Marks	0-10	10-20	20-30	30-40	40-50
Number of students	5	8	6	7	4

Draw a frequency polygon for this data.

Solution: Let us first draw a histogram for this data (Fig. 24.15)

Mark the mid points B, C, D, E and F of the tops of the rectangles as shown in Fig. 24.15. Here, the first class is 0-10. So, to find the class preceding 0-10, we extend the horizontal axis in the negative direction and find the mid point of the **imaginary** class (–10)-0. Let us

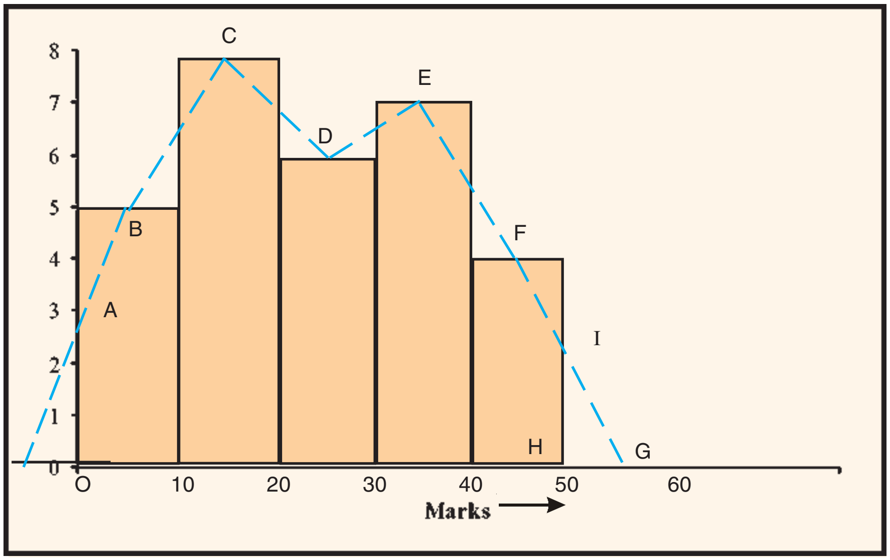


Fig. 24.15

join B to the mid point of the class (015010)-0. Let A be the mid point where this line segment meets the vertial axis. Let G be the mid point of the class 50-60 (succeeding the last class). Let the line segment FG intersects the length of the last rectangle at I (Fig. 24.15). Then OABCDEFIH is the required frequency polygon representing the given data.

Note: Why have we not taken the points before O and G? This is so because marks obtained by the students cannot go below 0 and beyond maximum marks 50. In the figure, extreme line segments are only partly drawn and then brought down vertically to 0 and 50.

Frequency polygon can also be drawn independently without drawing histogram. We will illustrate it through the following example.

Example 24.13: Draw a frequency polygon for the data given in Example 24.9, without drawing a histogram for the data.

Solution: To draw a frequency polygon without drawing a histogram, we go through the following steps.

Step 1: Draw two lines perpendicualar to each other.

Step 2: Find the class marks of the classes.

Here they are: $\frac{20+30}{2}, \frac{30+40}{2}, \frac{40+50}{2}, \frac{50+60}{2}, \frac{60+70}{2}$ and $\frac{70+80}{2}$

i.e. the class marks are 25, 35, 45, 55, 65 and 75 respectively.

Step 3: Plot the points B (25, 1), C(35, 3), D(45, 1), E(55, 6), F(65, 4) and G(75, 5), i.e., (class mark, frequency)

Step 4: Join the points B, C, D, E, F and G by line segments and complete the polygon as explained earlier.

The frequency polygon (ABCDEFGH) is given below:

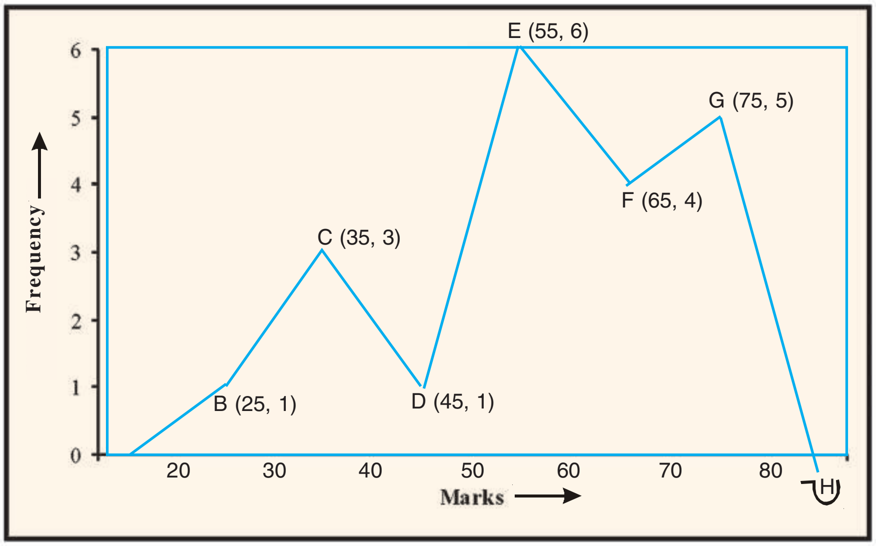


Fig. 24.16

Reading a Histogram

Consider the following example:

Example 24.14: Study the histogram given below and answer the following questions:

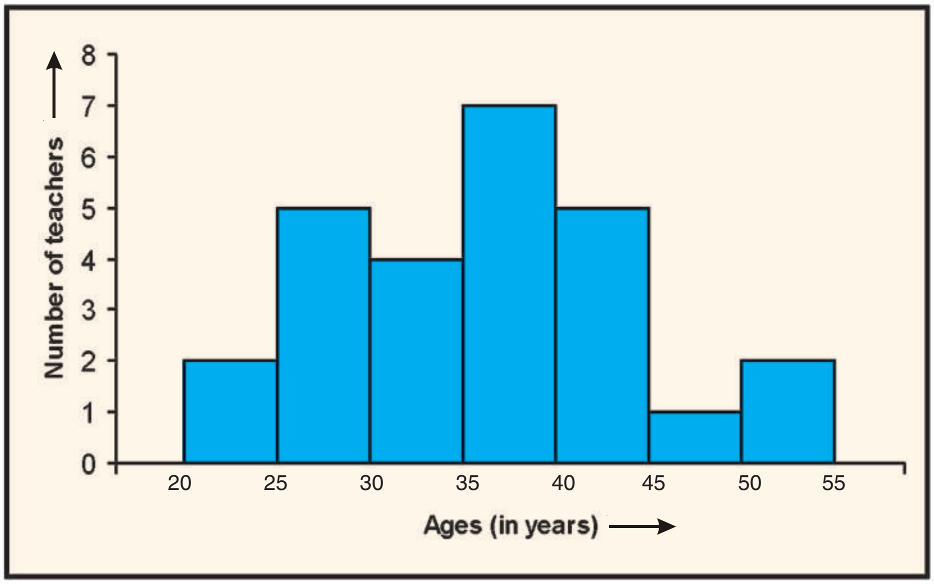


Fig. 24.17

(i) What is the number of teachers in the oldest and the youngest group in the school?

(ii) In which age group is the number of teachers maximum?

(iii) In which age group is the number of teachers 4?

(iv) In which two age groups, the number of teachers is the same?

Solution:

(i) Number of teachers in oldest and youngest group = 3 + 2 = 5

(ii) Number of teachers is the maximum in the age group 35-40.

(iii) In the age group 30-35, the number of teachers is 4.

(iv) Number of teachers is the same in the age groups 25-35 and 40-45. It is 4 in each group. In age groups 20-25 and 50-55, the number of teachers is same i.e., 2

CHECK YOUR PROGRESS 24.5

1. Fill in the blanks:

   (i) In a histogram, the class intervals are generally taken along _________.

   (ii) In a histogram, the class frequencies are generally taken along ________.

   (iii) In a histogram, the areas of rectangles are proportional to the ________ of the respective classes.

   (iv) A histogram is a graphical representation of a ___________.

2. The daily earnings of 26 workers are given below:

Daily earnings (in ₹)	150-200	200-250	250-300	300-350	350-400
Number of workers	4	8	5	6	3

   Draw a histogram to represent the data.

3. Draw a frequency polygon for the data in Question 2 above by

   (i) drawing a histogram

   (ii) without drawing a histogram

4. Observe the histogram given below and answer the following questions:

   (i) What information is given by the histogram?

   (ii) In which class (group) is the number of students maximum?

   (iii) How many students have the height of 145 cm and above?

   (iv) How many students have the height less than 140 cm?

(v) How many students have the height more than or equal to 140 but less than 155?

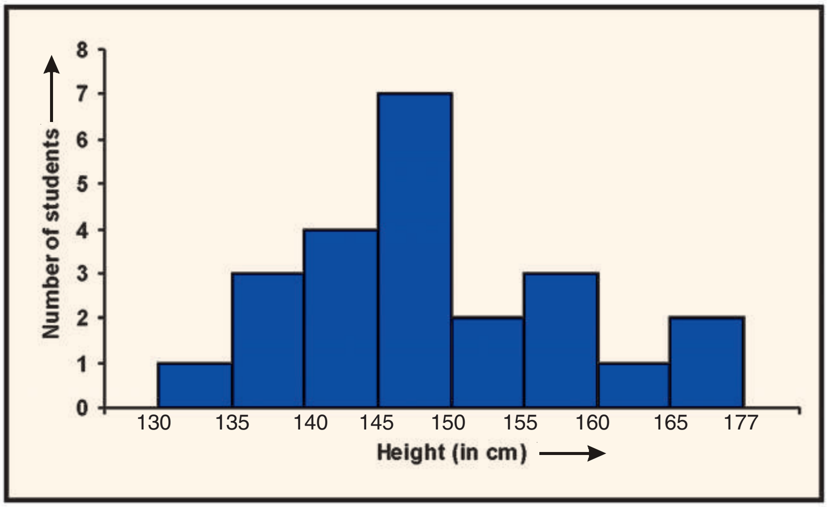


Fig. 24.18

LET US SUM UP

- Statistics is that branch of mathematics which deals with collection, organisation, analysis and interpretation of data.
- Statistics is used in both plural and singular sense.
- The data collected from the respondents "as it is" is called raw data.
- Data are said to be primary if the investigator himself collects it through his/her own designed tools.
- Data taken from other sources such as printed reports, and not collected by the experimenter himself, is called secondary data.
- The raw data arranged in ascending or decending order is called "arrayed data".
- When the arrayed data are arranged with frequencies, they are said to form a frequency table for ungrouped data or a ungrouped frequency distribution table.
- When the data are divided into groups/classes, they are called grouped data.
- The difference between the maximum and minimum observations occuring in the data is called the range of the raw data.
- The number of classes have to be decided according to the range of the data and size of class.

- In a class say 10-15, 10 is called the lower limit and 15 is called the upper limit of the class.
- The number of observations in a particualr class is called its frequency and the table showing classes with frequencies is called a frequency table.
- Sometimes, the classes have to be changed to make them continuous. In such case, the class limits are called true class limits.
- The total of frequency of a particular class and frequencies of all other classes preceding that class is called the cumulative frequency of that class.
- The table showing cumulative frequencies is called cumulative frequency table.
- A bar graph is a graphical representation of the numerical data by a number of bars (rectangles) of uniform width, erected horizontally or vertically with equal space between them.
- A histogram is a graphical representation of a grouped frequency distribution with continuous classes. In a histogram, the area of the rectangles are proportional to the corresponding frequencies.
- A frequency polygon is obtianed by first joining the mid points of the tops of the adjacent rectangles in the histogram and then joining the mid point of first rectangle to the mid point of the class preceding the lowest class and the the last mid point to the mid point of the class succeeding the highest class.
- A frequency polygon can also be drawn independently without drawing a histogram by using the class marks of the classes and respective frequencies of the classes.

TERMINAL EXERCISE

1. Fill in the blanks by appropriate words/phrases to make each of the following statements true:

   (i) When the data are condensed in classes of equal size with frequencies, they are called ________ data and the table is called _______ table.

   (ii) When the class limits are adjusted to make them continuous, the class limits are renamed as ________.

   (iii) The number of observations falling in a particular class is called its _______.

   (iv) The difference between the upper limit and lower limit of a class is called _________.

   (v) The sum of frequencies of a class and all classes prior to that class is called ________ frequency of that class.

(vi) Class size = Difference between ________ and _____ of the class.

(vii) The raw data arranged in ascending or descending order is called an _______ data.

(viii) The difference between the maximum and minimum observations occuring in the data is called the _________ of the raw data.

2. The number of TV sets in each of 30 households are given below:

1, 2, 2, 4, 2, 1, 1, 1, 2, 1, 3, 1, 1, 1, 3

1, 2, 2, 1, 2, 0, 3, 3, 1, 2, 1, ,1 0, 1, 1

Construct a frequency table for the data.

3. The number of vehicles owned by each of 50 families are listed below:

2, 1, 2, 1, 1, 1, 2, 1, 2, 1, 0, 1, 1, 2, 3, 1, 1, 1,

2, 2, 1, 1, 3, 1, 1, 2, 1, 0, 1, 2, 1, 2, 1, 1, 4, 1

3, 1, 1, 1, 2, 2, 2, 2, 1, 1, 3, 2, 1, 2

Construct a frequency distribution table for the data.

4. The weight (in grams) of 40 New Year's cards were found as:

10.4	6.3	8.7	7.3	8.8	9.1	6.7	11.1	14.0	12.2
11.3	9.4	8.6	7.1	8.4	10.0	9.1	8.8	10.3	10.2
7.3	8.6	9.7	10.9	13.6	9.8	8.9	9.2	10.8	9.4
6.2	8.8	9.4	9.9	10.1	11.4	11.8	11.2	10.1	8.3

Prepare a grouped frequency distribution using the class 5.5-7.5, 7.5-9.5 etc.

5. The lengths, in centimetres, to the nearest centimeter of 30 carrots are given below:

15	21	20	10	18	18	16	18	20	20
18	16	13	15	15	16	13	14	14	16
12	15	17	12	14	15	13	11	14	17

Construct a frequency table for the data using equal class sizes and taking one class as 10-12 (12 excluded).

6. The following is the distribution of weights (in kg) of 40 persons:

Weight	Number of persons
40-45	4
45-50	5
50-55	10
55-60	7
60-65	6
65-70	8
Total	**40**

(i) Determine the class marks of the classes 40-45, 45-50 etc.

(ii) Construct a cumulative frequency table.

7. The class marks of a distribution and the corresponding frequencies are given below:

Class marks	5	15	25	35	45	55	65	75
Frequency	2	6	10	15	12	8	5	2

Determine the frequency table and construct the cumulative frequency table.

8. For the following frequency table

Classes	Frequency
15-20	2
20-25	3
25-30	5
30-35	7
35-40	4
40-45	3
45-50	1
Total	**25**

(i) Write the lower limit of the class 15-20.

(ii) Write the class limits of the class 25-30.

(iii) Find the class mark of the class 35-40.

(iv) Determine the class size.

(v) Form a cumulative frequency table.

9. Given below is a cumulative frequency distribution table showing marks obtained by 50 students of a class.

Marks	**Number of students**
Below 20	15
Below 40	24
Below 60	29
Below 80	34
Below 100	50

Form a frequency table from the above data.

10. Draw a bar graph to represent the following data of sales of a shopkeeper:

Day	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Sales (in ₹)	16000	18000	17500	9000	85000	16500

11. Study the following bar graph and answer the following questions:

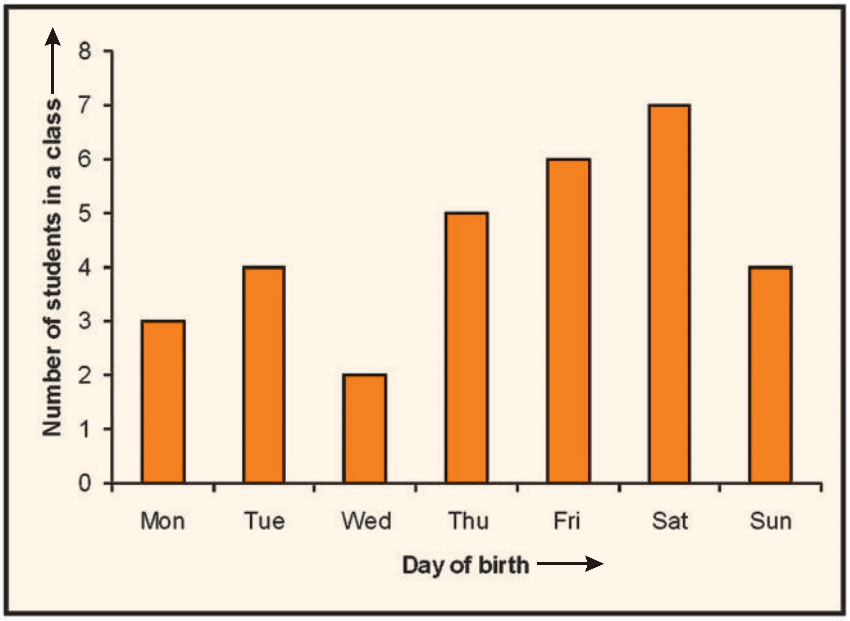


Fig. 24.19

(i) What is the information given by the bar graph?

(ii) On which day is number of students born the maximum?

(iii) How many more students were born on Thursday than that on Tuesday.

(iv) What is the total number of students in the class?

12. The times (in minutes) taken to complete a crossword at a competition were noted for 50 competitors are recorded in the following table:

Time (in minutes)	Number of competitors
20-25	8
25-30	10
30-35	9
35-40	12
40-45	6
45-50	5

(i) Construct a histogram for the data.

(ii) Construct a frequency polygon.

13. Construct a frequency polygon for tha data in question 12 without drawing a histogram.

14. The following histogram shows the number of literate females in the age group 10 to 40 (in years) in a town:

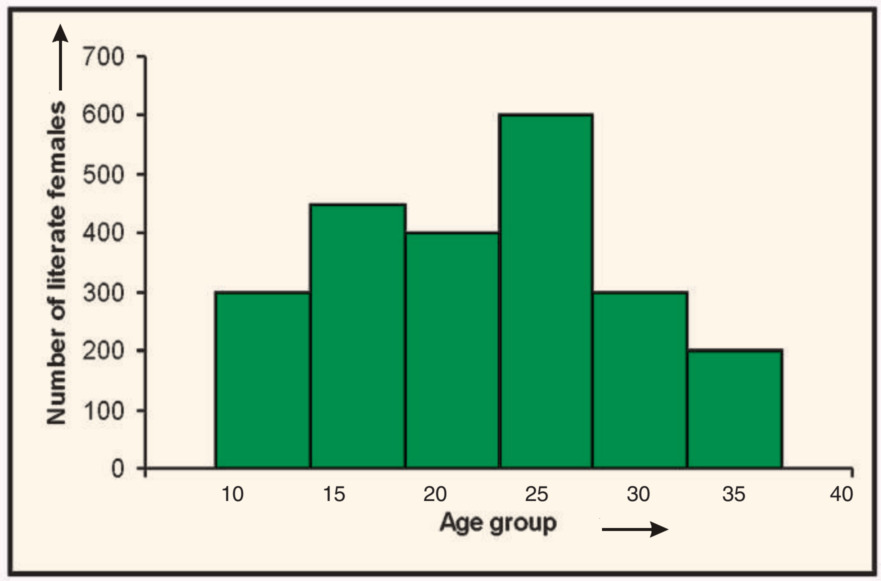


Fig. 24.20

Study the above histogram and answer the following questions:

(i) What was the total number of literate females in the town in the age group 10 to 40?

(ii) In which age group, the number of literate females was the highest?

(iii) In which two age groups was the number of literate females the same?

(iv) State true or false:

The number of literate females in the age group 25-30 is the sum of the numbers of literate females in the age groups 20-25 and 35-40.

Write the correct option:

15. The sum of the class marks of the classes 90-120 and 120-150 is

(A) 210 (B) 220 (C) 240 (D) 270

16. The range of the data

28, 17, 20, 16, 19, 12, 30, 32, 10 is

(A) 22 (B) 28 (C) 30 (D) 32

17. In a frequency distribution, the mid-value of a class is 12 and its width is 6. The lower limit of the class is:

(A) 6 (B) 9 (C) 12 (D) 18

18. The width of each of five continuous classes in a frequency distribution is 5 and the lower limit of the lowest (first) class is 10. The upper limit of the highest (last) class is

(A) 15 (B) 20 (C) 30 (D) 35

19. The class marks (in order) of a frequency distribution are 10, 15, 20, ... The class corresponding to the class mark 15 is

(A) 11.5-18.5 (B) 17.5-22.5

(C) 12.5-17.5 (D) 13.5-16.5

20. For drawing a frequency polygon of a continuous frequency distribution, we plot the points whose ordinates are the frequencies of the respective classes and abcissae are respectively:

(A) class marks of the classes (B) lower limits of the classes

(C) upper limits of the classes (D) upper limits of preceding classes

ANSWERS TO CHECK YOUR PROGRESS

24.1

1. (a) Classification, organisation, inferences (b) numerical data
   (c) primary (d) secondary
   (e) numerical data

2. Primary 3. Secondary

24.2

2. 21 cm

4

Marks	Number of students
0-10	1
10-19	2
20-29	1
30-39	2
40-49	5
50-59	6
60-69	6
70-79	4
80-89	2
90-99	1
Total	30

19 students secured more than 49 marks.

5.

Class interval	Frequency
210-230	2
230-250	5
250-270	2
270-290	2
290-310	4
310-330	6
330-350	2
350-370	2
370-390	0
390-410	3
Total	25

6. (a) 6 (b) 43 (c) 49

24.3

1. (i)

Classes	Frequency	Cumulative frequency
1-5	4	4
6-10	6	10
11-15	10	20
16-20	13	33
21-25	6	39
26-30	2	41
Total	41	

(ii)

Classes	Frequency	Cumulative frequency
0-10	3	3
10-20	10	13
20-30	24	37
30-40	32	69
40-50	9	78
50-60	7	85
Total	85	

2.

Heights (in cm)	Number of students	Cumulative frequency
110-120	14	14
120-130	30	44
13-140	60	104
140-150	42	146
150-160	14	160
Total	160	

140 students have heights less than 150.

24.4

1. (i) bars (ii) equal (iii) proportional
2. (i) 2 (ii) 6 (iii) Bus
3. (i) 6 (ii) Football (iii) Table tennis
4. (i) 5900 (ii) 2007 (iii) 2003 (iv) 2008

24.5

1. (i) Horizontal axis
   (ii) Vertical axis
   (iii) Frequency
   (iv) Continuous grouped frequency distribution
2. (i) Heights (in cm) of students
   (ii) 145-150
   (iii) 15
   (iv) 4
   (v) 13

ANSWERS TO TERMINAL EXERCISE

1. (i) group, frequency table (ii) true limits

(iii) frequency (iv) class size

(v) cumulative frequency (vi) upper limt, lower limit

(vii) arrayed (vii) range

2.

Number of TV sets	Number of hours
0	2
1	15
2	8
3	4
4	1
Total	30

3.

Numbre of vehicles	Number of families
0	2
1	27
2	16
3	4
4	1
Total	50

4.

Weights (in grams)	Number of cards
5.5-7.5	6
7.5-9.5	15
9.5-11.5	15
11.5-13.5	2
13.5-15.5	2
Total	40

5.

Length (in cm)	Number of carrots
10-12	2
12-14	5
14-16	9
16-18	6
18-20	4
20-22	4
Total	30

6. (i) 42.5

(ii)

Weight (in kg)	Number of persons	Cumulative frequency
40-45	4	4
45-50	5	9
50-55	10	19
55-60	7	26
60-65	6	32
65-70	8	40
Total	40	

7.

Class interval	Frequency	Cumulative frequency
0-10	2	2
10-20	6	8
20-30	10	18
30-40	15	33
40-50	12	45
50-60	8	53
60-70	5	58
70-80	2	60
Total	60	

8. (i) 15 (ii) Lower limit : 25, Upper limit: 30

(iii) 37.5 (iv) 5

(iv)

Classes	Frequency	Cumulative frequency
15-20	2	2
20-25	3	5
25-30	5	10
30-35	7	17
35-40	4	21
40-45	3	24
45-50	1	25
Total	25	

9.

Marks	No. of students (frequency)
0-20	15
20-40	9
40-60	5
60-80	5
80-100	16

10. (i) Days of birth of the students in a class

(ii) Saturday

(iii) 1

(iv) 31

11. (i) 2250 (ii) 25-30

(iii) 10-15 and 30-35 (iv) True

12. (C)

13. (A)

14. (B)

15. (D)

16. (C)

17. (A)

25

MEASURES OF CENTRAL TENDENCY

In the previous lesson, we have learnt that the data could be summarised to some extent by presenting it in the form of a frequency table. We have also seen how data were represented graphically through bar graphs, histograms and frequency polygons to get some broad idea about the nature of the data.

Some aspects of the data can be described quantitatively to represent certain features of the data. An average is one of such representative measures. As average is a number of indicating the representative or central value of the data, it lies somewhere in between the two extremes. For this reason, average is called a **measure of central tendency.**

In this lesson, we will study some common measures of central tendency, viz.

(i) Arithmetical average, also called mean

(ii) Median

(iii) Mode

OBJECTIVES

After studying this lesson, you will be able to

- *define mean of raw/ungrouped and grouped data;*
- *calculate mean of raw/ungrouped data and also of grouped data by ordinary and short-cut-methods;*
- *define median and mode of raw/ungrouped data;*
- *calculate median and mode of raw/ungrouped data.*

25.1 ARITHMETIC AVERAGE OR MEAN

You must have heard people talking about average speed, average rainfall, average height, average score (marks) etc. If we are told that average height of students is 150 cm, it does not mean that height of each student is 150 cm. In general, it gives a message that height of

students are spread around 150 cm. Some of the students may have a height less than it, some may have a height greater than it and some may have a height of exactly 150 cm.

25.1.1 Mean (Arithmetic average) of Raw Data

To calculate the mean of raw data, all the observations of the data are added and their sum is divided by the number of observations. Thus, the mean of n observations $x_1, x_2, ...x_n$ is

$$\frac{x_1 + x_2 + ... + x_n}{n}$$

It is generally denoted by $\overline{x}$. so

$$\overline{x} = \frac{x_1 + x_2 + ... + x_n}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i}{n} \qquad \text{(I)}$$

where the symbol "Σ" is the capital letter 'SIGMA' of the Greek alphabet and is used to denote summation.

To economise the space required in writing such lengthy expression, we use the symbol Σ, read as **sigma.**

In $\sum_{i=1}^{n} x_i$, i is called the index of summation.

Example 25.1: The weight of four bags of wheat (in kg) are 103, 105, 102, 104. Find the mean weight.

Solution: Mean weight ($\overline{x}$) $= \frac{103+105+102+104}{4}$ kg

$$= \frac{414}{4} \text{ kg} = 103.5 \text{ kg}$$

Example 25.2: The enrolment in a school in last five years was 605, 710, 745, 835 and 910. What was the average enrolment per year?

Solution: Average enrolment (or mean enrolment)

$$= \frac{605+710+745+835+910}{5} = \frac{3805}{5} = 761$$

Example 25.3: The following are the marks in a Mathematics Test of 30 students of Class IX in a school:

40	73	49	83	40	49	27	91	37	31
91	40	31	73	17	49	73	62	40	62
49	50	80	35	40	62	73	49	31	28

Find the mean marks.

Solution: Here, the number of observation (n) = 30

$x_1 = 40,\ x_2 = 73,\ ....,\ x_{10} = 31$

$x_{11} = 41,\ x_{12} = 40,\ ....,\ x_{20} = 62$

$x_{21} = 49,\ x_{22} = 50,\ ....,\ x_{30} = 28$

From the Formula (I), the mean marks of students is given by

$$\text{Mean} = (\bar{x}) = \frac{\sum_{i=1}^{30} x_i}{n} = \frac{40+73+....+28}{30} = \frac{1455}{30}$$

$$= 48.5$$

Example 25.4: Refer to Example 25.1. Show that the sum of $x_1 - \bar{x}$, $x_2 - \bar{x}$, $x_3 - \bar{x}$ and $x_4 - \bar{x}$ is 0, where x_i's are the weights of the four bags and $\bar{x}$ is their mean.

Solution: $x_1 - \bar{x} = 103 - 103.5 = -0.5,\ x_2 - \bar{x} = 105 - 103.5 = 1.5$

$x_3 - \bar{x} = 102 - 103.5 = -1.5,\ x_4 - \bar{x} = 104 - 103.5 = 0.5$

So, $(x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) + (x_4 - \bar{x}) = -0.5 + 1.5 + (-1.5) + 0.5 = 0$

Example 25.5: The mean of marks obtained by 30 students of Section A of Class X is 48, that of 35 students of Section B is 50. Find the mean marks obtained by 65 students in Class X.

Solution: Mean marks of 30 students of Section A = 48

So, total marks obtained by 30 students of Section A = 30 × 48 = 1440

Similarly, total marks obtained by 35 students of Section B = 35 × 50 = 1750

Total marks obtained by both sections = 1440 + 1750 = 3190

$$\text{Mean of marks obtained by 65 students} = \frac{3190}{65} = 49.1 \text{ approx.}$$

Example 25.6: The mean of 6 observations was found to be 40. Later on, it was detected that one observation 82 was misread as 28. Find the correct mean.

Solution: Mean of 6 observations = 40

So, the sum of all the observations = 6 × 40 = 240

Since one observation 82 was misread as 28,

therefore, correct sum of all the observations = 240 – 28 + 82 = 294

Hence, correct mean = $\dfrac{294}{6} = 49$

CHECK YOUR PROGRESS 25.1

1. Write formula for calculating mean of n observations x_1 , x_2 ..., x_n.
2. Find the mean of first ten natural numbers.
3. The daily sale of sugar for 6 days in a certain grocery shop is given below. Calculate the mean daily sale of sugar.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
74 kg	121 kg	40 kg	82 kg	70.5 kg	130.5 kg

4. The heights of 10 girls were measured in cm and the results were as follows:

   142, 149, 135, 150, 128, 140, 149, 152, 138, 145

   Find the mean height.

5. The maximum daily temperature (in °C) of a city on 12 consecutive days are given below:

32.4	29.5	26.6	25.7	23.5	24.6
24.2	22.4	24.2	23.0	23.2	28.8

   Calcualte the mean daily temperature.

6. Refer to Example 25.2. Verify that the sum of deviations of x_i from their mean ($\bar{x}$) is 0.
7. Mean of 9 observatrions was found to be 35. Later on, it was detected that an observation which was 81, was taken as 18 by mistake. Find the correct mean of the observations.
8. The mean marks obtained by 25 students in a class is 35 and that of 35 students is 25. Find the mean marks obtained by all the students.

25.1.2 Mean of Ungrouped Data

We will explain to find mean of ungrouped data through an example.

Find the mean of the marks (out of 15) obtained by 20 students.

12	10	5	8	15	5	2	8	10	5
10	12	12	2	5	2	8	10	5	10

This data is in the form of raw data. We can find mean of the data by using the formula (I), i.e., $\frac{\sum x_i}{n}$. But this process will be time consuming.

We can also find the mean of this data by first making a frequency table of the data and then applying the formula:

$$\text{mean} = \overline{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} \qquad \text{(II)}$$

where f_i is the frequency of the ith observation x_i.
Frequency table of the data is :

Marks (x_i)	Number of students (f_i)
2	4
5	5
8	3
10	5
12	2
15	1
	$\Sigma f_i = 20$

To find mean of this distribution, we first find $f_i x_i$, by multiplying each x_i with its corresponding frequency f_i and append a column of $f_i x_i$ in the frequency table as given below.

Marks (x_i)	Number of students (f_i)	$f_i x_i$
2	4	2×4 = 8
5	5	5× 5 = 25
8	3	3× 8 = 24
10	5	5 × 10 = 50
12	2	2 × 12 = 24
15	1	1 × 15 = 15
	$\Sigma f_i = 20$	$\Sigma f_i x_i = 146$

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{146}{20} = 7.3$$

Example 25.7: The following data represents the weekly wages (in rupees) of the employees:

Weekly wages (in ₹)	900	1000	1100	1200	1300	1400	1500
Number of employees	12	13	14	13	14	11	5

Find the mean weekly wages of the employees.

Solution: In the following table, entries in the first column are x_i's and entries in second columen are f_i's, i.e., corresponding frequencies. Recall that to find mean, we require the product of each x_i with corresponding frequency f_i. So, let us put them in a column as shown in the following table:

Weekly wages (in ₹) (x_i)	**Number of employees (f_i)**	**$f_i x_i$**
900	12	10800
1000	13	13000
1100	14	15400
1200	13	15600
1300	12	15600
1400	11	15400
1500	5	7500
	$\Sigma f_i = 80$	$\Sigma f_i \, x_i = 93300$

Using the Formula II,

$$\text{Mean weekly wages} = \frac{\sum f_i x_i}{\sum f_i} = ₹ \, \frac{93300}{80}$$

$$= ₹ \, 1166.25$$

Sometimes when the numerical values of x_i and f_i are large, finding the product f_i and x_i becomes tedius and time consuming.

We wish to find a **short-cut method**. Here, we choose an arbitrary constant a, also called the **assumed mean** and subtract it from each of the values x_i. The reduced value, $d_i = x_i - a$ is called the **deviation of x_i from a.**

Thus, $x_i = - a + d_i$

and $\quad f_i x_i = a f_i + f_i d_i$

$$\sum_{i=1}^{n} f_i x_i = \sum_{i=1}^{n} a f_i + \sum_{i=1}^{n} f_i d_i \quad \text{[Summing both sides over } i \text{ from } i \text{ to } r\text{]}$$

Hence $\quad \bar{x} = \sum f_i + \frac{1}{N}\sum f_i d_i$, where $\Sigma f_i = N$

$$\bar{x} = a + \frac{1}{N}\sum f_i d_i \qquad \text{(III)}$$

[since $\Sigma f_i = N$]

This meghod of calcualtion of mean is known as **Assumed Mean Method.**

In Example 25.7, the values x_i were very large. So the product $f_i x_i$ became tedious and time consuming. Let us find mean by **Assumed Mean Method.** Let us take assumed mean $a = 1200$

Weekly wages (in ₹) (x_i)	**Number of employees (f_i)**	**Deviations $d_i = x_i - 1200$**	**$f_i d_i$**
900	12	– 300	– 3600
1000	13	– 200	– 2600
1100	14	– 100	– 1400
1200	13	0	0
1300	12	100	+ 1200
1400	11	200	+ 2200
1500	5	300	+ 1500
	$\Sigma f_i = 80$		$\Sigma f_i d_i = -2700$

Using Formula III,

$$\text{Mean} \;= a + \frac{1}{N}\sum f_i d_i$$

$$= 1200 + \frac{1}{80}(-2700)$$

$$= 1200 - 33.75 = 1166.25$$

So, the mean weekly wages = ₹ 1166.25

Observe that the mean is the same whether it is calculated by Direct Method or by Assumed Mean Method.

Example 25.8: If the mean of the following data is 20.2, find the value of k

x_i	10	15	20	25	30
f_i	6	8	20	k	6

Solution: Mean $= \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{60+120+400+25k+180}{40+k}$

$$= \frac{760+25k}{40+k}$$

So, $\dfrac{760+25k}{40+k} = 20.2$ (Given)

or $760 + 25k = 20.2\ (40 + k)$

or $7600 + 250k = 8080 + 202k$

or $k = 10$

CHECK YOUR PROGRESS 25.2

1. Find the mean marks of the following distribution:

Marks	1	2	3	4	5	6	7	8	9	10
Frequency	1	3	5	9	14	18	16	9	3	2

2. Calcualte the mean for each of the following distributions:

(i)

x	6	10	15	18	22	27	30
f	12	36	54	72	62	42	22

(ii)

x	5	5.4	6.2	7.2	7.6	8.4	9.4
f	3	14	28	23	8	3	1

3. The wieghts (in kg) of 70 workers in a factory are given below. Find the mean weight of a worker.

Weight (in kg)	Number of workers
60	10
61	8
62	14
63	16
64	15
65	7

4. If the mean of following data is 17.45 determine the value of p:

x	15	16	17	18	19	20
f	3	8	10	p	5	4

25.1.3 Mean of Grouped Data

Consider the following grouped frequency distribution:

Daily wages (in ₹)	Number of workers
150-160	5
160-170	8
170-180	15
180-190	10
190-200	2

What we can infer from this table is that there are 5 workers earning daily somewhere from ₹ 150 to ₹ 160 (not included 160). We donot know what exactly the earnings of each of these 5 workers are

Therefore, to find mean of the grasped frequency distribution, we make the following assumptions:

Frequency in any class is centred at its class mark or mid point

Now, we can say that there are 5 workers earning a daily wage of ₹ $\frac{150+160}{2}$ = ₹ 155 each, 8 workers earning a daily wage of ₹ $\frac{160+170}{2}$ = ₹ 165, 15 workers aerning a daily wage of ₹ $\frac{170+160}{2}$ = ₹ 175 and so on. Now we can calculate mean of the given data as follows, using the Formula (II)

Daily wages (in ₹)	Number of workers (f_i)	Class marks (x_i)	$f_i x_i$
150-160	5	155	775
160-170	8	165	1320
170-180	15	175	2625
180-190	10	185	850
190-200	2	195	390
	$\Sigma f_i = 40$		$\Sigma f_i x_i = 6960$

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{6960}{40} = 174$$

So, the mean daily wage = ₹ 174

This method of calculate of the mean of grouped data is Direct Method.

We can also find the mean of grouped data by using Formula III, i.e., by **Assumed Mean Method** as follows:

We take assumed mean a = 175

Daily wages (in ₹)	Number of workers (f_i)	Class marks (x_i)	Deviations $d_i = x_i - 175$	$f_i d_i$
150-160	5	155	– 20	– 100
160-170	8	165	– 10	– 80
170-180	15	175	0	0
180-190	10	185	+ 10	100
190-200	2	195	+ 20	40
	$\Sigma f_i = 40$			$\Sigma f_i d_i = -40$

So, using Formula III,

$$\text{Mean} = a + \frac{1}{N}\sum f_i d_i$$

$$= 175 + \frac{1}{40}(-40)$$

$$= 175 - 1 = 174$$

Thus, the mean daily wage = ₹ 174.

Example 25.9: Find the mean for the following frequency distribution by (i) Direct Method, (ii) Assumed Mean Method.

Class	Frequency
20-40	9
40-60	11
60-80	14
80-100	6
100-120	8
120-140	15
140-160	12
Total	75

Solution: (i) **Direct Method**

Class	Frequency (f_i)	Class marks (x_i)	$f_i x_i$
20-40	9	30	270
40-60	11	50	550
60-80	14	70	980
80-100	6	90	540
100-120	8	110	880
120-140	15	130	1950
140-160	12	150	1800
	$\Sigma f_i = 75$		$\Sigma f_i x_i = 6970$

So, mean = $\dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{6970}{75} = 92.93$

(ii) **Assumed mean method**

Let us take assumed mean = $a = 90$

Class	Frequency (f_i)	Class marks (x_i)	Deviation $d_i = x_i - 90$	$f_i d_i$
20-40	9	30	– 60	– 540
40-60	11	50	– 40	– 440
60-80	14	70	– 20	– 280
80-100	6	90	0	0
100-120	8	110	+ 20	160
120-140	15	130	+ 40	600
140-160	12	150	+ 60	720
	N = $\Sigma f_i = 75$			$\Sigma f_i d_i = 220$

$$\text{Mean} = a + \frac{1}{\text{N}}\sum f_i d_i = 90 + \frac{220}{75} = 92.93$$

Note that mean comes out to be the same in both the methods.

In the table above, observe that the values in column 4 are all multiples of 20. So, if we divide these value by 20, we would get smaller numbers to multiply with f_i.

Note that, 20 is also the class size of each class.

So, let $u_i = \dfrac{x_i - a}{h}$, where a is the assumed mean and h is the class size.

Now we calculate u_i in this way and then $u_i f_i$ and can find mean of the data by using the formula

$$\text{Mean} = \bar{x} = a + \left(\frac{\sum f_i U_i}{\sum f_i}\right) \times h \qquad \text{(IV)}$$

Let us find mean of the data given in Example 25.9

Take $a = 90$. Here $h = 20$

Class	**Frequency** (f_i)	**Class marks** (x_i)	**Deviation** $d_i = x_i - 90$	$u_i = \frac{x_i - a}{h}$	$f_i u_i$
20-40	9	30	– 60	– 3	– 27
40-60	11	50	– 40	– 2	– 22
60-80	14	70	– 20	– 1	– 14
80-100	6	90	0	0	0
100-120	8	110	+ 20	1	8
120-140	15	130	+ 40	2	30
140-160	12	150	+ 60	3	36
	$\Sigma f_i = 75$				$\Sigma f_i u_i = 11$

Using the Formula (IV),

$$\text{Mean} = \bar{x} = a + \left(\frac{\sum f_i u_i}{\sum f_i}\right) \times h = 90 + \frac{11}{75} \times 20$$

$$= 90 + \frac{220}{75} = 92.93$$

Calculating mean by using Formula (IV) is known as **Step-deviation Method.**

Note that mean comes out to be the same by using Direct Method, Assumed Method or Step Deviation Method.

Example 25.10: Calcualte the mean daily wage from the following distribution by using Step deviation method.

Daily wages (in ₹)	150-160	160-70	170-180	180-190	190-200
Numbr of workers	5	8	15	10	2

Solution: We have already calculated the mean by using Direct Method and Assumed Method. Let us find mean by Step deviation Method.

Let us take $a = 175$. Here $h = 10$

Daily wages (in ₹)	**Number of workers (f_i)**	**Class marks (x_i)**	**Deviation $d_i = x_i - 90$**	$u_i = \frac{x_i - a}{h}$	$f_i u_i$
150-160	5	155	– 20	– 2	– 10
160-170	8	165	– 10	– 1	– 8
170-180	15	175	0	0	0
180-190	10	185	10	1	10
190-200	2	195	20	2	4
	$\Sigma f_i = 40$				$\Sigma f_i u_i = -4$

Using Formula (IV),

$$\text{Mean daily wages} = a + \left(\frac{\sum f_i u_i}{\sum f_i}\right) \times h = 175 + \frac{-4}{40} \times 10 = ₹\ 174$$

Note: Here again note that the mean is the same whether it is calculated using the Direct Method, Assumed mean Method or Step deviation Method.

CHECK YOUR PROGRESS 25.3

1. Following table shows marks obtained by 100 students in a mathematics test

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of students	12	15	25	25	17	6

Calculate mean marks of the students by using Direct Method.

2. The following is the distribution of bulbs kept in boxes:

Number of bulbs	50-52	52-54	54-56	56-58	58-60
Number of boxes	15	100	126	105	30

Find the mean number of bulbs kept in a box. Which method of finding the mean did you choose?

3. The weekly observations on cost of living index in a certain city for a particular year are given below:

Cost of living index	140-150	150-160	160-170	170-180	180-190	190-200
Number of weeks	5	8	20	9	6	4

Calculate mean weekly cost of living index by using Step deviation Method.

4. Find the mean of the following data by using (i) Assumed Mean Method and (ii) Step deviation Method.

Class	150-200	200-250	250-300	300-350	350-400
Frequency	48	32	35	20	10

25.2 MEDIAN

In an office there are 5 employees: a superviosor and 4 workers. The workers draw a salary of ₹ 5000, ₹ 6500, ₹ 7500 and ₹ 8000 per month while the supervisor gets ₹ 20000 per month.

$$\text{In this case mean (salary)} = ₹\,\frac{5000+6500+7500+8000+20000}{5}$$

$$= ₹\,\frac{47000}{5} = ₹\ 9400$$

Note that 4 out of 5 employees have their salaries much less than ₹ 9400. The mean salary ₹ 9400 does not given even an approximate estimate of any one of their salaries.

This is a weakness of the mean. It is affected by the **extreme** values of the observations in the data.

This weekness of mean drives us to look for another average which is unaffected by a few extreme values. Median is one such a measure of central tendency.

Median is a measure of central tendency which gives the value of the middle-most observation in the data when the data is arranged in ascending (or descending) order.

25.2.1 Median of Raw Data

Median of raw data is calculated as follows:

(i) Arrange the (numerical) data in an ascending (or descending) order

(ii) When the number of observations (n) is **odd**, the median is the value of $\left(\frac{n+1}{2}\right)$th observation.

(iii) When the number of observations (n) is **even**, the median is the mean of the $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th observations.

Let us illustrate this with the help of some examples.

Example 25.11: The weights (in kg) of 15 dogs are as follows:

9, 26, 10 , 22, 36, 13, 20, 20, 10, 21, 25, 16, 12, 14, 19

Find the median weight.

Solution: Let us arrange the data in the **ascending (or descending)** order:

$$9,\ 10,\ 10,\ 12,\ 13,\ 14,\ 16,\ \underset{\text{Median}}{\underset{\uparrow}{\textcircled{19}}},\ 20,\ 20,\ 21,\ 22,\ 25,\ 36$$

Here, number of observations = 15

So, the median will be $\left(\frac{n+1}{2}\right)$th, i.e., $\left(\frac{15+1}{2}\right)$th, i.e., 8th observation which is 19 kg.

Remark: The median weight 19 kg conveys the information that 50% dogs have weights less than 19 kg and another 50% have weights more then 19 kg.

Example 25.12: The points scored by a basket ball team in a series of matches are as follows:

16, 1, 6, 26, 14, 4, 13, 8, 9, 23, 47, 9, 7, 8, 17, 28

Find the median of the data.

Solution: Here number of observations = 16

So, the median will be the mean of $\left(\frac{16}{2}\right)$th and $\left(\frac{16}{2}+1\right)$th, i.e., mean of 6th and 9th observations, when the data is arranged in ascending (or descending) order as:

$$1,\quad 4,\quad 6,\quad 7,\quad 8,\quad 8,\quad \underset{\text{8th term}}{\underset{\uparrow}{9}},\quad 9,\quad \underset{\text{9th term}}{\underset{\uparrow}{13}},\quad 14,\quad 16,\quad 17,\quad 23,\quad 26,\quad 28,\quad 47$$

So, the median $= \frac{9+13}{2} = 11$

Remark: Here again the median 11 conveys the information that the values of 50% of the observations are less than 11 and the values of 50% of the observations are more than 11.

25.2.2 Median of Ungrouped Data

We illustrate caluculation of the median of ungrouped data through examples.

Example 25.13: Find the median of the following data, which gives the marks, out of 15, obtaine by 35 students in a mathematics test.

Marks obtained	3	5	6	11	15	14	13	7	12	10
Number of Students	4	6	5	7	1	3	2	3	3	1

Solution: First arrange marks in ascending order and prepare a frequency table as follows:

Marks obtained	3	5	6	7	10	11	12	13	14	15
Number of Students (frequency)	4	6	5	3	1	7	3	2	3	1

Here $n = 35$, which is odd. So, the median will be $\left(\frac{n+1}{2}\right)$th, i.e., $\left(\frac{35+1}{2}\right)$th, i.e., 18th observation.

To find value of 18th observation, we prepare cumulative frequency table as follows:

Marks obtained	Number of students	Cumulative frequency
3	4	4
5	6	10
6	5	15
7	3	18
10	1	19
11	7	26
12	3	29
13	2	31
14	3	34
15	1	35

From the table above, we see that 18th observation is 7

So, Median = 7

Example 25.14: Find the median of the following data:

Weight (in kg)	40	41	42	43	44	45	46	48
Number of students	2	5	7	8	13	26	6	3

Solution: Here $n = 2 + 5 + 7 + 8 + 13 + 26 + 6 + 3 = 70$, which is even, and weight are already arranged in the ascending order. Let us prepare cumulative frequency table of the data:

Weight (in kg)	Number of students (frequency)	Cumulative frequency	
40	2	2	
41	5	7	
42	7	14	
43	8	22	
44	13	35	35th observation
45	26	61	36th observation
46	6	67	
48	3	70	

Since n is even, so the median will be the mean of $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th observations, i.e., 35th and 36th observations. From the table, we see that

35 the observation is 44

and 36th observation is 45

So, Median $= \frac{44+45}{2} = 44.5$

CHECK YOUR PROGRESS 25.4

1. Following are the goals scored by a team in a series of 11 matches

   1, 0 , 3, 2, 4, 5, 2, 4, 4, 2, 5

   Determine the median score.

2. In a diagnostic test in mathematics given to 12 students, the following marks (out of 100) are recorded

   46, 52, 48, 39, 41, 62, 55, 53, 96, 39, 45, 99

   Calculate the median for this data.

3. A fair die is thrown 100 times and its outcomes are recorded as shown below:

Outcome	1	2	3	4	5	6
Frequency	17	15	16	18	16	18

Find the median outcome of the distributions.

4. For each of the following frequency distributions, find the median:

(a)

x_i	2	3	4	5	6	7
f_i	4	9	16	14	11	6

(b)

x_i	5	10	15	20	25	30	35	40
f_i	3	7	12	20	28	31	28	26

(c)

x_i	2.3	3	5.1	5.8	7.4	6.7	4.3
f_i	5	8	14	21	13	5	7

25.3 MODE

Look at the following example:

A company produces readymade shirts of different sizes. The company kept record of its sale for one week which is given below:

size (in cm)	90	95	100	105	110	115
Number of shirts	50	125	190	385	270	28

From the table, we see that the sales of shirts of size 105 cm is maximum. So, the company will go ahead producing this size in the largest number. Here, 105 is nothing but the **mode** of the data. Mode is also **one of the measures of central tendency.**

The observation that occurs most frequently in the data is called mode of the data.

In other words, the observation with maximum frequency is called mode of the data.

The readymade garments and shoe industries etc, make use of this measure of central tendency. Based on mode of the demand data, these industries decide which size of the product should be produced in large numbers to meet the market demand.

25.3.1 Mode of Raw Data

In case of raw data, it is easy to pick up mode by just looking at the data. Let us consider the following example:

Example 25.15: The number of goals scored by a football team in 12 matches are:

1, 2, 2, 3, 1, 2, 2, 4, 5, 3, 3, 4

What is the modal score?

Solution: Just by looking at the data, we find the frequency of 2 is 4 and is more than the frequency of all other scores.

So, mode of the data is 2, or modal score is 2.

Example 25.16: Find the mode of the data:

9, 6, 8, 9, 10, 7, 12, 15, 22, 15

Solution: Arranging the data in increasing order, we have

6, 7, 8, 9, 9, 10, 12, 15, 15, 22

We find that the both the observations 9 and 15 have the same maximum frequency 2. So, both are the modes of the data.

Remarks: 1. In this lesson, we will take up the data having a single mode only.

2. In the data, if each observation has the same frequency, then we say that the data does not have a mode.

25.3.2 Mode of Ungrouped Data

Let us illustrate finding of the mode of ungrouped data through an example

Example 25.17: Find the mode of the following data:

Weight (in kg)	40	41	42	43	44	45	46	48
Number of Students	2	6	8	9	10	22	13	5

Solution: From the table, we see that the weight 45 kg has maximum frequency 22 which means that maximum number of students have their weight 45 kg. So, the mode is 45 kg or the modal weight is 45 kg.

CHECK YOUR PROGRESS 25.5

1. Find the mode of the data:

   5, 10, 3, 7, 2, 9, 6, 2, 11, 2

2. The number of TV sets in each of 15 households are found as given below:

   2, 2, 4, 2, 1, 1, 1, 2, 1, 1, 3, 3, 1, 3, 0

   What is the mode of this data?

3. A die is thrown 100 times, giving the following results

Outcome	1	2	3	4	5	6
Frequency	15	16	16	15	17	20

Find the modal outcome from this distribution.

4. Following are the marks (out of 10) obtained by 80 students in a mathematics test:

Marks obtained	0	1	2	3	4	5	6	7	8	9	10
Number of students	5	2	3	5	9	11	15	16	9	3	2

Determine the modal marks.

LET US SUM UP

- Mean, median and mode are the measures of central tendency.

- Mean (Arithmetic average) of raw data is givne by $\overline{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$

  where $x_1, x_2 ...., x_n$ are n observations.

- Mean of ungrouped data is given by $\overline{x} = \dfrac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \dfrac{\sum f_i x_i}{N}$

  where f_i is the frequency of the ith observation x_i.

- Mean of ungrouped data can also be found by using the formula $\overline{x} = a + \dfrac{1}{N}\sum f_i d_i$

  where $d_i = x_i - a$, a being the assumed mean

Mean of grouped data

(i) To find mean of the grouped frequency distribution, we take the assumption:

Frequency in any class is centred at its class mark or mid point.

(ii) **Driect Method**

$$\overline{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

where x_i's are the class marks and f_i are the corresponding freqeucies of x_i's.

(iii) **Assumed Mean Method**

$$\overline{x} = a + \frac{\sum_{i=1}^{n} f_i d_i}{\mathrm{N}}$$

where a is the assumed mean, and $d_i = x_i - a$.

(iv) **Step deviation method**

$$\overline{x} = a + \left(\frac{\sum_{i=1}^{n} f_i u_i}{\sum_{i=1}^{n} f_i} \right) \times h$$

where a is the assumed mean, $u_i = \dfrac{x_i - a}{h}$ and h is the class size.

- Median is a measure of central tendency which gives the value of the middle most obseration in the data, when the data is arranged in ascending (or descending) order.
- **Median of raw data**

  (i) When the number of observations (n) is odd, the median is the value of $\left(\frac{n+1}{2}\right)$th observation.

  (ii) When the number of observations (n) is even, the median is the mean of the $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th observations.

- **Median of ungrouped data**

  Median of ungrouped data can be found from the cumulative frequency table (arranging data in increasing or decreasing order) using (i) and (ii) above.

- The value of observation with maximum frequency is called the mode of the data.

TERMINAL EXERCISE

1. Find the mean of first five prime numbers.

2. If the mean of 5, 7, 9, x, 11 and 12 is 9, find the value of x.

3. Following are the marks obtained by 9 students in a class

   51, 36, 63, 46, 38, 43, 52, 42 and 43

   (i) Find the mean marks of the students.

   (ii) What will be the mean marks if a student scoring 75 marks is also included in the class.

4. The mean marks of 10 students in a class is 70. The students are divided into two groups of 6 and 4 respectively. If the mean marks of the first group is 60, what will be the mean marks of the second group?

5. If the mean of the observations $x_1, x_2....,x_n$ is $\overline{x}$, show that $\sum_{i=1}^{n} (x_1 - \overline{x}) = 0$

6. There are 50 numbers. Each number is subtracted from 53 and the mean of the numbers so obtained is found to be – 3.5. Determine the mean of the given numbers.

7. Find the mean of the following data:

(a)

x_i	5	9	13	17	22	25
f_i	3	5	12	8	7	5

(b)

x_i	16	18	28	22	24	26
f_i	1	3	5	7	5	4

8. Find the mean of the following data

(a)

Classes	10-20	20-30	30-40	40-50	50-60	60-70
Frequencies	2	3	5	7	5	3

(b)

Classes	100-200	200-300	300-400	400-500	500-600	600-700
Frequencies	3	5	8	6	5	3

(c) The ages (in months) of a group of 50 students are as follows. Find the mean age.

Age	156-158	158-160	160-162	162-164	164-166	166-168
Number of students	2	4	8	16	14	6

9. Find the median of the following data:

   (a) 5, 12, 16, 18, 20, 25, 10

   (b) 6, 12, 9, 10, 16, 28, 25, 13, 15, 17

   (c) 15, 13, 8, 22, 29, 12, 14, 17, 6

10. The following data are arranged in ascending order and the median of the data is 60. Find the value of x.

    26, 29, 42, 53, x, $x + 2$, 70, 75, 82, 93

11. Find the median of the following data:

(a)

x_i	25	30	35	45	50	55	65	70	85
f_i	5	14	12	21	11	13	14	7	3

(b)

x_i	35	36	37	38	39	40	41	42
f_i	2	3	5	4	7	6	4	2

12. Find the mode of the following data:

    (a) 8, 5, 2, 5, 3, 5, 3, 1

    (b) 19, 18, 17, 16, 17, 15, 14, 15, 17, 9

13. Find the mode of the following data which gives life time (in hours) of 80 bulbs selected at random from a lot.

Life time (in hours)	300	500	700	900	1100
Number of bulbs	10	12	20	27	11

14. In the mean of the following data is 7, find the value of p:

x_i	4	p	6	7	9	11
f_i	2	4	6	10	6	2

15. For a selected group of people, an insurance company recorded the following data:

Age (in years)	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Number of deaths	2	12	55	95	71	42	16	7

Determine the mean of the data.

16. If the mean of the observations: $x + 1$, $x + 4$, $x + 5$, $x + 8$, $x + 11$ is 10, the mean of the last three observations is

    (A) 12.5 (B) 12.2 (C) 13.5 (D) 14.2

17. If each observation in the data is increased by 2, than their mean

    (A) remains the same   (B) becomes 2 times the original mean

    (C) is decreased by 2   (D) is increased by 2

18. Mode of the data: 15, 14, 19, 20, 14, 15, 14, 18, 14, 15, 17, 14, 18 is

    (A) 20   (B) 18   (C) 15   (D) 14

ANSWERS TO CHECK YOUR PROGRESS

25.1

1. $\sum_{i=1}^{n} x_i/n$ 2. 5.5   3. 86.33 kg

4. 142.8 cm   5. 25.68°C   7. 42

8. 29.17

25.2

1. 5.84   2. (i) 18.99   (ii) 6.57

3. 11.68   4. 10

25.3

1. 28.80   2. 55.19   3. 167.9   4. 244.66

25.4

1. 3   2. 50   3. 4

4. (a) 4   (b) 30   (c) 5.8

25.5

1. 2   2. 1   3. 6   4. 7

ANSWERS TO TERMINAL EXERCISE

1. 5.6   2. 10   3. (i) 46   (ii) 48.9

4. 85   6. 56.5   7. (a) 15.775   (b) 21.75

8. (a) 42.6   (b) 396.67   (c) 163 months (approx)

9. (a) 16   (b) 14   (c) 14

10. 59   11. (a) 45   (b) 24   12. (a) 5   (b) 17

13. 900   14. 5   15. 39.86 years   16. (A)

17. (D)   18. D

26

INTRODUCTION TO PROBABILITY

In our day to day life, we sometimes make the statements:

(i) It may rain today

(ii) Train is likely to be late

(iii) It is unlikely that bank made a mistake

(iv) Chances are high that the prices of pulses will go down in next september

(v) I doubt that he will win the race.

and so on.

The words **may**, **likely**, **unlikely**, **chances**, **doubt** etc. show that the event, we are talking about , is **not certain to occur**. It may or may not occur. Theory of probability is a branch of mathematics which has been developed to deal with situations involving uncertainty.

The theory had its beginning in the 16th century. It originated in the games of chance such as throwing of dice and now probability is used extensively in biology, economics, genetics, physics, sociology etc.

OBJECTIVES

After studying this lesson, you will be able to

- *understand the meaning of a random experiment;*
- *differentiate between outcomes and events of a random experiment;*
- *define probability P(E) of occurrence of an event E;*
- *determine* $P(\overline{E})$ *if P(E) is given;*
- *state that for the probability P(E),* $0 \le P(E) \le 1$;
- *apply the concept of probability in solving problems based on tossing a coin throwing a die, drawing a card from a well shuffled deck of playing cards, etc.*

EXPECTED BACKGROUND KNOWLEDGE

We assume that the learner is already familiar with

- the term associated with a coin, i.e., head or tail
- a die, face of a die, numbers on the faces of a die
- playing cards - number of cards in a deck, 4- suits of 13 cards-spades, hearts, diamonds and clubs. The cards in each suit such as king, queen, jack etc, are face cards.
- Concept of a ratio/fraction/decimal and operations on them.

26.1 RANDOM EXPERIMENT AND ITS OUTCOMES

Observe the following situations:

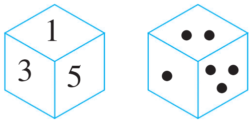
(1) Suppose we toss a coin. We know in advance that the coin can only land in one of two possible ways that is either Head (H) up or Tail (T) up.

(2) Suppose we throw a die. We know in advance that the die can only land in any one of six different ways showing up either 1, 2, 3, 4, 5 or 6.

(3) Suppose we plant 4 seeds and observe the number of seeds germinated after three days. The number of germinated seeds could be either 0, 1, 2, 3, or 4.

When we speak of a coin, we assume it to be fair in the sense that it is symmetrical so that there is no reason for it to land more often on a particular side.

A die is a well balanced cube with its six faces marked with numbers (or dots) from 1 to 6, one number on one face



In the above situations, tossing a coin, throwing a die, planting seeds and observing the germinated seeds, each is an example of a random experiment

In (1), the possible outcomes of the random experiment of tossing a coin are: Head and Tail.

In (2), the possible outcomes of the experiment are: 1, 2, 3, 4, 5, 6

In (3), the possible outcomes are: 0, 1, 2, 3, 4.

A random experiment always has more than one possible outcomes. When the experiment is performed only one outcome out of all possible outcomes comes out. Moreover, we can not predict any particular outcome before the experiment is performed. Repeating the experiment may lead to different outcomes.

Some more examples of random experiments are:

(i) drawing a ball from a bag containing identical balls of different colours without looking into the bag.

(ii) drawing a card at random from a well suffled deck of playing cards

we will now use the word experiment for random experiment throughout this lesson

A deck of playing cards consists of 52 cards which are divided into four suits of 13 cards each-spades (♠) hearts (♥) diamonds (♦) and clubs (♣). Spades and clubs are of black colour and others are of red colour. The cards in each suit are ace, king, queen, jack, 10, 9, 8, 7, 6, 5, 4, 3, and 2. Cards of kings, queens and jacks are called **face cards.**

CHECK YOUR PROGRESS 26.1

1. Which of the following is a random experiment?

   (i) Suppose you guess the answer to a multiple choice question having four options A, B, C, and D, in which only one is correct.

   (ii) The natural numbers 1 to 20 are written on separate slips (one number on one slip) and put in a bag. You draw one slip without looking into the bag.

   (iii) You drop a stone from a height

   (iv) Each of Hari and John chooses one of the numbers 1, 2, 3, independently.

2. What are the possible outcomes of random experiments in Q. 1 above?

26.2 PROBABILITY OF AN EVENT

Suppose a coin is tossed **at random**. We have two possible outcomes, Head (H) and Tail (T). We may assume that each outcome H or T is as likely to occur as the other. In other words, we say that the two outcomes H and T are **equally likely.**

Similarly, when we throw a die, it seems reasonable to assume that each of the six faces (or each of the outcomes 1, 2, 3, 4, 5, 6) is just as likely as any other to occur. In other words, we say that the six outcomes 1, 2, 3, 4, 5 and 6 are **equally likely**.

Tossed **at random** means that the coin is allowed to fall freely without any **bias** or interference.

Before we come to define probability of an event, let us understand the meaning of word **Event**. One or more outcomes constitute an event of an experiment. For example, in throwing a die an event could be "the die shows an even number". This event corresponds to three different outcomes 2, 4 or 6. However, the term event also often used to describe a single outcome. In case of tossing a coin, "the coin shows up a head" or "the coin shows up a tail" each is an event, the first one corresponds to the outcome H and the other to the outcome T. If we write the event E: "the coin shows up a head" If F : " the coin shows up a tail" E and F are called elementary events. An event having only one outcome of the experiment is called an elementary event.

The probability of an event E, written as P(E), is defined as

$$P(E) = \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes of the experiment}}$$

assuming the outcomes to be equally likely.

In this lesson, we will take up only those experiments which have equally likely outcomes.

To find probability of some events, let us consider following examples:

Example 26.1: A coin is tossed once. Find the probability of getting (i) a head, (ii) a tail.

Solution: Let E be the event "getting a head"

Possible outcomes of the experiment are : Head (H), Tail (T)

Number of possible outcomes = 2

Number of outcomes favourable to E = 1 (i.e., Head only)

So, probability to E = P(E) = P (getting a head) = P(head)

$$= \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes of the experiment}}$$

$$= \frac{1}{2}$$

Similarly, if F is the event "getting a tail", then

$$P(F) = \frac{1}{2}$$

Example 26.2: A die is thrown once. What is the probability of getting a number 3?

Solution: Let E be the event "getting a number 3".

Possible outcomes of the experiment are: 1, 2, 3, 4, 5, 6

Number of possible outcomes = 6

Number of outcomes favourable to E = 1 (i.e., 3)

$$\text{So, } P(E) = P(3) = \frac{1}{6}\begin{array}{l} \longleftarrow \text{Number of outcomes favourable to E} \\ \longleftarrow \text{Number of all possible outcomes} \end{array}$$

Example 26.3: A die is thrown once. Determine the probability of getting a number other than 3?

Solution: Let F be the event "getting a number other than 3" which means "getting a number 1, 2, 4, 5, 6".

Possible outcomes are : 1, 2, 3, 4, 5, 6

Number of possible outcomes = 6

Number of outcomes favourable to F = 5 (i.e., 1, 2, 4, 5, 6)

$$\text{So, } P(F) = \frac{5}{6}$$

Note that event F in Example 26.3 is the same as event 'not E' in Example 26.2.

Example 26.4: A ball is drawn at random from a bag containing 2 red balls, 3 blue balls and 4 black balls. What is the probability of this ball being of (i) red colour (ii) blue colour (iii) black colour (iv) not blue colour?

Solution:

(i) Let E be the event that the drawn ball is of red colour

$$\text{Number of possible outcomes of the experiment} = \underset{\text{(Red)}}{2} + \underset{\text{(Blue)}}{3} + \underset{\text{(black)}}{4} = 9$$

Number of outcomes favourable to E = 2

$$\text{So, } P(\text{Red ball}) = P(E) = \frac{2}{9}$$

(ii) Let F be the event that the ball drawn is of blue colour

$$\text{So, } P(\text{Blue ball}) = P(F) = \frac{3}{9} = \frac{1}{3}$$

(iii) Let G be the event that the ball drawn is of black colour

$$\text{So } P\ (\text{Black ball}) = P(G) = \frac{4}{9}$$

(iv) Let H be the event that the ball drawn is not of blue colour.

Here "ball of not blue colour" means "ball of red or black colour)

Therefore, number of outcomes favourable to H = 2 + 4 = 6

$$\text{So, } P(H) = \frac{6}{9} = \frac{2}{3}$$

Example 26.5: A card is drawn from a well shuffled deck of 52 playing cards. Find the probability that it is of (i) red colour (ii) black colour

Solution: (i) Let E be the event that the card drawn is of red colour.

Number of cards of red colour = 13 + 13 = 26 (diamonds and hearts)

So, the number of favourable outcomes to E = 26

Total number of cards = 52

$$\text{Thus, } P(E) = \frac{26}{52} = \frac{1}{2}$$

(ii) Let F be the event that the card drawn is of black colour. Number of cards of black colour = 13 + 13 = 26

$$\text{So } P(F) = \frac{26}{52} = \frac{1}{2}$$

Example 26.6: A die is thrown once. What is the probability of getting a number (i) less than 7? (ii) greater than 7?

Solution: (i) Let E be the event "number is less than 7".

Number of favourable outcomes to E = 6 (since every face of a die is marked with a number less than 7)

$$\text{So, } P(E) = \frac{6}{6} = 1$$

(ii) Let F be the event "number is more than 7"

Number of outcomes favourable to F = 0 (since no face of a die is marked with a number more than 7)

$$\text{So, } P(F) = \frac{0}{6} = 0$$

CHECK YOUR PROGRESS 26.2

1. Find the probability of getting a number 5 in a single throw of a die.
2. A die is tossed once. What is the probability that it shows:

   (i) a number 7?

   (ii) a number less than 5?
3. From a pack of 52 cards, a card is drawn at random. What is the probability of this card to be a king?
4. An integer is chosen between 0 and 20. What is the probability that this chosen integer is a prime number?
5. A bag contains 3 red and 3 white balls. A ball is drawn from the bag without looking into it. What is the probability of this ball to be of (i) red colour (ii) white colour?
6. 3 males and 4 females appear for an interview, of which one candidate is to be selected. Find the probability of selection of a (i) male candidate (ii) female candidate.

26.3 MORE ABOUT PROBABILITY

Probability has many interesting properties. We shall explain these through some examples:

Observation 1: In Example 26.6 above,

(a) Event E is sure to occur, since every number on a die is always less than 7. Such an event which is sure to occur is called a sure (or certain) event. Probability of a sure event is taken as 1.

(b) Event F is impossible to occur, since no number on a die is greater than 7. Such an event which is impossible to occur is called an impossible event. Probability of an impossible event is taken as 0.

(c) From the definition of probability of an event E, P(E) cannot be greater than 1, since numerator being the number of outcomes favourable to E cannot be greater than the denominator (number of all possible outcomes).

(d) both the numerator and denominator are natural numbers, so P(E) cannot be negative.

In view of (a), (b), (c) and (d), P(E) takes any value from 0 to 1, i.e.,

$$0 \le P(E) \le 1$$

Observation 2: In Example 26.1, both the events getting a head (H) and getting a tail (T) are elementary events and

$$P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$$

Similarly, in the experiment of throwing a die once, elementary events are getting the numbers 1, 2, 3, 4, 5 or 6 and also

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

Observe that the sum of the probabilities of all the elementary events of an experiment is one.

Observation 3: From Examples 26.2 and 26.3,

Probability of getting 3 + Probability of getting a number other than 3 $= \frac{1}{6} + \frac{5}{6} = 1$

i.e. $P(3) + P(\text{not } 3) = 1$

or $P(E) + P(\text{not } E) = 1$ ...(1)

Similarly, in Example 26.1

P(getting a head) = $P(E) = \frac{1}{2}$

P(getting a tail) = $P(F) = \frac{1}{2}$

So, $P(E) + P(F) = \frac{1}{2} + \frac{1}{2} = 1$

So, $P(E) + P(\text{not } E) = 1$ [getting a tail means getting no head] ...(2)

From (1) and (2), we see that for any event E,

$$P(E) + P(\text{not } E) = 1$$

or $P(E) + P(\overline{E}) = 1$ [We denote 'not E' by $\overline{E}$]

Event $\overline{E}$ is called **complement** of the event E or E and $\overline{E}$ are called **complementary events**.

In general, it is true that for an event E

$$P(E) + P(\overline{E}) = 1$$

Example 26.7: If $P(E) = \frac{2}{7}$, what is the probability of 'not E'?

Solution: $P(E) + P(\text{not } E) = 1$

So, $\quad P(\text{not } E) = 1 - P(E) = 1 - \frac{2}{7} = \frac{5}{7}$

Example 26.8: What is the probability that the number 5 will not come up in single throw of a die?

Solution: Let E be the event "number 5 comes up on the die"

Then we have to find P(not E) i.e. $P(\overline{E})$

Now $\quad P(E) = \frac{1}{6}$

So, $\quad P(\overline{E}) == 1 - \frac{1}{6} = \frac{5}{6}$

Example 26.9: A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability that this card is a face card.

Solution: Number of all possible outcomes = 52

Number of outcomes favourable to the Event E "a face card" = 3 × 4 = 12

[Kings, queens, and jacks are face cards]

So, $P(\text{a face card}) = \frac{12}{52} = \frac{3}{13}$

Example 26.10: A coin is tossed two times. What is the probability of getting a head each time?

Solution: Let us write H for Head and T for Tail.

In this expreiment, the possible outcomes will be: HH, HT, TH, TT

HH means Head on both the tosses

HT means Head on 1st toss and Tail on 2nd toss.

TH means Tail on 1st toss and Head on 2nd toss.

TT means Tail on both the tosses.

So, the number of possible outcomes = 4

Let E be the event "getting head each time". This means getting head in both the tosses, i.e. HH.

Therefore, $P(HH) = \frac{1}{4}$

Example 26.11: 10 defective rings are accidentally mixed with 100 good ones in a lot. It is not possible to just look at a ring and tell whether or not it is defective. One ring is drawn at random from this lot. What is the probability of this ring to be a good one?

Solution: Number of all possible outcomes = 10 + 100 = 110

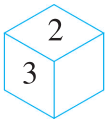
Number of outcomes favourable to the event E " ring is good one" = 100

$$\text{So, } P(E) = \frac{100}{110} = \frac{10}{11}$$

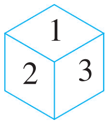
Example 26.12: Two dice, one of black colour and other of blue colour, are thrown at the same time. Write down all the possible outcomes. What is the probability that same number appear on both the dice?

Solution: All the possible outcomes are as given below, where the first number in the bracket is the number appearing on black coloured die and the other number is on blue die.

Blue coloured die



Black coloured die



	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

So, the number of possible outcomes = 6 × 6 = 36

The outcomes favourable to the event E : "Same number appears on both dice". are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6).

So, the number of outcomes favourable to E = 6.

$$\text{Hence, } P(E) = \frac{6}{36} = \frac{1}{6}$$

CHECK YOUR PROGRESS 26.3

1. Complete the following statements by filling in blank spaces:

   (a) The probability of an event is always greater than or equal to _______ but less than or equal to _______

(b) The probability of an event that is certain to occur is ________. Such an event is called ________

(c) The probability of an event which cannot occur is _________. Such an event is __________

(d) The sum of probabilities of two complementary events is _________

(e) The sum of probabilities of all the elementary events of an experiment is ______

2. A die is thrown once. What is the probability of getting

(a) an even number

(b) an odd number

(c) a prime number

3. In Question 2 above, verify:

P(an even number) + P(an odd number) = 1

4. A die is thrown once. Find the probability of getting

(i) a number less than 4

(ii) a number greater than or equal to 4

(iii) a composite number

(iv) a number which is not composite

5. If P(E) = 0.88, what is the probability of 'not E'?

6. If P($\overline{E}$) = 0, find P(E).

7. A card is drawn from a well shuffled deck of 52 playing cards. Find the probability that this card will be

(i) a red card (ii) a black card

(iii) a red queen (iv) an ace of black colour

(v) a jack of spade (vi) a king of club

(vii) not a face card (viii) not a jack of diamonds

8. A bag contains 15 white balls and 10 blue balls. A ball is drawn at random from the bag. What is the probability of drawing

(i) a ball of not blue colour (ii) a ball not of white colour

9. In a bag there are 3 red, 4 green and 2 blue marbles. If a marble is picked up at random what is the probability that it is

(i) not green? (ii) not red? (iii) not blue?

10. Two different coins are tossed at the same time. Write down all possible outcomes. What is the probability of getting head on one and tail on the other coin?

11. In Question 10 above, what is the probability that both the coins show tails?

12. Two dice are thrown simultaneously and the sum of the numbers appearing on them is noted. What is the probability that the sum is

    (i) 7 (ii) 8 (iii) 9 (iv) 10 (v) 12

13. 8 defective toys are accidentally mixed with 92 good ones in a lot of identical toys. One toy is drawn at random from this lot. What is the probability that this toy is defective?

LET US SUM UP

- A random experiment is one which has more than one outcomes and whose outcome is not exactly predictable in advance before performig the experiment.
- One or more outcomes of an experiment constitute an event.
- An event having only one outcome of the experiment is called an elementary event.
- Probability of an event E, P(E), is defined as

  $$\text{P(E)} = \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes of the experiment}},$$ When the outcomes are equally likely
- $0 \leq \text{P(E)} \leq 1$
- If P(E) = 0, E is called an impossible event. If P(E) = 1, E is called a sure or certain event.
- The sum of the probabilities of all the elementary events of an experiment is 1.
- $\text{P(E)} + \text{P}(\overline{\text{E}}) = 1$, where E and $\overline{\text{E}}$ are complementary events.

TERMINAL EXERCISE

1. Which of the following statements are True (T) and which are False (F):

   (i) Probability of an event can be 1.01

   (ii) If P(E) = 0.08, then $\text{P}(\overline{\text{E}}) = 0.02$

(iii) Probability of an impossible event is 1

(iv) For an event E, $0 \leq P(E) \leq 1$

(v) $P(\overline{E}) = 1 + P(E)$

2. A card is drawn from a well shuffled deck of 52 cards. What is the probability that this card is a face card of red colour?

3. Two coins are tossed at the same time. What is the probability of getting atleast one head? [Hint: P(atleast one head) = 1 – P(no head)]

4. A die is tossed two times and the number appearing on the die is noted each time. What is the probability that the sum of two numbers so obtained is

   (i) greater than 12? (ii) less than 12?

   (iii) greater than 11? (iv) greater than 2?

5. Refer to Question 4 above. What is the probability that the product of two number is 12?

6. Refer to Question 4 above. What is the probability that the difference of two numbers is 2?

7. A bag contains 15 red balls and some green balls. If the probability of drawing a green ball is $\frac{1}{6}$, find the number of green balls.

8. Which of the following can not be the probability of an event?

   (A) $\frac{2}{3}$ (B) – 1.01 (C) 12% (D) 0.3

9. In a single throw of two dice, the probability of getting the sum 2 is

   (A) $\frac{1}{9}$ (B) $\frac{1}{18}$ (C) $\frac{1}{36}$ (D) $\frac{35}{36}$

10. In a simultaneous toss of two coins, the probability of getting one head and one tail is

   (A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$

ANSWERS TO CHECK YOUR PROGRESS

26.1

1. (i), (ii) and (iii) 2. (i) A, B, C, D (ii) 1, 2, 3, ..., 20

(iii) (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)

26.2

1. $\frac{1}{6}$ 2. (i) 0 (ii) $\frac{2}{3}$ 3. $\frac{1}{13}$ 4. $\frac{8}{19}$

5. (i) $\frac{3}{8}$ (ii) $\frac{5}{8}$ 6. (i) $\frac{3}{7}$ (ii) $\frac{4}{7}$

26.3

1. (a) 0, 1 (b) 1, sure or certain event (c) 0, impossible event
(d) 1 (e) 1

2. (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{2}$

4. (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{3}$ (iv) $\frac{2}{3}$

5. 0.12 6. 1

7. (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{26}$ (iv) $\frac{1}{26}$ (v) $\frac{1}{52}$ (vi) $\frac{1}{52}$
(vii) $\frac{10}{13}$ (viii) $\frac{51}{52}$

8. (i) $\frac{3}{5}$ (ii) $\frac{2}{5}$

9. (i) $\frac{5}{9}$ (ii) $\frac{2}{3}$ (iii) $\frac{7}{9}$

10. HH, HT, TH, TT, $\frac{1}{2}$

11. $\frac{1}{4}$ 12. (i) $\frac{1}{6}$ (ii) $\frac{5}{36}$ (iii) $\frac{1}{9}$ (iv) $\frac{1}{12}$ (v) $\frac{1}{36}$

13. $\frac{2}{25}$

ANSWERS TO TERMINAL EXERCISE

1. (i) F (ii) T (iii) F (iv) T (v) F

2. $\frac{3}{26}$ 3. $\frac{3}{4}$ 4. (i) 0 (ii) 1 (iii) $\frac{1}{36}$ (iv) 1

5. $\frac{1}{9}$ 6. $\frac{2}{9}$ 7. 3 8. (B) 9. (C) 10. (C)